UDC 624.04 DOI: 10.31650/2786-6696-2022-2-37-43

ANALYTICAL METHOD FOR CALCULATING ANNULAR PLATES ON A VARIABLE ELASTIC BASE

[1]**Surianinov M.H.,** DSc., Professor,
sng@ogasa.org.ua, ORCID: 0000-0003-2592-5221
[1]**Krutii Y.S.,** DSc., Professor,
yurii.krutii@gmail.com, ORCID: 0000-0001-7105-3087
[1]**Karnaukhova A.S.,** PhD, Associate Professor,
karnauhovaanna@ukr.net, ORCID: 0000-0002-3542-4300
[1]**Klymenko O.M.,** Graduate student,
bolshajaakula@gmail.com, ORCID: 0000-0002-7445-6057
[1]*Odesa State Academy of Civil Engineering and Architecture*
Didrikhsona str, 4, Odesa, 65029, Ukraine

Abstract. The paper considers the application of the method of direct integration to calculations of annular plates and slabs on a continuous variable elastic base. Ring-shaped plates with variable geometric and mechanical parameters are increasingly used. Not only the elastic base, but also the plate thickness and cylindrical stiffness can be variable parameters here. The need for an analytical method for calculating such structures raises no doubts, since it makes it possible to evaluate the accuracy of finite-element analysis. To date, there are no proposals in the literature regarding a general analytical method for the calculation of annular plates on a variable elastic base.

A detailed description of the algorithm of the direct integration method is not given in the paper, and all the calculation formulas for the annular plate are taken from the authors' already published article. The results of numerical implementation of this algorithm for specific examples are considered: a concrete plate, which is rigidly pinch on the inner contour, and its outer contour is free, and a steel plate, which is rigidly pinch on the outer contour, and its inner contour is free.

To estimate the results of calculation by the author's method, computer modeling of the considered structures in PC LIRA-SAPR and their calculations by the finite-element method have been executed.

The foundation reaction is described by Winkler model with a variable bedding factor. In the first case a bed factor is assumed constant, and in the second case it changes under the linear law. Calculations have shown that discrepancy between deflections calculated by the finite-element method and the author's method does not exceed 1 %, and the results of radial and circumferential moments calculation differ more considerably, amounting to 10 %. The authors explain this difference by the inaccuracy of the numerical analysis associated with the semi-automatic method of constructing a finite-element mesh, which should be made finer. The densification of the mesh in the manual mode of its partitioning significantly reduces the discrepancy between the results of calculating the deflections, radial and circumferential bending moments by the finite-element method and the author's method.

Keywords: direct integration method, annular slab, elastic foundation, Winkler model, variable bedding factor, finite element method, PC LIRA-SAPR.

Introduction. Ring-shaped plates with variable geometric and mechanical parameters are increasingly being used. Variable parameters here may be not only the elastic basis, but also the plate thickness and cylindrical stiffness.

However, in spite of the considerable number of studies devoted to plate bending, their absolute majority is related to plates which have constant parameters, and only some authors consider variable parameters. Studies of plates of variable stiffness (thickness) predominate among such works. A number of questions in the theory and practical calculations of variable thickness plates remain unresolved. Of great interest is the development of analytical methods of calculations.

Recent research analysis. A relatively small number of works [1-3] are devoted to the study of the neo-symmetric bending of a flexible plate on an elastic basis. The approach based on the symbiosis of the orthogonal polynomial and Ritz methods proposed in [2] by S. V. Bosakov. The results were obtained for the case of a concentrated force eccentrically applied to a circular plate lying on an elastic half-space. Generalization of this solution makes it possible to obtain a solution for an arbitrary load acting on a circular or circular plate.

The authors of [4] believe that the two-parameter elastic base model demonstrates a more realistic ground reaction behavior than the one-parameter Winkler model. Circular plates are modeled as a set of individual beam elements connected together in the radial and tangential directions. As a numerical method, the grid method has been applied.

In the thesis of E. R. Telegulova [5] obtained an analytical solution of the problem of determining the ultimate load for a circular in plan reinforced concrete plate, which lies on an elastic basis (Fuss-Winkler model).

B. B. Grosman in [6] presents the results of studies of antisymmetric deformation of circular plates of constant cross-section, made of orthotropic and isotropic material, which lie on an elastic basis, the properties of which are described by the Winkler model.

From the works of foreign authors, we note the article [7], the basis of which was a design work that requires less rigid treatment of circular plates than it is possible on the basis of standard formulas. The issues of dynamics are considered. The transfer matrix method used here allows the plate thickness and pressure to be any function of the radius, and the boundary conditions can be any. The natural frequencies and waveforms can be determined by including inertia conditions in the matrices. The limitation is that the method covers only axisymmetric deformations and waveforms.

The subject of the paper [8] is a universal software for reinforced concrete ring and circular slabs reinforced in radial and tangential directions. Such slabs are used for roofing and ceiling structures of buildings. The software can be used to design and calculate axisymmetrically loaded circular and circular reinforced concrete slabs with arbitrary supports and spans. Based on the user's choice, the software offers solutions that not only comply with all design principles, including limit state calculations.

Above-ground and suspended tanks are used to store water, liquid oil, petroleum products and similar liquids [9]. The strength analysis of such tanks is about the same regardless of the chemical nature of the product. All tanks are designed as crack-free structures to eliminate any leakage. In this study, an attempt was made to obtain the optimal design of reinforced concrete water tanks at the given characteristics, ensuring the efficiency of geometric forms in terms of functional requirements, assessing the economic efficiency of each of the options. The practical result of the work was the development of tank design program in spreadsheets Microsoft Excel.

Finite element analysis of annular and circular plate is carried out in [10]. Different ratios of inner radius to outer radius are considered to investigate effective geometric characteristics of circular plates. Critical moments and deflections in both annular and circular plates are determined. A finite element analysis of the plates using the ANSYS software was performed to evaluate the results.

The purpose of this work is to analytically and numerically investigate the bending of circular plates on a variable elastic base.

Research materials and methods. The analytical method of direct integration and computer modeling in PC LIRA-SAPR [11-12] with subsequent calculations by the finite element method are used. Two examples are considered: a concrete slab that is rigidly pinch on the inner contour and its outer contour is free and a steel slab that is rigidly pinch on the outer contour and its inner contour is free.

Research results. Consider a circular plate of constant cylindrical stiffness D, lying on a variable elastic base and under the action of a continuously distributed arbitrary transverse load (Fig. 1). Let us denote by a and b the radii of the outer and inner circles of the plate, by r the radial coordinate ($0 \le r \le a$), by $q(r)$ the arbitrary transverse load, and by $R(r)$ the elastic base reaction. There will be internal forces in the plate: radial bending moment M_r, circumferential bending moment M_θ and radial transverse force Q_r (Fig. 1).

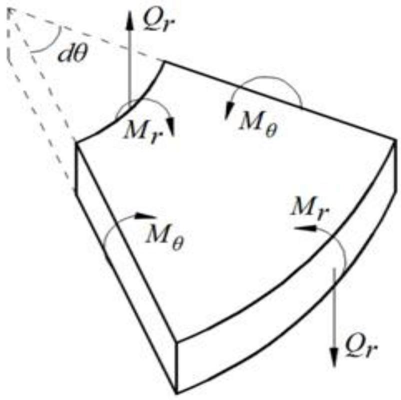

Fig. 1. Forces in the ring plate

Bedding ratio $k(r)$ and load $q(r)$ we will represent in a form [13]:

$$k(r) = k_0\left(A_0 + A_1\left(\frac{r}{a}\right) + A_2\left(\frac{r}{a}\right)^2 + ... + A_s\left(\frac{r}{a}\right)^s\right); \tag{1}$$

$$q(r) = q_0\left(B_0 + B_1\left(\frac{r}{a}\right) + B_2\left(\frac{r}{a}\right)^2 + ... + B_p\left(\frac{r}{a}\right)^p\right), \tag{2}$$

where k_0, q_0 – values of bedding factor and load at some characteristic point of the plate.

When a circular plate bends, its deflection function $w(r)$ is defined as [13]:

$$w(r) = \frac{q_0 a^4}{D} W(r); \tag{3}$$

$$W(r) = \lambda_1 X_1(r) + \lambda_2 X_2(r) + \lambda_3 Y_1(r) + \lambda_4 Y_2(r) + X_3(r), \tag{4}$$

where $W(r)$ – dimensionless function, and λ_n – arbitrary dimensionless constants.

$$Y_n(r) = X_n(r)\ln\frac{r}{a} + Z_n(r) \quad (n = 1, 2). \tag{5}$$

For rotation angle and internal forces M_r, M_θ, Q_r we will have [13]:

$$\frac{dw}{dr} = \frac{q_0 a^3}{D}\tilde{W}(r); \qquad \frac{d^2 w}{dr^2} = \frac{q_0 a^2}{D}\widehat{W}(r); \qquad \frac{d^3 w}{dr^3} = \frac{q_0 a}{D}\hat{W}(r); \tag{6}$$

$$M_r = -q_0 a^2\left(\widehat{W}(r) + \mu\frac{a}{r}\tilde{W}(r)\right); \tag{7}$$

$$M_\theta = -q_0 a^2\left(\mu\widehat{W}(r) + \frac{a}{r}\tilde{W}(r)\right); \tag{8}$$

$$Q_r = -q_0 a\left(\hat{W}(r) + \frac{a}{r}\widehat{W}(r) - \left(\frac{a}{r}\right)^2\tilde{W}(r)\right). \tag{9}$$

It follows from the formulas that the determination of deflections and internal forces in the plate, one way or another, is reduced to the calculation of the values of the functions $X_n(r), (n = 1, 2, 3), Z_n(r), (n = 1, 2)$, as well as their dimensionless derivatives $\tilde{X}_n(r),\ \widehat{X}_n(r),\ \hat{X}_n(r), (n = 1, 2, 3);\ Z_n(r),\ Z_n(r),\ Z_n(r), (n = 1, 2)$. To represent these functions, series are used, which is described in detail in [14].

Consider examples of calculations, varying the boundary conditions and the law of change in the reaction of an elastic foundation.

Example 1. Consider a concrete slab ($E = 1,5 \cdot 10^7 kPa,\ \mu = 1/6$) of thickness of $h = 0,12m$, outer radius $a = 1,8m$, inner radius $b = 0,9m$, which is under the action of a uniformly distributed constant load $q = 80kPa$. The plate is rigidly pinched on the inner contour and the outer contour is free.

The bedding factor (Fig. 2) is constant: $k(r) = const = 5000kH / m^3$.

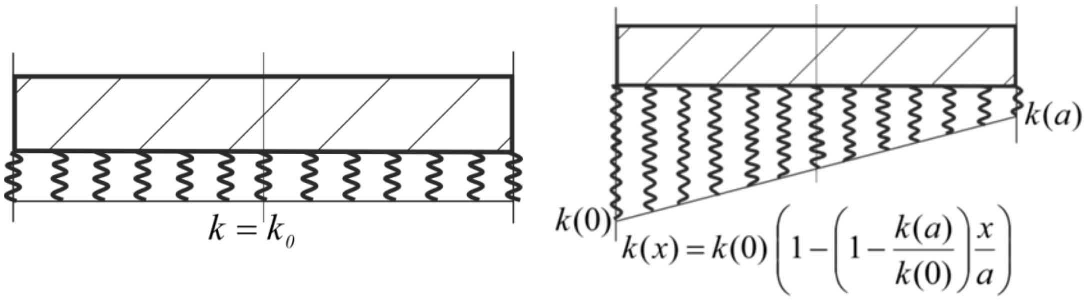

Fig. 2. Laws of change in the bed coefficient

The results of calculations by the author's method (AM) and by the finite element method (FEM) in PC LIRA-SAPR are shown in Table 1, and their graphical interpretation is shown in Fig. 3.

Table 1 – Concrete slab calculation results

r, м	w, м		Discrepancy (%)	M_r		Discrepancy (%)	M_θ		Discrepancy (%)
	FEM	AM		FEM	AM		FEM	AM	
0,9	0,0000	0,0000	0,000	-38,953	-39,086	0,341	-6,4918	-6,5143	0,346
0,9429	0,0209	0,0210	0,351	-33,679	-33,785	0,313	-7,1959	-7,2321	0,500
0,9857	0,0795	0,0798	0,355	-29,019	-29,110	0,312	-7,6406	-7,6944	0,699
1,0286	0,1703	0,1709	0,352	-24,902	-24,980	0,312	-7,8864	-7,9583	0,904
1,0714	0,2886	0,2896	0,352	-21,236	-21,328	0,430	-7,9776	-8,0689	1,132
1,1143	0,4299	0,4314	0,354	-17,993	-18,096	0,568	-7,9512	-8,0619	1,373
1,1571	0,5906	0,5928	0,373	-15,130	-15,241	0,731	-7,8355	-7,9657	1,635
1,2	0,7673	0,7703	0,384	-12,602	-12,720	0,925	-7,6528	-7,8033	1,928
1,2429	0,9573	0,9611	0,400	-10,380	-10,502	1,159	-7,4224	-7,5930	2,247
1,2857	1,1579	1,1626	0,405	-8,4350	-8,5580	1,440	-7,1571	-7,3496	2,619
1,3286	1,3667	1,3726	0,432	-6,7404	-6,8628	1,784	-6,8694	-7,0853	3,047
1,3714	1,5822	1,5893	0,449	-5,2763	-5,3953	2,206	-6,5680	-6,8098	3,550
1,4143	1,8024	1,8110	0,472	-4,0236	-4,1366	2,732	-6,2594	-6,5311	4,160
1,4571	2,0252	2,0361	0,533	-2,9660	-3,0703	3,397	-5,9483	-6,2558	4,915
1,5	2,2512	2,2637	0,552	-2,0888	-2,1817	4,260	-5,6377	-5,9890	5,866
1,5429	2,4772	2,4926	0,619	-1,3934	-1,4578	4,417	-5,3838	-5,7352	6,127
1,5857	2,7026	2,7222	0,721	-0,8425	-0,8872	5,038	-5,1196	-5,4977	6,877
1,6286	2,9255	2,9517	0,887	-0,4295	-0,4592	6,469	-4,8844	-5,2794	7,482
1,6714	3,1476	3,1808	1,044	-0,1520	-0,1647	7,701	-4,6352	-5,0825	8,800
1,7143	3,3689	3,4090	1,177	0,0046	0,0050	8,605	-4,5383	-4,9088	7,547
1,7571	3,5882	3,6364	1,324	0,0540	0,0575	6,154	-4,3865	-4,7596	7,838
1,8	3,8014	3,8627	1,587	0,0000	0,0000	0,000	-4,2246	-4,6361	8,877

The analysis of Table 1 shows that the values of deflections practically coincide during the calculation of the ring plate (under the above conditions) by the author's method and by the finite element method in PC LIRA-SAPR, and the discrepancy of the bending moment values reaches 8 %.

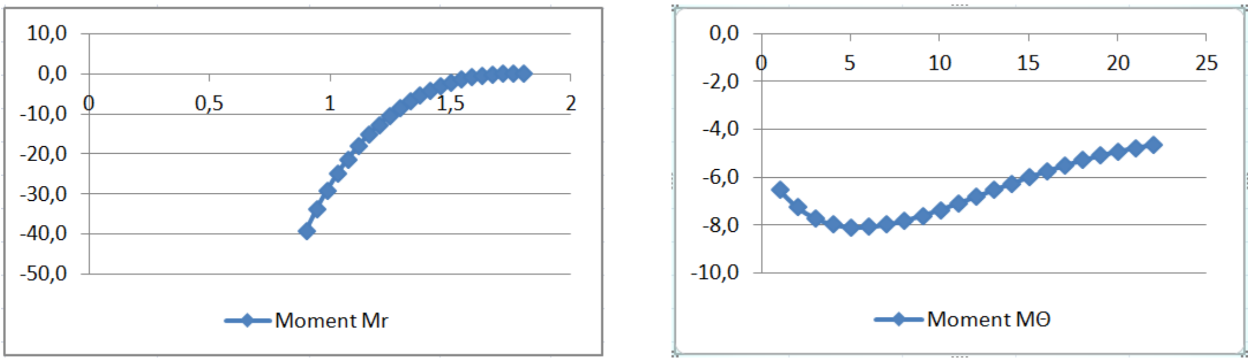

Fig. 3. Radial and circumferential bending moments

Example 2. Consider a steel plate ($E = 2,0 \cdot 10^8 kPa,\ \mu = 0,33$) of thickness of $h = 0,03m$, outer radius $a = 0,9m$, inner radius $b = 0,45m$, which is under the action of a uniformly distributed constant load $q = 30kPa$. The plate is rigidly pinched along the outer contour and the inner contour is free. The bedding coefficient varies according to the linear law: $k(b) = 4000kH / m^3$, $k(a) = 5000kH / m^3$ (Fig. 2). The results of calculations by the author's method (AM) and by the finite element method (FEM) in PC LIRA-SAPR are shown in Table 2, and their graphical interpretation is shown in Fig. 4.

Table 2 – Steel plate calculation results

r, м	w, м		Discre-pancy (%)	M_r		Discre-pancy (%)	M_θ		Discre-pancy (%)
	FEM	AM		FEM	AM		FEM	AM	
0,45	0,0113	0,0114	0,301	0,0000	0,0000	0,000	0,0102	0,0104	1,583
0,4714	0,0111	0,0112	0,457	0,0036	0,0037	1,521	0,0107	0,0110	2,233
0,4929	0,0109	0,0110	0,317	0,1125	0,0128	2,329	0,0133	0,0138	3,361
0,5143	0,0108	0,0108	0,337	0,0247	0,0255	3,167	0,0174	0,0182	4,486
0,5357	0,0105	0,0105	0,369	0,0388	0,0405	4,077	0,0226	0,0239	5,614
0,5571	0,0103	0,0103	0,412	0,0535	0,0564	5,111	0,0282	0,0302	6,773
0,5786	0,0100	0,0100	0,364	0,0672	0,0718	6,357	0,0340	0,0369	7,974
0,6	0,0096	0,0096	0,276	0,0790	0,0858	7,940	0,0610	0,0434	8,252
0,6214	0,0091	0,0091	0,328	0,0882	0,0970	9,110	0,0398	0,0493	8,644
0,6429	0,0086	0,0086	0,398	0,0945	0,1043	9,379	0,0490	0,0540	9,205
0,6643	0,0080	0,0080	0,487	0,0964	0,1067	9,673	0,0513	0,0570	10,07
0,6857	0,0073	0,0073	0,601	0,0932	0,1032	9,642	0,0520	0,0578	10,06
0,7071	0,0066	0,0066	0,742	0,0832	0,0927	10,19	0,0501	0,0560	10,47
0,7286	0,0057	0,0057	0,909	0,0668	0,0744	10,23	0,0461	0,0511	9,729
0,75	0,0048	0,0048	0,544	0,0427	0,0474	9,887	0,0391	0,0427	8,389
0,7714	0,0039	0,0039	0,369	0,0101	0,0112	9,405	0,0276	0,0304	9,191
0,7929	0,0030	0,0030	0,303	-0,0317	-0,0349	9,074	0,0126	0,0138	8,525
0,8143	0,0021	0,0021	0,272	-0,0835	-0,0912	8,410	-0,0066	-0,0072	8,599
0,8357	0,0013	0,0013	0,201	-0,1459	-0,1579	7,585	-0,0300	-0,0331	9,506
0,8571	0,0006	0,0006	0,000	-0,2203	-0,2350	6,235	-0,0586	-0,0638	8,210
0,8786	0,0002	0,0002	0,000	-0,3058	-0,3225	5,180	-0,0926	-0,0994	6,874
0,9	0,0000	0,0000	0,000	-0,3959	-0,4202	5,778	-0,1313	-0,1400	6,224

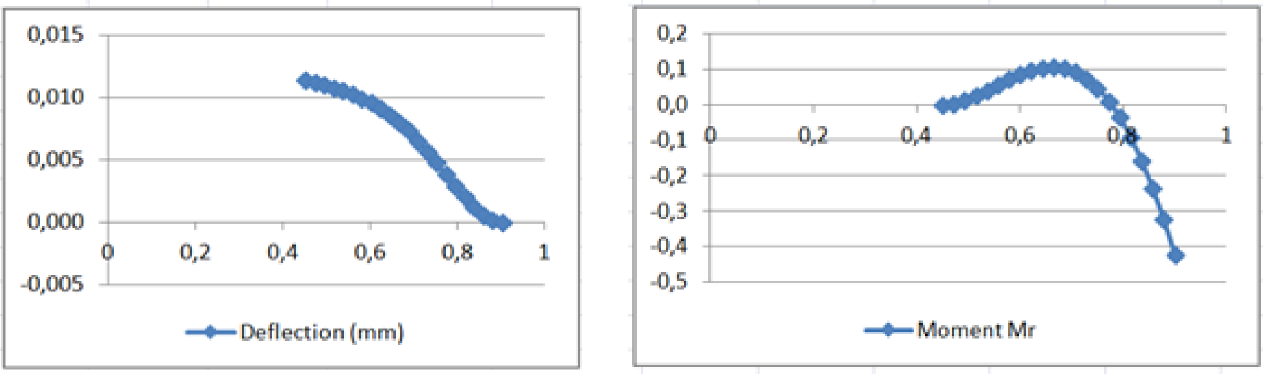

Fig. 4. Deflections and radial bending moments

Here the deflections calculated by the two methods practically do not differ from each other, and the discrepancy between the bending moments is greater than in the previous example (the maximum discrepancy is 10.47%).

Conclusions. The results of calculations by the finite element method in PC LIRA-SAPR, which are given in Tables 1 and 2, were obtained with a semi-automatic breakdown of the finite element grid. This means that a partitioning step of 0,1r was chosen along the radius, and the partitioning along the circle of the plate was carried out automatically. With this approach the discrepancy in the results of calculating the deflections of FEM and the author's method was insignificant, while the difference in the results of calculating the radial and circumferential moments turned out to be quite significant. This effect is well known but, as the authors of [15] have shown, when the mesh is thickened in the circumferential direction, the results obtained by the two methods in determining the radial and circumferential bending moments become considerably closer. This becomes particularly important if the design of the slab and its reinforcement is based on modeling and finite element analysis using engineering computer programs.

Thus, the high accuracy of application of the analytical method proposed by the authors – the method of direct integration – for calculations of building structures in the form of circular plates and slabs, which lie on a continuous variable elastic base, is shown.

References

[1] S.A. Ambartsumian, D. V. Peshtmaldzhian, "K teoryy ortotropnykh obolochek y plastyn", *Yzvestyia AN Arm. SSR seryia fyz.-mat nauk*, vol. 1 (12), pp. 43-59, 1959.

[2] S.V. Bosakov, "K reshenyiu neosesymmetrychnoi kontaktnoi zadachy dlia kruhloi plastynky", *Vestnyk Brestskoho hosudarstvennoho tekhnycheskoho unyversyteta*, vol. 1 (85), pp. 83-87, 2014.

[3] *Razvytye teoryy kontaktnykh zadach v SSSR*, pod red. L. A. Halyna, M.: Nauka, 1976.

[4] Abdulhalim Karasin, Polat Gülkan, Gultekin Aktas, "Finite grid solution for circular plates on elastic foundations", *KSCE Journal of Civil Eng*., vol. 19 (4), pp. 1157-1163, 2014.

[5] E.R. Telehulova, "Nesushchaia sposobnost plyt, lezhashchykh na deformyruemom osnovanyy", avtoreferat dyssertatsyy na soyskanye uchenoi stepeny k.f.-m.n.: 01.02.04, Kazan, 2009.

[6] V.R. Hrosman, "Nekotorye voprosu statyky kruhlykh ortotropnykh y yzotropnykh plastyn», *Vestnyk MHSU*, vol. 7, pp. 65-68, 2012.

[7] A.W. Crook, "A transfer matrix method for calculating the elastic behaviour of annular plates", *The Journal of Strain Analysis for Engineering Design*, vol. 26 (1), pp. 65-73, 1991. doi:10.1243/03093247V261065.

[8] J. Vaskova, P. Matečková, "Software for Design and Assessment of Rotationally Symmetrically Loaded Reinforced Concrete Slabs in the Shape of Circle or Ring", *In Applied Mechanics and Materials*, vol. 749, pp. 368–372, 2015. https://doi.org/10.4028/www.scientific.net/amm.749.368.

[9] Yogesh Rana, Abbas Jamani, "Comparative Study of Annular Raft Foundation & Solid Circular Raft Foundation for Different Diameter of Water Tank", *International Research Journal of Engineering and Technology*, vol. 05 (04), pp. 3428-3436, 2018.

[10] Subhani Shaik, M, Manivannan. «Analysis Of Annular Raft Foundation using Finite

Element Method», *Proceedings of the First International Conference on Combinatorial and Optimization* 10.4108/eai.7-12-2021.2314553, 2021.

[11] D.A. Horodetskyi, M.S. Barabash, R.Iu. Vodopianov i dr., *Prohrammnyi kompleks LYRA-SAPR 2015,* Uchebnoe posobye; pod red. akademyka RAASN A.S. Horodetskoho, M., 2015.

[12] M.S. Barabash, P.M. Kiriaziev, O.I. Lapenko, M.A. Romashkina, *Osnovy kompiuternoho modeliuvannia*, K., NAU, 2019.

[13] Yu.S. Krutii, "*Rozrobka metodu rozviazannia zadach stiikosti i kolyvan deformivnykh system zi zminnymy neperervnymy parametramy*" dys. … d-ra. tekhn. nauk: 01.02.04, Odesa, 2016.

[14] Y.S. Krutii, M.G. Surianinov, G.S. Karnaukhova, "Calculation Method for Axisymmetric Bending of Circular and Annular Plates on a Changeable Elastic Bed, Part 1. Analytical Relations", *Strength of Materials*, vol. 53(2), pp. 247–257, 2021.

[15] G.S. Karnauhova, D.O. Kirichenko, "*Krugli pliti na pruzhniy osnovi zi zminnim koefitsientom posteli*. Mehanika ta matematichni metodi", vol. 2, pp. 46-61, 2020.

АНАЛІТИЧНИЙ МЕТОД РОЗРАХУНКУ КІЛЬЦЕВИХ ПЛИТ НА ЗМІННІЙ ПРУЖНІЙ ОСНОВІ

[1]**Сур'янінов М.Г.,** д.т.н., професор,
sng@ogasa.org.ua, ORCID: 0000-0003-2592-5221
[1]**Крутій Ю.С.,** д.т.н., професор,
yurii.krutii@gmail.com, ORCID: 0000-0001-7105-3087
[1]**Карнаухова Г.С.,** к.т.н., доцент,
karnauhovaanna@ukr.net, ORCID: 0000-0002-3542-4300
[1]**Клименко О.М.,** аспірант,
bolshajaakula@gmail.com, ORCID: 0000-0002-9551-9475
[1]*Одеська державна академія будівництва та архітектури*
вул. Дідріхсона, 4, м. Одеса, 65029, Україна

Анотація. У статті розглядається застосування методу прямого інтегрування до розрахунків кільцевих пластин та плит на безперервній змінній пружній основі. Пластини кільцевої форми зі змінними геометричними та механічними параметрами знаходять все ширше застосування. Змінними параметрами тут можуть виступати не тільки пружна основа, а й товщина пластини, циліндрична жорсткість. Необхідність аналітичного методу розрахунку подібних конструкцій не викликає сумнівів, оскільки дозволяє оцінити точність скінчено-елементного аналізу. На сьогоднішній день у літературі пропозиції щодо загального аналітичного методу розрахунку кільцевих пластин на змінній пружній основі відсутні.

Детальний виклад алгоритму методу прямого інтегрування у роботі не наводиться, проте розрахункові формули для кільцевої пластини взяті з вже опублікованої статті авторів. Розглядаються результати чисельної реалізації цього алгоритму для конкретних прикладів: бетонна плита, яка жорстко затиснена за внутрішнім контуром, а її зовнішній контур – вільний і сталева плита, яка жорстко затиснена за зовнішнім контуром, а її внутрішній контур – вільний.

Для оцінки результатів розрахунку авторським методом виконано комп'ютерне моделювання розглянутих конструкцій у ПК ЛІРА-САПР та їх розрахунки методом скінчених елементів.

Реакція основи описується моделлю Вінклера зі змінним коефіцієнтом постелі. У першому випадку коефіцієнт постелі прийнято постійним, а у другому випадку він змінюється за лінійним законом. Виконані розрахунки показали, що розбіжність у результатах обчислення прогинів методом скінчених елементів і авторським методом вбирається у 1 %, а результати обчислення радіальних і окружних моментів відрізняються значно, досягаючи 10 %. Цю відмінність автори пояснюють неточністю чисельного аналізу, пов'язаної з напівавтоматичним методом побудови скінчено-елементної сітки, яку слід робити дрібнішою. Згущення сітки в ручному режимі її розбиття суттєво знижує розбіжність між результатами обчислення прогинів, радіальних та окружних моментів, що згинають методом скінчених елементів і авторським методом.

Ключові слова: метод прямого інтегрування, кільцева плита, пружна основа, модель Вінклера, змінний коефіцієнт постелі, метод скінчених елементів, ПК ЛІРА-САПР.

Стаття надійшла до редакції 2.12.2022