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ОДЕСЬКА ДЕРЖАВНА АКАДЕМІЯ БУДІВНИЦТВА ТА АРХІТЕКТУРИ

О.М. ПИСАРЕНКО

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ЕЛЕКТРОМАГНЕТИЗМ, ОПТИКА ТА КВАНТОВА ФІЗИКА

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У навчальному посібнику розглянуті основні явища електромагнетизму, оптики і фізики мікросвіту з експериментальної та аналітичної точок зору. В посібнику також наведені питання для самоперевірки, приклади розв’язання задач, умови задач для самостійного розв’язання, а також довідкові дані, необхідні для їх вирішення. Весь навчальний матеріал наведено англійською мовою.

Навчальний посібник призначений для студентів освітньо-професійної програми підготовки освітньо-кваліфікаційного ступеня Бакалавр спеціальності 192 Будівництво та цивільна інженерія.

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PREFACE

Study guide “Physics course. Electromagnetism, optics and quantum physics" is intended for students of higher educational institutions of specialty #192 "Construction and Civil Engineering". This study guide aims to provide an up-to-date and comprehensive coverage of the core curriculum in physics specified in the current Odessa State Academy of Civil Engineering and Architecture syllabus.

The guide covers topics related to electricity, magnetism, optics, heat radiation and quantum physics. It builds from concrete experiments to more abstract understanding. Elements of the study guide include the following:

- fundamental concepts of physics
- test questions
- problem-solving examples
- problems
- appendices.

The problem-solving examples at the end of each chapter are provided to clarify concepts and to guide students in the analytical approach to the solutions of problems. In order to unify conceptual, analytical and calculation skills within the learning process, the International System of Units is used in study guide.

Most of the chapters are relatively independent, but some necessary background is established in certain key chapters. Chapter 7 provides an introduction to the quantitative law of electromagnetic induction. The generalized form of this law is included in the Maxwell’s equations presented in Chapter 8. Chapter 10 and Chapter 11 introduce the idea of secondary waves interaction (Huygens-Fresnel principle).

By the end of study guide “Physics course. Electromagnetism, optics and quantum physics" students will be able to:

- apply principles and concepts of physics to explain various phenomena
- construct models and simulations to describe and explain natural phenomena
- use mathematics as a precise method for showing relationships
- solve problems by applying physics principles and laws
- select and use appropriate technological instruments to collect data,
- analyze data, check it for accuracy and construct reasonable conclusions
- use precise scientific language in oral and written communication.

Physics is the science that studies the simplest and, at the same time, the most general patterns of natural phenomena, the properties and structure of matter.

The most simple forms of matter motion (mechanical, thermal, electromagnetic) are part of more complex movements (chemical and biological).

Physics has common objects and research methods with other natural sciences, as a result of which the following areas of knowledge have emerged: physical chemistry, chemical physics, chemical thermodynamics, astrophysics, biophysics, geophysics.

Mathematics is the basis of modern physics. The mathematical apparatus is widely used in the processing and generalization of experimental results. The electromagnetic field theory, statistical theory, thermodynamics, the theory of relativity, as well as quantum mechanics could not be developed without mathematics.

Physics is the basis of modern scientific and technological progress. The successful development of such areas of technology as: mechanical transport, electrical engineering, electronics, heat engineering, automation and remote control, construction equipment, modern technology, semiconductor and computing technology is strongly dependent on knowledge of physical laws and phenomena.

Physics is of great importance in the development of all areas of the economy. This fact determines the place of the physics course in the curriculums of higher education, especially in the curriculums of higher technical educational institutions. Acquaintance with the main physical phenomena, their mechanisms, laws and practical application can be postulated as the goal of studying physics. Achieving this goal is the physical basis for the study of general technical and special disciplines. Proper understandings of the nature of physical phenomena are particularly important in the practice of engineering.

Course of general physics refers to the experimental knowledge, and one of its main tasks is to represent knowledge as a result of observation, experiment, reflection and generalization of the experience. Therefore, in general, the course statement must be inductive. However, this does not preclude the use of the deductive method of presentation.

The model nature of physical theories, various methods for determining physical quantities and concepts, features of measuring physical quantities, the correct choice of units of measurement and systems of units occupy a significant place in this study guide.

It is well known that theoretical knowledge is useless without the ability to use it to solve practical problems. Therefore, the acquisition of problem solving skills is an integral part of studying the course of general physics. Currently, there are a sufficient number of collections of physical problems, but, unfortunately, there are practically no manuals intended for training in methods of solving problems. The material located at the end of each chapter of study guide is intended to remove the indicated disadvantage. This material is divided into three blocks. The first block contains test questions on the theoretical information that is present in the chapter. Examples of solving typical problems are included in the second block. The third block contains a number of problems for independent solution. These tasks are accompanied only by short answers. It is worth noting that in the theoretical part, the descriptions of experiments and in the methods of solving problems, the SI system is mainly used, which is convenient from a practical point of view.

The appendices placed at the end of the textbook are, on the one hand, an illustrative addition to the laws and phenomena that are described in the physics course, and on the other hand, have a reference character necessary for successful problem solving.

CHAPTER 1. ELECTRIC FIELD IN VACUUM

1.1. Coulomb Law

Electrostatics is a section of electricity that studies the interaction and properties of electric charge systems, which are fixed relative to their chosen reference frame. Two types of electric charges, in particular positive and negative charges, exist in nature. French physicist Charles François de Cisternay du Fay (1698 – 1739) established the existence of two types of electricity: glass and resin, which were manifested when rubbing glass about silk and resin on the wool. A positive charge, for example, appears on glass rubbing the skin, and a negative charge appears on amber, rubbed with wool. Two objects that carry the same type of charge repel each other, and two objects that carry opposite charges attract each other. Experiments carried out by American physicist Robert Andrews Milliken (1868 – 1953). He showed that the electric charge of any body consists of an entire number of elementary charges equal to 1.60×10^{-19} C. The smallest particle with a negative elemental charge is called the *electron*. The electron mass is 9.11×10^{-31} kg. The smallest stable part having a positive elemental charge is called the *proton*. Proton mass is 1.67×10^{-27} kg. Protons and electrons are the part of all atoms and molecules.

The assignment of bodies to the category of conductors, dielectrics, and semiconductors depends on the concentration of free charges in these bodies. Bodies in which the electric charge can move freely throughout its volume are called *conductors*. Conductors can be divided into two groups: 1) conductors of the first kind (metals), i.e. conductors in which the transfer of charges (free electrons) is not accompanied by chemical transformations; 2) conductors of the second kind (for example, molten salts, solutions of acids), i.e. conductors, in which the transfer of charges (positive and negative ions, electrons) leads to chemical changes. Bodies in which there are no free charges (e.g. glass, plastics) are called *dielectrics*. Semiconductors (e.g. germanium, silicon, selenium, tellurium, a large number of non-limiting substances, etc.) occupy an intermediate position between conductors and dielectrics. This classification of bodies is not absolute, since the ability of bodies to conduct electricity depends on conditions (for example, temperature, concentration of impurities, the presence of different types of radiation) in which they are located.

Charge conservation is one of the fundamental laws of nature: the algebraic sum of electrical charges of bodies or particles forming an electrically isolated system does not change with any processes occurring in this system.

The basic law of electric charges interaction (*Coulomb's law*) was found by French physicist Charles-Augustin de Coulomb (1736 – 1806) as a result of measuring the force of charged balls interaction with the help of torsion weights. S. Coulomb found that the interaction strength F between small charged balls is inversely proportional to the square of the distance between them r and depends on the size of their charges q_1 and q_2 (namely, directly proportional to the product of charges):

$$F = k_1 \frac{q_1 q_2}{r^2}, \qquad (1.1)$$

where k_1 is the proportionality factor ($k_1 > 0$).

Electric forces acting on the charges are central, that is, these forces are directed along one line that connects the charges. For charges of same sign, the inequality $q_1 q_2 > 0$ is performed, therefore, the force $\vec{F} = \vec{F}_{12}$ (Figure 1.1, a) corresponds to the case of mutual repulsion of charges, and the force $\vec{F} = \vec{F}_{21}$ (Figure 1.1, b) corresponds to the case of mutual attraction of charges with different signs.

Coulomb's law can be written in vector form. The repulsive force $\vec{F}_{12}$ between charges is equal to

$$\vec{F}_{12} = k_1 \frac{q_1 q_2}{r^2} \frac{\vec{r}_{12}}{r}, \qquad (1.2)$$

where $\vec{r}_{12}$ is a radius vector that connects charges q_1 and q_2, and $|\vec{r}_{12}| = r$.

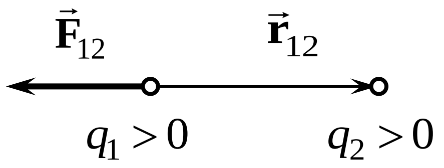


Figure 1.1, a. Repulsive force between like-charged objects.

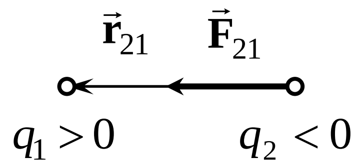


Figure 1.1, b. Attractive force between opposite charged object.

The attractive force $\vec{F}_{21}$ between charges q_1 and q_2 is

$$\vec{F}_{21} = k_1 \frac{q_1 q_2}{r^2} \frac{\vec{r}_{21}}{r}, \qquad (1.3)$$

where $\vec{r}_{21} = -\vec{r}_{12}$ is a radius vector that connects charges q_2 and q_1.

The Coulomb’s law is valid only for the interaction of point electric charges, that is, charged bodies whose linear dimensions can be neglected in comparison with the distance between them. In addition, the Coulomb's law expresses the power of interaction between fixed charges, that is, this law has electrostatic character (the force of interaction between two moving point charges is different from Coulomb's force). The Coulomb's law in the given mathematical form is also valid for the interaction of non-intersecting charged bodies of a globular shape under the condition of uniform distribution of charges in volume or on the surface of bodies.

Further experimental studies have shown that the force F and coefficient k_1 in the Coulomb's law depend on the properties of the medium, namely

$$k_1 = k / \varepsilon \,, \tag{1.4}$$

where k is the coefficient, which depends only on the choice of system units;

ε is the dimensionless quantity characterizing the electrical properties of the medium, which does not depend on the choice of the system of units and is called the *relative permittivity* of the medium ($\varepsilon = 1$ for vacuum).

So, the Coulomb's law can be rewritten in form

$$F = k \frac{q_1 q_2}{\varepsilon r^2} \,. \tag{1.5}$$

Formulas that include the value of ε, in contrast to the formula that expresses the Coulomb's law for vacuum, are not universal. These formulas are correct if the point charges q_1 and q_2 are in a homogeneous, infinite and isotropic gaseous or liquid dielectric. Charges in solid dielectrics are always located inside some cavities and the calculation of forces acting on charges is greatly complicated, in particular, for various shapes of cavities. The coefficient k in the formula of the Coulomb's law in the system of units SI is

$$k = 1 / 4\pi\varepsilon_0 \,. \tag{1.6}$$

The value ε_0 is called the *electric constant* ($\varepsilon_0 \approx 8.85 \times 10^{-12}$ C^2/ (N×m^2)). Coulomb's law in the case of using the system of units SI can be written as

$$F = \frac{1}{4\pi\varepsilon\varepsilon_0} \frac{q_1 q_2}{r^2} \,. \tag{1.7}$$

This form of Coulomb's law and all the laws of electrostatics arising from it, commonly used in electrical and radio engineering, is called the *rationalized system*.

1.2. Electrostatic Field Intensity

The placement of the electric charge Q_2 in the space that surrounds the charge Q_1 leads to the appearance of a Coulomb's force acting on the charge Q_2. It follows that there is a force field in the space surrounding the electric charges. According to the ideas of modern physics, the field really exists and, along with matter, is one of the forms of matter through which certain interactions are realized between macroscopic bodies or particles that make up the substance. In this case, we mean an electric field through which electric charges interact. We will consider electric fields, which are created by stationary electric charges and are called electrostatic fields.

Detection and experimental investigation of the electrostatic field is carried out using a *test charge*, i.e. such a charge, which does not distort the field under investigation (does not cause a redistribution of the charges creating the field). The placement of the test charge Q_0 in the field of charge Q results in a force F that acts on the charge Q_0 and is different at different points in the electrostatic field. This force, according to the Coulomb's law, is proportional to the test charge Q_0. Therefore, the ratio F/Q_0 does not depend on Q_0 and characterizes the electric field at the point where the test charge is located. This quantity is called the *electric field intensity* and is the electrostatic field power characteristic.

The electric field intensity at a given point is a physical quantity determined by the force acting on a unit positive charge placed at this point

$$E = F/Q_0. \tag{1.8}$$

The electric field intensity of a point charge located in a vacuum is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\frac{\vec{r}}{r}. \tag{1.9}$$

or in the scalar form

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}. \tag{1.10}$$

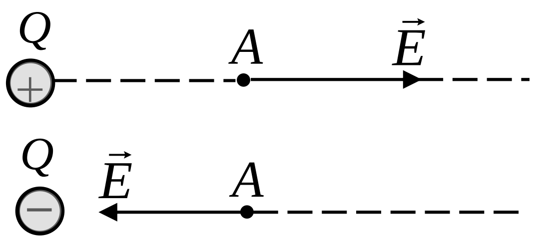


Figure 1.2. Electric field of point charges.

The direction of the vector $\vec{E}$ coincides with the direction of the force acting on the positive charge. If the field is created by a positive charge, the vector $\vec{E}$ is directed along the radius vector from the charge to the outer space (repulsion of the test positive charge); if the field is created by a negative charge, then the vector $\vec{E}$ is directed to the charge (Figure 1.2).

Graphically, the electrostatic field is represented by field lines. The lines tangent to which at each point coincide with the direction of the vector are called *electric field lines*. The field lines are assigned a direction that coincides with the direction of the vector $\vec{E}$.

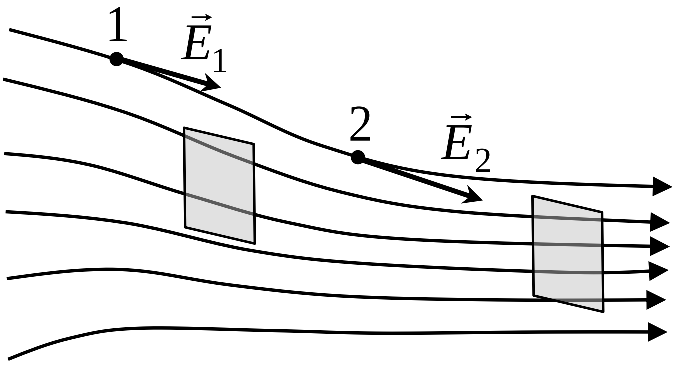


Figure 1.3. Module and direction of vector E in a non-uniform electric field.

Since at each given point in space the vector of electric field intensity has only one direction, the field lines never intersect. Field lines are parallel to the vector $\vec{E}$ for the homogeneous field (when the electric field intensity vector at any point is constant in magnitude and direction). Field lines coincide with radial lines that leave the charge if the field is created by a positive point charge. Field lines coincide with radial lines entering the charge if the field is created by a negative point charge.

The graphical method describing electric field is widely used in electrical engineering. Field lines are drawn with a certain density (Figure 1.3) in order to characterize not only the direction, but also the magnitude of the electrostatic field intensity [7].

The number of force lines that permeate a unit of surface area perpendicular to the lines of electric field intensity should be equal to the of vector $\vec{E}$ magnitude. The number of field lines penetrating the elementary area dS, the normal $\vec{n}$ of which forms an angle α with the vector $\vec{E}$, is

$$EdS\cos\alpha = E_n dS, \tag{1.11}$$

where E_n is the projection of the vector $\vec{E}$ onto the normal $\vec{n}$ to the area dS (Figure 1.4).

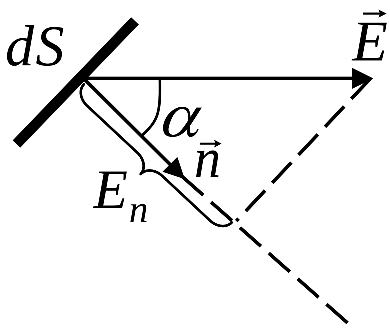


Figure 1.4. The flux of the electric field intensity vector.

The quantity

$$d\Phi_E = E_n dS = \vec{E}d\vec{S} \tag{1.12}$$

is called the *electric flux* through area dS where $d\vec{S} = dS\vec{n}$ is a vector whose magnitude is dS, and the direction coincides with the direction of the normal $\vec{n}$ to the area. Choice of the direction of the vector $\vec{n}$ (and therefore, $d\vec{S}$) is conditional, since it can be directed to either side. The electric flux is an algebraic quantity, because the sign of the flux depends not only on the configuration of the field $\vec{E}$, but also on the choice of the direction $\vec{n}$. The positive direction of the normal for closed surfaces coincides with the direction of the outer normal.

1.3. Superposition of Electrostatic Fields. Dipole Field

Let us consider a method for determining the value and direction of the electric field intensity vector $\vec{E}$ at each point of the electrostatic field produced by a system of stationary charges $Q_1, Q_2, ...Q_n$.

As a result of numerous experiments, it was found that the principle of the independence of the action of forces, which was considered in mechanics, is applicable to Coulomb's forces. The net force $\vec{F}$ acting on the test charge Q_0 is equal to the vector sum of the forces $\vec{F_i}$ applied to it from the side of each of the charges Q_i:

$$\vec{F} = \sum_{i=1}^{n} \vec{F}_i \, . \tag{1.13}$$

The following relationships are valid

$$\vec{F} = Q_0\vec{E}, \tag{1.14}$$

and

$$\vec{F}_i = Q_0\vec{E}_i, \tag{1.15}$$

where $\vec{E}$ is the intensity of the resulting field, and $\vec{E}_i$ is the field intensity created by the charge Q_i. Substituting the last expressions into the formula

$$\vec{F} = \sum_{i=1}^{n} \vec{F}_i \tag{1.16}$$

we get

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i \, . \tag{1.17}$$

The last formula expresses the <u>*superposition principle*</u> of electrostatic fields, according to which the intensity of the resultant field created by the system of charges is equal to the geometric sum of the field intensities created at a given point by each of the charges separately.

The superposition principle allows us to calculate the electrostatic fields of any fixed-charge system, since if charges are not point charges, then they can always be reduced to a set of point charges.

The superposition principle is applicable for calculating the electrostatic field of an electric dipole. Consider a system of two point charges whose modules are equal, and the signs are opposite, $(+Q, -Q)$. Let the distance l between these charges be much less than the distance between charges and considered points of the field. A system of such charges is called an <u>*electric dipole*</u>.

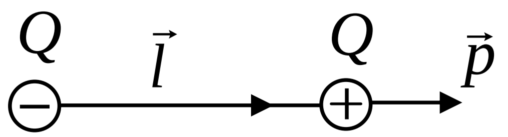


Figure 1.5. Electric dipole.

Vector directed along the axis of the dipole (the straight line passing through both charges) from the negative charge to the positive one and equal to the distance between them is called the <u>*displacement vector*</u> $\vec{l}$.

Vector

$$\vec{p} = |Q|\vec{l} \, , \tag{1.18}$$

coinciding with the displacement vector and equal to the product of the charge $|Q|$ on the displacement vector $\vec{l}$, is called the <u>*electric dipole moment*</u> (Figure 1.5).

According to the principle of superposition, the intensity $\vec{E}$ of the electric field of a dipole at an arbitrary point is equal to

$$\vec{E} = \vec{E}_+ + \vec{E}_- , \tag{1.19}$$

where $\vec{E}_+$ and $\vec{E}_-$ are the intensities of the electric fields produced by the positive and negative charges, respectively.

1.4. Gauss's Theorem for the Electrostatic Field in Vacuum

The calculation of the electric field strength of a system of electric charges using the principle of superposition of electrostatic fields can be considerably simplified using the theorem derived by German mathematician and physicist Johann Carl Friederich Gauss (1777 – 1855), which determines the electric flux through an arbitrary closed surface. The electric flux through a spherical surface of radius r enclosing the point charge Q, located in the centre of the sphere, is equal to

$$\Phi_E = \oint_S E_n dS = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{Q}{\varepsilon_0}. \tag{1.20}$$

This result is valid for a closed surface of any form. If we surround the sphere with an arbitrary closed surface, then each field line penetrating the sphere will pass through this surface. Suppose that a closed surface of an arbitrary shape covers a charge.

In this case, when any selected field line is crossed with the surface, it first enters the surface, and then exits it. An odd number of intersections in the calculation of the flux eventually reduces to a single intersection, since the flux is assumed to be positive if the field lines exit the surface, and negative for the lines entering the surface. For the case when the closed surface does not cover the charge, the flux through it is zero, since the number of field lines entering the surface is equal to the number of field lines emerging from it.

Thus, for a surface of any form, if it is closed and encloses a point charge Q, the flux of the vector $\vec{E}$ will be equal $\frac{Q}{\varepsilon_0}$, i.e.

$$\Phi_E = \oint_S \vec{E} d\vec{S} = \oint_S E_n dS = \frac{Q}{\varepsilon_0}. \tag{1.21}$$

The sign of the flux coincides with the sign of charge Q. Let us consider the general case of an arbitrary surface surrounding n charges. In accordance with the superposition principle, the intensity $\vec{E}$ of the electric field created by all charges is equal to the sum of the intensities $\vec{E}_i$ of the electric fields created by each charge separately

$$\vec{E} = \sum_i \vec{E}_i . \tag{1.22}$$

Therefore

$$\Phi_E = \oint_S \vec{E} d\vec{S} = \oint_S \left(\sum_i \vec{E}_i \right) d\vec{S} = \sum_i \oint_S \vec{E}_i d\vec{S} . \tag{1.23}$$

Each of the integrals in the sum is Q_i / ε_0.

Consequently

$$\oint_S \vec{E} d\vec{S} = \oint_S E_n dS = \frac{1}{\varepsilon_0} \sum_{i=1}^{n} Q_i . \tag{1.24}$$

The last formula expresses the <u>*Gauss's theorem*</u> for an electrostatic field in a vacuum: the electric flux in vacuum through an arbitrary closed surface is equal to the algebraic sum of the charges contained inside this surface divided by ε_0.

Electric charges can be distributed with a certain volume density

$$\rho = \frac{dQ}{dV}, \tag{1.25}$$

which varies in different places of space. The total charge enclosed within a volume V with side surface S is

$$\sum_i Q_i = \int_V \rho dV. \tag{1.26}$$

The Gauss's theorem can be written as follows

$$\oint_S \vec{E} d\vec{S} = \oint_S E_n dS = \frac{1}{\varepsilon_0} \int_V \rho dV. \tag{1.27}$$

1.5. Gauss's Theorem Applications

<u>The electric field of a uniformly charged infinite plane</u>**.** Consider the case of the infinite plane (Figure 1.6) charged with a constant surface charge density $+\rho$ ($\rho = dQ/dS$ is the charge distributed per unit surface). Field lines are perpendicular to this plane and are directed away from it in both directions. We construct a cylinder whose bases are parallel to the charged plane, and the axis is perpendicular to it. Generating lines of this cylinder are parallel to the electric field intensity lines ($\cos\alpha = 0$), therefore the electric flux through the cylinder's side surface is zero, and the total flux through the cylinder is equal to the sum of the fluxes through its bases (the base areas are equal and for base E_n coincides with E), i.e. is equal to $2ES$. The charge enclosed inside the constructed cylindrical surface is ρS.

According to the Gauss's theorem we get

$$2ES = \sigma S / \varepsilon_0. \tag{1.28}$$

Hence

$$E = \sigma / (2\varepsilon\varepsilon_0). \tag{1.29}$$

Consequently, the intensity E does not depend on the length of the cylinder, that is, the electric field intensity at any distance is the same in magnitude. The electric field of a uniformly charged plane is homogeneous.

<u>The electric field of two infinite parallel planes, with charges of opposite signs</u>. Consider the case when charges with opposite signs are uniformly distributed on two parallel planes. The surface charge densities are $+\sigma$, and $-\sigma$. The electric field of

such planes can be found as a superposition of electric fields created by each of the planes separately. The electric field intensities of left and right planes are subtracted (field lines are directed towards each other), therefore the net electric field intensity is $E = 0$. Electric field intensity between the planes is

$$E = E_{+} + E_{-} \tag{1.30}$$

and the net electric field intensity is

$$E = \sigma / \varepsilon_0 . \tag{1.31}$$

Thus, the net electric field intensity in the region between the planes is described by the formula $E = \sigma / \varepsilon_0$, and outside the volume bounded by the planes is zero.

The electric field of a uniformly charged spherical surface**.** Let us **c**onsider a spherical surface with a radius of R. The charge Q with a surface density of $+\sigma$ is uniformly distributed on this surface. Uniform charge distribution on the surface causes the appearance of a spherically symmetric electric field. Therefore, the field lines are directed along the radius. We construct a sphere of radius r with a centre in the charged sphere. Consider the case $r > R$, then the whole charge Q that generates the electric field is located on the surface. According to the Gauss's theorem, we get

$$4\pi r^2 E = Q / \varepsilon_0 , \tag{1.32}$$

then

$$E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}, \ (r \ge R). \tag{1.33}$$

The electric field decreases with distance r ($r > R$) according to the same law as for a point charge. For the case of small distances $r' < R$, the closed surface does not contain any charges inside, so there is no electrostatic field inside the uniformly charged spherical surface ($E = 0$).

The electric field of a volume-charged sphere. Consider the case of a sphere with radius R. Sphere contains a charge Q which uniformly distributed over its volume. The volume density of the charge is $\rho = dQ / dV$. The electric field intensity outside the sphere depends on the distance to the same law as the electric field intensity of a charged spherical surface.

The sphere of radius $r' < R$ covers the charge

$$Q' = 4/3\pi r'^3 \rho . \tag{1.34}$$

Therefore, according to the Gauss's theorem, we get

$$4\pi r'^2 E = Q' / \varepsilon_0 = 4/3\pi r^3 \rho / \varepsilon_0 . \tag{1.35}$$

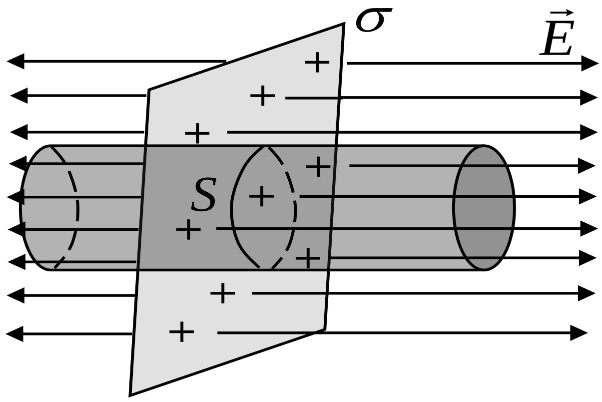


Figure 1.6. The field of a uniformly charged infinite plane.

Taking into account that

$$\rho = Q/(4/3\pi R^3), \tag{1.36}$$

we obtain

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^3}r', \ (r' \le R). \tag{1.37}$$

<u>The electric field of a uniformly charged infinite cylinder (filament)</u>. An infinite cylinder of radius R is charged uniformly with a linear charge density of τ ($\tau = dQ/dt$ is the charge per unit length). Taking into account the electric field symmetry we get that the field lines will be directed along the radii of the circular sections of the cylinder with the same density in all directions relative to the axis of the cylinder. We construct a closed surface in the form of a coaxial cylinder with radius r and height l. The flux of vector $\vec{E}$ through the ends of the coaxial cylinder is zero (ends of the cylinder are parallel to the field lines), and through the side surface the flux of the vector $\vec{E}$ is equal to $2\pi rlE$. According to the Gauss's theorem, for $r > R$ we have

$$2\rho rlE = \tau l/\varepsilon_0, \tag{1.38}$$

hence

$$E = \frac{1}{2\pi\varepsilon_0}\frac{\tau}{r}, \ (r \ge R). \tag{1.39}$$

Electric charges are not located inside a closed surface for the case then $r < R$, therefore, in this region $E = 0$.

1.6. Circulation of the Electrostatic Field Intensity Vector

The Coulomb's force performs the work if a point charge Q_0 moves along the arbitrary trajectory in the electrostatic field of the point charge Q from point 1 to point 2 along the arbitrary trajectory. The work of the force $\vec{F}$ in the case of an elementary displacement $d\vec{l}$ of the charge is

$$dA = \vec{F}d\vec{l} = Fdl\cos\alpha = \frac{1}{4\pi\varepsilon_0}\frac{QQ_0}{r^2}DL\cdot\cos\alpha\,. \qquad (1.40)$$

Since

$$dl\cos\alpha = dr\,, \qquad (1.41)$$

then

$$dA = \frac{1}{4\pi\varepsilon_0}\frac{QQ_0}{r^2}dr\,. \qquad (1.42)$$

The work of the Coulomb's force in the case of the displacement of the charge Q_0 from point 1 to point 2 is equal to

$$A_{12} = \int_{r_1}^{r_2} dA = \frac{QQ_0}{4\pi\varepsilon_0}\int_{r_1}^{r_2}\frac{dr}{r^2} = \frac{1}{4\pi\varepsilon_0}\left(\frac{QQ_0}{r_1} - \frac{QQ_0}{r_2}\right). \qquad (1.43)$$

This work does not depend on the trajectory of displacement, but is determined only by the positions of the initial and final points. Consequently, the electrostatic field of a point charge is potential, and the electrostatic forces are conservative. It follows from the last formula that the work performed when the electric charge is moved in an external electrostatic field along any closed path L is zero

$$\oint_L dA = 0\,. \qquad (1.44)$$

Let us consider the displacement of a single point positive charge in an electrostatic field. Then the elementary work of the field forces on the path $d\vec{l}$ is

$$\vec{E}d\vec{l} = E_l dl\,, \qquad (1.45)$$

where $E_l = E\cos\alpha$ is the projection of the vector $\vec{E}$ on the direction of the elementary displacement.

In this case, formula $\vec{E}d\vec{l} = E_l dl$ can be written in the form

$$\oint_L \vec{E}d\vec{l} = \oint_L E_l dl = 0\,. \qquad (1.46)$$

The integral

$$\oint_L \vec{E}d\vec{l} = \oint_L E_l dl \qquad (1.47)$$

is called the *circulation of electric field intensity* vector. Consequently, the circulation of the vector of the intensity of the electrostatic field along any closed contour is zero. A force field with property

$$\oint_L \vec{E}d\vec{l} = \oint_L E_l dl = 0 \qquad (1.48)$$

is called potential. The property of the potentiality of the electrostatic field leads to the fact that the lines of the electrostatic field intensity can not be closed, they begin and end on charges (respectively positive or negative) or move to infinity.

1.7. Potential of Electrostatic Field

The body in the potential force field (the electrostatic field) has potential energy. The electrostatic field forces are associated with potential energy and perform work. The work of conservative forces performs by the loss of potential energy. Therefore, the work of the forces of the electrostatic field can be represented as the difference in the potential energies possessed by a point charge Q_0 at the initial and final points of the charge Q field

$$A_{12} = \frac{1}{4\pi\varepsilon_0}\frac{QQ_0}{r_1} - \frac{1}{4\pi\varepsilon_0}\frac{QQ_0}{r_2} = U_1 - U_2. \qquad (1.49)$$

Therefore the potential energy of charge Q_0 in the electric field generated by charge Q is equal to

$$U = \frac{1}{4\pi\varepsilon_0}\frac{QQ_0}{r} + C. \qquad (1.50)$$

The potential energy is not uniquely determined, but accurate to an arbitrary constant C. Under the assumption that when the charge is removed to infinity ($r \to \infty$), the potential energy vanishes ($U \to 0$) and $C = 0$. The potential energy of charge Q_0 in the electric field generated by charge Q at a distance r is equal to

$$U = \frac{1}{4\pi\varepsilon_0}\frac{QQ_0}{r}. \qquad (1.51)$$

In the case of charges of the same sign $Q_0Q > 0$ and the potential energy of their interaction (repulsion) is positive. In the case of charges with opposite signs $Q_0Q < 0$ and the potential energy of their interaction (attraction) is negative.

Let us consider the field created by the system of n point charges $Q_1, Q_2, ..., Q_n$. We'll place the charge Q_0 in this area. The work of the electrostatic charges $Q_1, Q_2, ..., Q_n$ performed over the charge Q_0 is equal to the algebraic sum of the work of the forces due to each of the charges separately. Therefore, the potential energy U of the charge Q_0 located in this field is equal to the sum of its potential energies U_i, created by each of the charges separately

$$U = \sum_{i=1}^{n} U_i = Q_0 \sum_{i=1}^{n} \frac{Q_i}{4\pi\varepsilon_0 r_i}. \quad (1.52)$$

Consequently, the ratio U/Q_0 does not depend on Q_0 and is therefore the energy characteristic of the electrostatic field. This characteristic is called a *potential*

$$\varphi = \frac{U}{Q_0}. \quad (1.53)$$

The potential at some point of the electrostatic field is a physical quantity determined by the potential energy of a single positive charge placed at this point. The potential of the electrostatic field created by the point charge Q is

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}. \quad (1.54)$$

The work done by the forces of the electrostatic field for the case when the charge Q_0 moves from point 1 to point 2 can be represented as

$$A_{12} = U_1 - U_2 = Q_0(\varphi_1 - \varphi_2). \quad (1.55)$$

This work is equal to the product of the transported charge by the potential difference at the initial and final points. The potential difference between the two points 1 and 2 in the electrostatic field is determined by the work done by the field forces when the unit positive charge moves from point 1 to point 2. The displacement of charge Q_0 from point 1 to point 2 is accompanied by the work of field forces, which is equal to

$$A_{12} = \int_1^2 Q_0 \vec{E} d\vec{l}. \quad (1.56)$$

Equating the expressions for the work, we obtain a formula for the potential difference

$$\varphi_1 - \varphi_2 = \int_1^2 \vec{E} d\vec{l} = \int_1^2 E_l dl, \quad (1.57)$$

where integration can be performed along any line connecting the initial and final points, since the work of the forces of the electrostatic field does not depend on the trajectory of displacement.

The potential of the electrostatic field for an infinitely distant point is zero. The work done by the forces of the electrostatic field for the case of electric charge Q_0 displacement from an arbitrary point to infinity, is

$$A_\infty = Q_0 \varphi \,. \qquad (1.58)$$

Thus, the potential is a physical quantity determined by the work that is done when the point positive charge moves from a given point to infinity. This work is numerically equal to the work done by external forces (with the direction against the forces of the electrostatic field) and is accompanied by the displacement of a single positive charge from infinity to a given point of the field.

The field of a system of several charges generates a potential that is equal to the algebraic sum of the potentials of the fields created by each of the charges separately

$$\varphi = \sum_{i=1}^{n} \varphi_i = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{Q_i}{r_i} \,. \qquad (1.59)$$

1.8. Intensity and Potential of Electrostatic Field

Let us find the relationship between the electrostatic field intensity,(power characteristic) and the potential (energy characteristic).

Let us consider the case when points 1 and 2 are located on the x-axis. The distance between these points is small $x_2 - x_1 = dx$. The work done by Coulomb's force in the case when a single point positive charge moves from point 1 to point 2 is equal to $E_x dx$. The same work is equal

$$\varphi_1 - \varphi_2 = -d\varphi \,. \qquad (1.60)$$

Equating both expressions, we can write

$$E_x = -\partial \varphi / \partial x \,, \qquad (1.61)$$

where the symbol of the partial derivative emphasizes that differentiation is performed only with respect to X . We repeat the analogous arguments for the x-axis and y-axis. Then we can find the vector

$$\vec{E} = -\left(\frac{\partial \varphi}{\partial x} \vec{i} + \frac{\partial \varphi}{\partial y} \vec{j} + \frac{\partial \varphi}{\partial z} \vec{k} \right) \,, \qquad (1.62)$$

where $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors of the coordinate axes. It follows from the definition of the gradient that

$$\vec{E} = -grad\varphi \tag{1.63}$$

or

$$\vec{E} = -\nabla\varphi, \tag{1.64}$$

that is, the electric field intensity is equal to the gradient of the potential with a minus sign. The minus sign is determined by the fact that the directions of the vector $\vec{E}$ of the electrostatic field intensity and the direction of the potential decrease coincide.

A graphical representation of the distribution of the electrostatic field potential, as in the case of a gravitational field, is made using equipotent surfaces. Surfaces at all points of which the potential has the same value are called *equipotent surfaces*.

A point charge generates an electrostatic field with potential

$$\varphi = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q}{r}. \tag{1.65}$$

Consequently, the equipotent surfaces of the electrostatic field, in this case, have the form of concentric spheres. On the other hand, the field lines in the case of a point charge are radial lines. The above leads to the conclusion that the field lines in the case of a point charge are perpendicular to equipotent surfaces.

Field lines are always normal to equipotent surfaces. Indeed, all points of the equipotent surface have the same potential, so the work that is done when the charge moves along this surface is zero, i.e. the electrostatic forces acting on the charge are always directed along the normal to equipotent surfaces.

Each charge and each system of charges assumes the possibility of constructing an infinite number of equipotent surfaces. However, they are usually carried out so that the potential differences between any two adjacent equipotent surfaces are the same. Then the density of equipotent surfaces clearly characterizes the electric field intensity at different points. The arrangement of equipotent surfaces with a higher density corresponds to an increase in the electric field intensity.

The arrangement of the field lines of the electrostatic field makes it possible to construct equipotent surfaces and, conversely, from the known location of the equipotent surfaces, the magnitude and direction of the electric field intensity can be determined at each point of the field.

1.9. Calculation of the Potential Difference

The interrelation between the field intensity and the potential makes it possible to find the potential difference between two arbitrary points of this field from the known field intensity.

1. The electric field of a uniformly charged infinite plane is determined by the formula

$$E = \frac{\sigma}{2\varepsilon_0}, \qquad (1.66)$$

where σ is the *surface charge density*. Consider two points that are at distances x_1 and x_2 from the charged plane. The potential difference between these points is

$$\varphi_1 - \varphi_2 = \int_{x_1}^{x_2} E dx = \int_{x_1}^{x_2} \frac{\sigma}{2\varepsilon_0} dx = \frac{\sigma}{2\varepsilon_0}(x_2 - x_1). \qquad (1.67)$$

2. The electric field of two infinite parallel planes with charges of different signs is determined by the formula

$$E = \frac{\sigma}{\varepsilon_0}, \qquad (1.68)$$

where σ is the surface charge density.

The potential difference between the charged planes is equal

$$\varphi_1 - \varphi_2 = \int_0^d E dx = \int_0^d \frac{\sigma}{\varepsilon_0} dx = \frac{\sigma}{\varepsilon_0} d, \qquad (1.69)$$

where d is the distance between the planes.

3. The field of a uniformly charged spherical surface with radius R and the net charge Q outside the sphere ($r > R$) is calculated according to the formula

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}. \qquad (1.69)$$

The potential difference between two points located at distances r_1 and r_2 from the centre of the sphere ($r_1 > R, r_2 > R$) is equal to

$$\varphi_1 - \varphi_2 = \int_{r_1}^{r_2} E dr = \int_{r_1}^{r_2} \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \qquad (1.70)$$

If we assume that $r_1 = r$ and $r_2 \to \infty$, then the field potential outside the spherical surface is given by the expression

$$\varphi = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r}. \qquad (1.71)$$

Inside a spherical surface, the potential is everywhere the same and equal to

$$\varphi = \frac{Q}{4\pi\varepsilon_0 R}. \qquad (1.72)$$

4. The electric field of a volume-charged sphere with radius R and charge Q is characterized by a potential difference between two points located at distances r_1' and r_2' from the centre of the sphere ($r_1' < R, r_2' < R$). This potential difference is

$$\varphi_1 - \varphi_2 = \int_{r_1'}^{r_2'} E dr = \frac{Q}{8\pi\varepsilon_0 R^3}\left(r_2'^2 - r_1'^2\right). \qquad (1.73)$$

5. The electric field of a uniformly charged infinite cylinder with radius R at distances $r > R$ is given by the formula

$$E = \frac{1}{2\pi\varepsilon_0}\frac{\tau}{r}, \qquad (1.74)$$

where τ is the _linear charge density_. Consequently, the potential difference between two points located at distances r_1 and r_2 from the axis of the charged cylinder ($r_1 > R$, $r_2 > R$) is equal to

$$\varphi_1 - \varphi_2 = \int_{r_1}^{r_2} E dr = \frac{\tau}{2\pi\varepsilon_0}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{\tau}{2\pi\varepsilon_0}\ln\frac{r_2}{r_1}. \qquad (1.75)$$

Test questions

1. What is the difference between electrostatics and electrodynamics?
2. Name the two groups into which the conductors can be divided.
3. What type of charges can not be present in dielectrics?
4. Formulate the law of conservation of electric charge.
5. How will the interaction force between two electric charges in a vacuum change if the distance between them has decreased by 0.25 times?
6. Does the value of relative dielectric constant depend on the choice of a system of units?
7. Indicate the functional dependence of the modulus of the electric field intensity vector generated by a point electric charge on the distance.
8. Specify the method for constructing the direction of the electric field lines.
9. Give the definition of the electric field flux.
10. Formulate the principle of superposition of electrostatic fields.
11. Give examples of physical factors that prevent the convergence of the charges of the opposite sign in the electric dipole.
12. Formulate the Gauss's theorem for the electric field intensity.
13. Calculate the electric field intensity of a uniformly charged infinite plane.

14. Under what conditions can the electric field between two parallel plates of finite size be considered homogeneous?
15. Explain why the dependence of the electric field intensity on the distance will be different inside and outside the charged spherical surface.
16. Plot the electrostatic field intensity depending on the distance to the volume-charged sphere.
17. Write a formula for the electrostatic field intensity of a uniformly charged infinite cylinder.
18. Give a definition of a potential force field.
19. How does the potential of a point electric charge depend on the distance?

Draw the mutual arrangement of the electric field lines and the equipotential surfaces that are generated by a point electric charge.

Problem-solving examples

Problem 1.1

Problem description. Three identical positive charges $Q_1 = Q_2 = Q_3 = 2\,nC$ are located at the vertices of an equilateral triangle. The Q_4 charge is located in the centre of the triangle. The force of attraction of the charge Q_4 balances the force of mutual repulsion of the charges that are at the vertices of the triangle. Find the amount of charge Q_4.

Known quantities: $Q_1 = Q_2 = Q_3 = 2\,nC$.

Quantities to be calculated: Q_4.

Problem solution. All three charges, located at the vertices of the triangle, are in the same conditions. Therefore, to solve the problem, it suffices to find out which charge should be placed in the centre of the triangle so that one of the three charges, for example, Q_1, is in equilibrium..

In accordance with the principle of superposition, any other charge acts independently of the others. Consequently, the charge Q_1 will be in equilibrium if the vector sum of the forces acting on it is zero:

$$\vec{F}_2 + \vec{F}_3 + \vec{F}_4 = \vec{F} + \vec{F}_4 = 0. \qquad (1.1.1)$$

Where $\vec{F}_2$, $\vec{F}_3, \vec{F}_4$ are the forces with which charges Q_2, Q_3, Q_4 act on the charge Q_1;

$\vec{F}$ is the resultant of forces $\vec{F}_2$ and $\vec{F}_3$.

Since forces $\vec{F}$ and $\vec{F}_4$ are directed along one straight line, the vector equality can be replaced by a scalar sum

$$F - F_4 = 0 \text{ or } F_4 = F. \tag{1.1.2}$$

Expressing in the last equality F through F_2 and F_3, and taking into account that $F_3 = F_2$, we get

$$F_4 = F_2\sqrt{2(1+\cos\alpha)}. \tag{1.1.3}$$

Applying the Coulomb's law and considering that $Q_2 = Q_3 = Q_1$, we find

$$\frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_4}{r_1^{\,2}} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1^{\,2}}{r^2}\sqrt{2(1+\cos\alpha)}, \tag{1.1.4}$$

and

$$Q_4 = \frac{Q_1 r_1^{\,2}}{r^2}\sqrt{2(1+\cos\alpha)}, \tag{1.1.5}$$

where r_1 is the distance between charges Q_1 and Q_4;

r is the side of the triangle.

The ratio

$$r_1 = \frac{(r/2)}{\cos 30^0} = r/\sqrt{3}, \quad \cos\alpha = \cos 60^0 = 1/2 \tag{1.1.6}$$

holds for distances r and r_1 in an equilateral triangle

Then we get

$$Q_4 = Q_1/\sqrt{3}. \tag{1.1.7}$$

We insert the numerical values into last formula: $Q_4 = 1.16 \ nC$.

Answer. The charge Q_4 is 1.16 nC.

Problem 1.2

Problem description. Two long filaments are located at a distance of 12 cm from each other. Charges of the same sign are placed on the filaments. The linear charge density on the filaments is 10^{-7} C / cm.
Find the magnitude and direction of the strength of the resulting electric field at a point at a distance of 12 cm from each filament.

Known quantities: $r = 12 \ cm$; $\tau_1 = \tau_2 = 10^{-7} \ C/cm$; $a = 12 \ cm$.

Quantities to be calculated: $\vec{E}$.

Problem solution. We denote the linear charge density on the filaments as τ. So, we get $\tau_1 = \tau_2 = \tau$ for each filament.

Consequently, the intensity of each filament at point C, which is 12 cm away from each filament, is equal to

$$E_1 = E_2 = \frac{\tau}{2\pi \varepsilon\varepsilon_0 a}, \quad (1.2.1)$$

where a is the distance from the filament to point C;

ε is the relative dielectric constant of the medium;

ε_0 is the electrical constant.

According to the superposition principle, the resultant field strength is

$$\vec{E} = \vec{E}_1 + \vec{E}_2. \quad (1.2.2)$$

The line on which the vector $\vec{E}$ lies is perpendicular to the plane B passing through both filaments. Consider a triangle whose plane is perpendicular to the plane B. Two vertices of this triangle are on the filaments, and the third vertex coincides with point C. Such a triangle is equilateral with a side equal to the distance r between the filaments: $r = a$.

This implies

$$E = \sqrt{3}\frac{\tau}{2\pi \varepsilon\varepsilon_0\, a} = 2.6 \times 10^6 \ V/m. \quad (1.2.3)$$

Answer. The line on which the vector $\vec{E}$ lies is perpendicular to the plane B passing through both filaments. The magnitude of the electric field vector is $E = 2.6 \times 10^6 \ V/m$.

Problem 1.3

Problem description. The positive charges $Q_1 = 3\mu C$ and $Q_2 = 20\, nC$ are in vacuum at a distance of $r_1 = 1.5\, m$ from each other. Calculate the work A', which must be done to bring the charges closer to the distance $r_2 = 1\, m$.

Known quantities: $Q_1 = 3\mu C$, $Q_2 = 20\, nC$, $r_1 = 1.5\, m$, $r_2 = 1\, m$.

Quantities to be calculated: A'.

Problem solution. Suppose that the first charge remains stationary, while the second one moves in the field created by the first charge, approaching it from distance r_1 to distance r_2.

The work A' performed by an external force when a charge Q moves from one point of a field with a potential of φ_1 to another, whose potential is φ_2, is equal in magnitude and opposite in sign to work A, which make the field forces when the charge moves between the same points:

$$A' = -A. \tag{1.3.1}$$

The work of the field forces to move the charge is equal to

$$A = Q(\varphi_1 - \varphi_2). \tag{1.3.2}$$

Then the work A' of external forces can be written as

$$A' = -Q(\varphi_1 - \varphi_2) = Q(\varphi_2 - \varphi_1). \tag{1.3.3}$$

The potentials of the start and end points are equal

$$\varphi_1 = \frac{Q_1}{4\pi\varepsilon_0 r_1},\ \varphi_2 = \frac{Q_2}{4\pi\varepsilon_0 r_2}. \tag{1.3.4}$$

Substituting the expressions φ_1 and φ_2 into the formula for work A' and considering that for a given case the transferred charge is $Q = Q_2$, we get

$$A' = \frac{Q_1 Q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \tag{1.3.5}$$

We substitute numerically $A' = 180\ \mu J$.

Answer.The work A' is 180 μJ.

Problems

Problem A

Problem description. Two identical conducting balls are at a distance equal to $r = 30\,cm$. The balls have an electric charge. The force of attraction of the balls is $F_1 = 90\ \mu N$. After the balls were brought into contact and removed from each other to the previous distance, the strength of their mutual repulsion became equal to

$F_2 = 160\ \mu N$. Calculate the charges Q_1 and Q_2, which were on the balls before their contact. The diameter of the balls can be considered much smaller than the distance between them.

Answer. $Q_1 = 9 \times 10^{-8} C$, $Q_2 = -1 \times 10^{-8} C$.

Problem B

Problem description. A thin filament, the length of which is $L = 20 cm$, is uniformly charged with a linear density of $\tau = 10 nC/m$. At a distance of $a = 10 cm$ from the filament, a point charge $Q = 1 nC$ is located opposite its center. Calculate the force acting on this charge from the side of the charged filament.

Answer. $F = 1.27 \times 10^{-6} N$.

Problem C

Problem description. The distance between two long thin wires arranged parallel to each other is $d = 16 cm$. The wires are uniformly charged with opposite charges with a linear density of $|\tau| = 150 \mu C/m$. What is the electric field strength at a point at a distance of $a = 10 cm$ both from the first and from the second wire?

Answer. $E = 4.32 \times 10^{7} V/m$.

Problem D

Problem description. A flat square plate with a side length of $a = 10 cm$ is located at some distance from an infinite uniformly charged ($\sigma = 1 \mu C/m^2$) plane. The plane of the plate makes an angle of $\beta = 30°$ with the lines of force of the electric field. Find the displacement flux ψ through this plate.

Answer. $\psi = 2.5 \times 10^{-9} C$.

Problem E

Problem description. A metal ball with a diameter of $d = 2 cm$ is negatively charged to potential $\varphi = 150 V$. How many electrons are on the surface of the ball?

Answer. $N = 1.04 \times 10^{9}$.

CHAPTER 2. DIELECTRICS IN ELECTRIC FIELD

2.1. Types of Dielectrics

The dielectric consists of atoms and molecules. Since the positive charge of all the nuclei of the molecule is equal to the total charge of the electrons, the molecule as a whole is electrically neutral. The molecule can be considered as an electric dipole if the positive charges of the nuclei of the molecule are replaced by the total charge $+Q$ located in the "centre of gravity" of the positive charges, and the charges of all electrons are replaced by the total negative charge $-Q$ located in the "centre of gravity" of the negative charges.

The first type of dielectrics includes substances $(N_2, H_2, O_2, CO_2, CH_4, ...)$ whose molecules have a symmetrical structure, that is, the "centres of gravity" of positive and negative charges coincide in the absence of an external electric field, and consequently the dipole moment $\vec{p}$ of the molecule is zero. The molecules of such dielectrics are called *nonpolar molecules*. Under the influence of an external electric field, the charges of nonpolar molecules are displaced in opposite directions (positive in the direction of the electric field intensity, and negative against the direction of the electric field intensity) and the dipole moment of the molecule becomes nonzero.

The second type of dielectrics includes substances $(H_2O, NH_3, SO_2, CO, ...)$ whose molecules have an asymmetric structure, that is, the "centres of gravity" of the positive and negative charges do not coincide. Thus, in the absence of an external electric field, these molecules have a dipole moment. The molecules of such dielectrics are called *polar molecules*. The dipole moments of polar molecules due to thermal motion are oriented in space randomly and their resultant moment is zero in the absence of an external field. If such a dielectric is placed in an external field, then the forces of this field will tend to rotate the dipoles along the field and a nonzero resultant moment arises.

The third type of dielectrics includes substances $(NaCl, KCl, KBr, ...)$ whose molecules have an ionic structure. *Ionic crystals* have spatial lattices with a regular alternation of ions of different signs. The behaviour of individual molecules is not considered for such crystals. Ionic crystals can be considered only as a system of two ion sub lattices inserted one into another. In the presence of an external electric field, some deformation of the crystal lattice of the ionic crystal or relative displacement of the sub lattices occurs. This displacement is the reason for the appearance of dipole moments.

Thus, placing all three types of dielectrics in an external electric field leads to the appearance of a nonzero resultant electrical moment of dielectric, or, in other words, leads to the polarization of dielectric. The process of dipoles orientation or the process of the appearance of field-oriented dipoles under the action of an electric field is called *polarization*.

Accordingly, three types of dielectrics are distinguished by three types of polarization:

a) electronic or *deformation polarization* of a dielectric with nonpolar molecules, associated with the appearance of an induced dipole moment in atoms due to deformation of electronic orbits;
b) dipole or *orientation polarization* of a dielectric with polar molecules, associated with the orientation of the existing dipole moments of molecules over the field. Thermal motion prevents the complete orientation of the molecules, but as a result of the combined effect of both factors (electric field and thermal motion), a predominant orientation of the dipole moments of molecules over the field arises. This orientation increases with increasing electric field intensity and with temperature decreasing;
c) ion polarization of dielectrics with ionic crystal lattices, associated with the displacement of the sub lattice with positive ions along the field. The sub lattice with negative ions is displaced against the field. Such a displacement leads to the appearance of dipole moments.

2.2. Polarization. Electric Field Intensity in Dielectric

The dielectric polarization is associated with nonzero dipole moment

$$p_V = \sum_i \vec{p}_i \,, \tag{2.1}$$

where $\vec{p}_i$ is the dipole moment of one molecule. Polarization is defined as the dipole moment of the unit volume of dielectric

$$\vec{P} = \frac{\vec{p}_V}{V} = \sum_i \frac{\vec{p}_i}{V} \,. \tag{2.2}$$

It follows from experiment that for a large class of dielectrics (with the exception of ferroelectrics) the polarization $\vec{P}$ depends linearly on the electric field intensity $\vec{E}$. Let us consider the case of an isotropic dielectric located in a weak external electric field. In this case, for the polarization, we get

$$\vec{P} = \chi\varepsilon_0\vec{E} \,, \tag{2.3}$$

where χ is the *dielectric susceptibility* of the substance, which characterizes the properties of the dielectric; χ is a dimensionless quantity.

The value of χ is always positive. For most dielectrics (solid and liquid) the susceptibility is equal to several units (although, for alcohol $\chi \sim 25$, for water $\chi \sim 80$).

Consider two infinite parallel planes with charges of opposite signs. These planes create a uniform external electrostatic field with a intensity of $\vec{E}_0$. We insert a plate from a homogeneous dielectric in the field $\vec{E}_0$, and arrange it as shown in

Figure 2.1. Under the action of the field, the dielectric is polarized, that is, the charges are shifted. Positive charges are displaced in the direction of the electric field, and negative charges are displaced against the field. As a result, on the right side of the dielectric, which is located near the negative plane, an excess of positive charge with a surface density $+\sigma'$ is formed, and a negative charge with a surface density of $-\sigma'$ appears on the left side.

These uncompensated charges, which appear as a result of the polarization of the dielectric, are called *bound charges*. The surface density of bound charges σ' is less than the surface density of free charges σ on the planes of the capacitor. As a result, not all field $\vec{E}$ is compensated by the dielectric charge field. Some of the force lines pass through the dielectric, while the other part of the force lines terminates on bound charges.

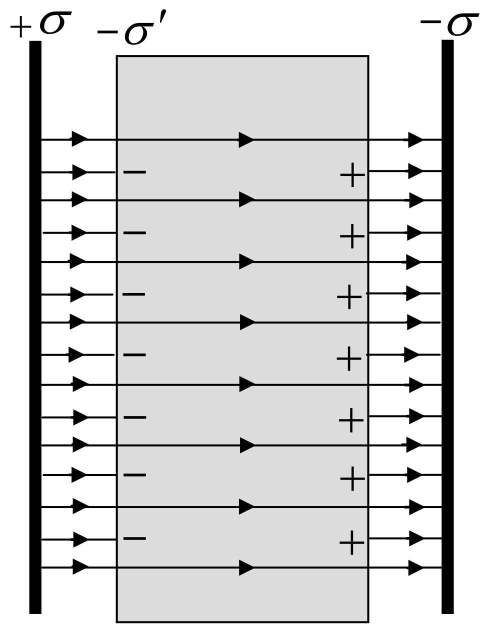


Figure 2.1. Electric field created by two infinite charged planes.

Consequently, the polarization of the dielectric causes a decrease in the electric field intensity in it in comparison with the original external field. The electric field strength outside the dielectric is

$$\vec{E} = \vec{E}_0 . \qquad (2.4)$$

Thus, the appearance of bound charges leads to the appearance of an additional electric field (the field created by the bound charges), which is directed against the external field (the field created by free charges) and weakens it. The resulting field inside the dielectric is

$$E = E_0 - E' . \qquad (2.5)$$

The field created by two infinite charged planes is equal

$$E' = \sigma' / \varepsilon_0 \ , \tag{2.6}$$

therefore

$$E = E_0 - \frac{\sigma}{\varepsilon_0} . \tag{2.7}$$

Let us determine the surface density of bound charges $+\sigma'$. The total dipole moment of the dielectric plane is equal

$$PV = PSd \ , \tag{2.8}$$

where S is the area of the plane;

d is its thickness.

On the other hand, the total dipole moment is equal to the product of the bound charge of each plane

$$Q' = \sigma' S \tag{2.9}$$

by the distance d between them, i.e.

$$PV = \sigma' Sd . \tag{2.10}$$

Thus,

$$PSd = \sigma' Sd \tag{2.11}$$

or

$$\sigma' = P , \tag{2.12}$$

i.e. the surface density of the bound charges σ' is equal to the polarization P.

Then for the intensity we get

$$E = E_0 - \chi E \tag{2.13}$$

from which the intensity of the resulting field inside the dielectric is

$$E = \frac{E_0}{1+\chi} = \frac{E_0}{\varepsilon} . \tag{2.14}$$

The dimensionless quantity

$$\varepsilon = 1 + \chi \tag{2.15}$$

is called the *dielectric permittivity* of the medium. The value ε shows how many times the field is weakened by a dielectric and quantitatively characterizes the property of a dielectric to polarize in an electric field.

2.3. Gauss Theorem for Electrostatic Field in Dielectric

The electrostatic field intensity depends on the properties of the medium. In a homogeneous isotropic medium, the electric field intensity E is inversely proportional to ε. The intensity vector $\vec{E}$, passing through the dielectric boundary, undergoes an abrupt change, thereby creating inconvenience in the calculation of electrostatic fields. Therefore, in addition to the intensity vector, the electric field is also characterized by the electric displacement vector. The electric displacement vector for an electrically isotropic medium is

$$\vec{D} = \varepsilon_0 \varepsilon \vec{E} . \tag{2.16}$$

The vector of *electric displacement* can be expressed as

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} . \tag{2.17}$$

Find out the factors that affect the electrical displacement. Bound charges appear in the dielectric in the presence of an external electrostatic field created by a system of free electric charges. Consequently, an additional field of bound charges is superimposed on the electrostatic field of free charges in the dielectric. The resulting field in the dielectric is described by the intensity vector $\vec{E}$, and therefore it depends on the properties of the dielectric. Vector $\vec{D}$ describes the electrostatic field created by free charges. Bound charges in dielectric can cause a redistribution of free charges that create an electric field. Therefore, vector $\vec{D}$ characterizes the electrostatic field created by free charges, but with such a distribution in space as is observed in the presence of a dielectric. Similarly to field $\vec{E}$, the field $\vec{D}$ is represented by electric displacement lines, the direction and density of which are determined exactly as for field lines. Lines of vector $\vec{E}$ can start and end on any charges both on free and on bonded ones, while the lines of vector $\vec{D}$ can start only on free charges. The lines of vector $\vec{D}$ are not interrupted in the regions where the bonded charges are located. For an arbitrary closed surface S, the flux of the vector $\vec{D}$ through this surface is

$$\vec{\Phi}_D = \oint_S \vec{D} d\vec{S} = \oint_S D_n dS . \tag{2.18}$$

The Gauss's theorem for the electrostatic field in a dielectric can be represented in the form

$$\oint_S \vec{D} d\vec{S} = \oint_S D_n dS = \sum_{i=1}^{n} Q_i . \tag{2.19}$$

The flux of the displacement vector of the electrostatic field in a dielectric through an arbitrary closed surface is equal to the algebraic sum of free electric charges that are inside this surface. In this form, the Gauss's theorem is valid for an electrostatic field for both homogeneous and isotropic media. Vectors $\vec{D}$ and $\vec{E}$ are not collinear in anisotropic dielectrics [1].

For a vacuum

$$D_n = \varepsilon_0 E_n, \tag{2.20}$$

and $\varepsilon = 1$ in the case then the flux of the intensity vector $\vec{E}$ through an arbitrary closed surface is equal to

$$\oint_S \varepsilon_0 E_n dS = n \sum_{i=1} Q_i \ . \tag{2.21}$$

Since the electric field sources in medium are both free and bound charges, the Gauss's theorem for field $\vec{E}$ can be written in the most general form as

$$\oint_S \varepsilon_0 \vec{E} dS = \oint_S \varepsilon_0 E_n dS = \sum_{i=1}^{n} Q_i + \sum_{i=1}^{k} Q_{ib} \ , \tag{2.22}$$

where $\sum_{i=1}^{n} Q_i$, and $\sum_{i=1}^{k} Q_{i_b}$ are, respectively, the algebraic sums of free and bound charges enclosed by a closed surface.

However, this formula is unacceptable for describing the field $\vec{E}$ in a dielectric, since it expresses the properties of an unknown field $\vec{E}$ through bound charges, which, in turn, are determined by it. This again proves the advisability of introducing an electric displacement vector.

2.4. Interface between Two Dielectric Media

Consider the relationship between vectors $\vec{E}$ and $\vec{D}$ at the interface between two homogeneous isotropic dielectrics (whose dielectric permittivity are ε_1 and ε_2) in the absence of free charges.

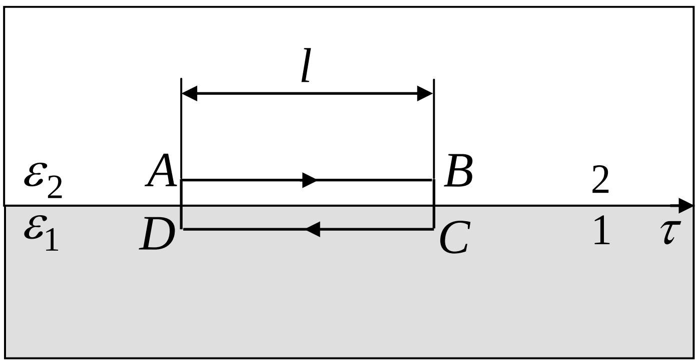


Figure 2.2. Interface between Two Dielectric Media.

Let us construct near the interface between dielectrics 1 and 2 a small closed rectangular contour ABCDA with perimeter $2(l+h)$ (Figure 2.2). According to the circulation theorem for vector $\vec{E}$, formula

$$\int_{ABCDA} \vec{E}d\vec{l} = 0 \tag{2.23}$$

holds, whence

$$E_{2\tau}l - E_{1\tau}l = 0 \tag{2.24}$$

(the signs of the integrals over AB and CD are different, since the integration paths are opposite, and the integrals over the segments $BC = h$ and $DA = h$ are negligibly small). Therefore,

$$E_{1\tau} = E_{2\tau}\,. \tag{2.25}$$

Replacing the projections of the vector $\vec{E}$ by the projections of the vector $\vec{D}$ divided by $\varepsilon_0\varepsilon$, we obtain

$$\frac{D_{1\tau}}{D_{2\tau}} = \frac{\varepsilon_1}{\varepsilon_2}\,. \tag{2.26}$$

We construct a straight cylinder of negligible height at the interface between two dielectrics. One cylinder is based at the first dielectric, and the other is based at the second dielectric. The bases ΔS are so small that within each of them the vector $\vec{D}$ is the same. According to the Gauss's theorem

$$D_{2n}\Delta S - D_{1n}\Delta S = 0 \tag{2.27}$$

(normals $\vec{n}$ and $\vec{n}'$ to the bases of the cylinder are directed into opposite direction). Therefore

$$D_{1n} = D_{2n}\,. \tag{2.28}$$

Replacing the projections of the vector $\vec{D}$ by the projections of the vector $\vec{E}$ multiplied by $\varepsilon_0\varepsilon$, we obtain

$$\frac{E_{1n}}{E_{2n}} = \frac{\varepsilon_2}{\varepsilon_1}\,. \tag{2.29}$$

Thus, when passing through the interface between two dielectric media, the tangential component of the vector $\vec{E}$ (E_τ) and the normal component of the vector $\vec{D}$ (D_n) change continuously, and the normal component of the vector $\vec{E}$ (E_n) and the tangential component of the vector $\vec{D}$ (D_τ) are discontinuous.

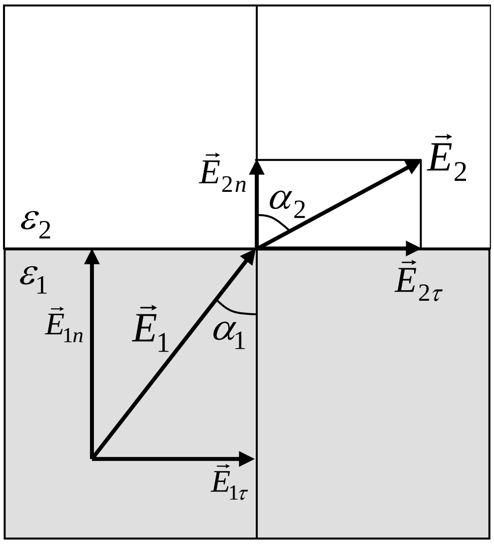


Figure 2.3. Refraction of electric field lines.

From the conditions for the components of the vectors $\vec{E}$ and $\vec{D}$ it follows that the lines of these vectors undergo a break (refracted).

Let us find the relationship between the angles α_1 and α_2 provided that $\varepsilon_1 > \varepsilon_2$ (Figure 2.3). Formulas:

$$E_{2\tau} = E_{1\tau} \tag{2.30}$$

and

$$\varepsilon_2 E_{2n} = \varepsilon_1 E_{1n} \tag{2.31}$$

are valid in this case.

We decompose the vectors $\vec{E}_1$ and $\vec{E}_2$ into tangential and normal components near the interface. From Figure 2.3 follows that

$$\frac{tg\alpha_2}{tg\alpha_1} = \frac{E_{2\tau} / E_{2n}}{E_{1\tau} / E_{1n}}. \tag{2.32}$$

Taking into account the above conditions, we obtain the law of refraction of force lines of electric field intensity $\vec{E}$ (and electric field displacement $\vec{D}$ lines)

$$\frac{tg\alpha_2}{tg\alpha_1} = \frac{\varepsilon_2}{\varepsilon_1}. \tag{2.33}$$

Lines $\vec{E}$ and $\vec{D}$ are removed from the normal when entering a dielectric with a larger permittivity.

Test questions

1. Do gravity centers of positive and negative charges of non-polar molecules coincide in the presence of an external electric field?
2. List a few examples of nonpolar molecules.

3. Does the dipole moment modulus of polar molecules depend on the intensity of an external electric field?
4. List a few examples of polar molecules.
5. Specify the reason that the dipole moments of polar molecules are oriented arbitrarily in the absence of an external electric field.
6. Describe the ionic crystals.
7. List a few examples of ionic crystals.
8. Is it possible to consider ionic crystals as a structure consisting of only one crystal lattice?
9. Specify the cause of the appearance of dipole moments in ionic crystals.
10. Give the definition of polarization.
11. What type of polarization is a consequence of the appearance of an induced dipole moment in atoms due to deformation of electronic orbit?
12. Describe the orientation polarization.
13. Indicate competing processes when the resulting dipole moment appears in orientational polarization.
14. What type of polarization is caused by the mutual displacement of the sublattices of positive and negative ions?
15. Write a formula for determining the polarization.
16. Specify the dimension, sign and characteristic values of the dielectric susceptibility.
17. Specify the relative position of the vector of the external electric field intensity and the vector of the electric field intensity caused by the bound charges.
18. Indicate how the relative dielectric permittivity and dielectric susceptibility are interrelated.
19. Formulate the Gauss theorem for electrostatic field in dielectric.
20. Specify the conditions for the electric field strength and the electric displacement vector at the interface of two dielectrics.

Problem-solving examples

Problem 2.1

<u>*Problem description*</u>. The distance between the iodine atom and the alpha particle is $r = 1 nm$. The atom has an induced dipole electric moment $p = 1.9 \times 10^{-32} C \times m$. Determine the polarizability β of the iodine atom.

<u>*Known quantities*</u>: $r = 1 nm$, $p = 1.9 \times 10^{-32} C \times m$.

<u>*Quantities to be calculated*</u>: β.

<u>*Problem solution*</u>. Polarizability can be determined from the formula

$$\beta = \frac{p}{\varepsilon_0 E_L}, \tag{2.1.1}$$

where p is the induced dipole moment of the atom;
E_L is the intensity of the local electric field in which the atom is located.

The local electric field is created by a α-particle. The strength of the local electric field is

$$E_L = E = \frac{2|e|}{4\pi\varepsilon_0 r^2}. \tag{2.1.2}$$

Then for polarizability we can write

$$\beta = \frac{2\pi r^2 p}{|e|}. \tag{2.1.3}$$

And finally, with numbers $\beta = 7.47 \times 10^{-30}\ m^3$.

<u>*Answer*</u>. The polarizability of the iodine atom is $\beta = 7.47 \times 10^{-30}\ m^3$.

Problem 2.2

<u>*Problem description*</u>. Endless plate charged with surface charge density $\sigma = 12\,nC/m^2$. There is air on one side of the plate, and oil on the other ($\varepsilon = 2$). Determine the electric field in oil and in air.

<u>*Known quantities*</u>: $\sigma = 12\,nC/m^2$, $\varepsilon = 2$.

<u>*Quantities to be calculated*</u>: E_0, E.

<u>*Problem solution*</u>. Since the plate is in contact with different substances, we first determine the electric displacement vector $\vec{D}$, and then, knowing the relationship between the electric field strength $\vec{E}$ and the electric displacement $\vec{D}$, we determine the electric field strength in different substances. We introduce the following notation: $\vec{D}_0$ is the vector of electrical displacement in air, $\vec{D}$ is the vector of electrical displacement in oil. We can write for these vectors:

$$\vec{D}_0 = \varepsilon_0 \vec{E}_0,\ \vec{D} = \varepsilon\varepsilon_0 \vec{E}_0,$$

where $\vec{E}_0$ is the electric field strength in the air.

We write the Gauss's theorem for the vector $\vec{D}$:

$$\oint_S \vec{D}d\vec{S} = Q_T \ , \qquad (2.2.1)$$

where Q_T is total charge.

As a free Gauss's surface, we choose a cylinder whose axis of symmetry is perpendicular to the plate. The flux of the vector $\vec{D}$ is determined only by the flow through the ends of the cylinder, and the flow through the side surface of the cylinder is zero ($\vec{D}d\vec{S}_S = DdS_S \cos 90^0 = 0$):

$$\oint_S \vec{D}d\vec{S} = D_0 S + DS = (D_0 + D)S = (\varepsilon_0 E_0 + \varepsilon_0 \varepsilon E_0)S \, . \qquad (2.2.2)$$

The total charge is determined by the surface charge density

$$Q_T = \int_S \sigma dS = \sigma S \, . \qquad (2.2.3)$$

Equating the right sides of the last two equations, we get

$$(\varepsilon_0 E_0 + \varepsilon_0 \varepsilon E_0)S = \sigma S \, . \qquad (2.2.4)$$

Then

$$(1 + \varepsilon)\varepsilon_0 E_0 = \sigma \, . \qquad (2.2.5)$$

This implies

$$E_0 = \frac{\sigma}{\varepsilon_0 (1 + \varepsilon)} \, . \qquad (2.2.6)$$

The electric field strength in oil is

$$E = \frac{E_0}{\varepsilon} = \frac{\sigma}{\varepsilon_0 (1 + \varepsilon)\varepsilon} \, . \qquad (2.2.7)$$

For the given numerical values we get $E_0 = 452.4 \, V/m$, $\quad E = 226.2 \, V/m$.

<u>*Answer*</u>. The electric field strength in air is $E_0 = 452.4 \, V/m$. The electric field strength in oil is $E = 226.2 \, V/m$.

Problem 2.3

<u>*Problem description*</u>. Ebonite solid ball of radius $R = 5 cm$ contains a charge evenly distributed throughout the volume with a bulk density of $\rho = 15 nC/m^3$. Determine

the electric field strength E, the displacement of the electric field D at the following points: 1) at a distance of $r_1 = 3cm$ from the center of the ball, 2) on the surface of the ball ($r_2 = R$), 3) at a distance of $r_3 = 10\ cm$ from the center of the ball.

<u>*Known quantities*</u>: $R = 5cm$, $\rho = 15nC/m^3$, $r_1 = 3cm$, $r_3 = 5\ cm$, $r_3 = 10\ cm$.

<u>*Quantities to be calculated*</u>: D_1, D_2, D_3, E_1, E_{2d}, E_{2V}, E_3.

<u>*Problem solution*</u>. Sources of the electric field are both foreign (Q_f) and bound (Q_b) electric charges. Therefore, to apply the Gauss's theorem, it is necessary to know the values Q_f and Q_b. The flux of vector $\vec{D}$ through a closed surface depends only on the algebraic sum of foreign charges that encompass the closed surface. The lines of vector $\vec{D}$ are not interrupted when they cross the interface between two dielectrics. Therefore, to calculate the electric field in a dielectric, it is convenient to use the Gauss theorem for the vector $\vec{D}$:

$$\oint_S \vec{D}d\vec{S} = \oint_S D_n dS = \sum Q_{in}, \tag{2.3.1}$$

where Q_{in} is internal charge.

We'll take a spherical surface of radius r as a closed surface

$$S = 4\pi r^2. \tag{2.3.2}$$

For the case when $r < R$, the charge covered by a sphere of radius r is

$$Q_{in} = \frac{4\pi r^3 \rho}{3}. \tag{2.3.3}$$

In this case

$$D = \left(\frac{4\pi r^3 \rho}{3}\right) / \left(4\pi r^2\right) = \frac{\rho r}{3}. \tag{2.3.4}$$

The displacement of the electric field at points $r_1 < R$ and $r_2 = R$ is equal, respectively,

$$D_1 = \frac{\rho r_1}{3} \text{ and } D_2 = \frac{\rho R}{3}. \tag{2.3.5}$$

Since $r_3 > R$, the charge Q_{in} is completely covered by a sphere of radius r_3. For this case, the displacement D of the electric field is

$$D_3 = \left(\frac{4\pi R^3 \rho}{3}\right) / \left(4\pi {r_3}^2\right) = \frac{\rho R^3}{3{r_3}^2}. \qquad (2.3.6)$$

The relationship between the electric field strength E and the displacement of the electric field D is

$$\vec{D} = \varepsilon\varepsilon_0 \vec{E}. \qquad (2.3.7)$$

The electric field at point $r_1 < R$ is 4

$$E_1 = \frac{\rho r_1}{3\varepsilon\varepsilon_0}. \qquad (2.3.8)$$

For cases where point $r_2 \leq R$ is in the dielectric and in vacuum, the electric field is

$$E_{2d} = \left(\frac{4\pi R^3 \rho}{3}\right) / \left(\varepsilon\varepsilon_0 4\pi R^2\right) = \frac{\rho R}{3\varepsilon\varepsilon_0} \text{ (dielectric)},$$

$$E_{2V} = \frac{\rho R}{3\varepsilon_0} \text{ (vacuum)}. \qquad (2.3.9)$$

During the transition from a dielectric to a vacuum, an abrupt change in the electric field strength occurs by a factor of ε. Therefore, for case $r_3 > R$ we get

$$E_3 = \frac{\rho R^3}{3\varepsilon_0 {r_3}^2}. \qquad (2.3.10)$$

We substitute numerically:

$D_1 = 1.5 \times 10^{-10}\ C/m^2$, $D_2 = 2.49 \times 10^{-10}\ C/m^2$, $D_3 = 6.25 \times 10^{-12}\ C/m^2$, $E_1 = 5.7\ V/m$, $E_{2d} = 9.41\ V/m$, $E_{2V} = 28.24\ V/m$, $E_3 = 7.05\ V/m$.

Answer. The displacements of the electric field are $D_1 = 1.5 \times 10^{-10}\ C/m^2$, $D_2 = 2.49 \times 10^{-10}\ C/m^2$, and $D_3 = 6.25 \times 10^{-12}\ C/m^2$. The electric field intensities are $E_1 = 5.7\ V/m$, $E_{2d} = 9.41\ V/m$, $E_{2V} = 28.24\ V/m$, $E_3 = 7.05\ V/m$.

Problems

Problem A

Problem description. A plane-parallel glass plate ($\varepsilon = 7$) with a thickness of $d = 1.5 mm$ and an area of $S = 200\, cm^2$ was placed perpendicular to the field in a uniform electrostatic field of intensity $E = 700 V/m$. Determine the electrical displacement inside the plate and the polarization of the glass.

Answer. $D = 6.2 \times 10^{-9}\, C/m^2$, $P = 5.31 \times 10^{-9}\, C/m^2$.

Problem B

Problem description. At a height of $L = 1cm$ above the kerosene ($\varepsilon = 2$) surface a point charge $Q = 314 nC$ hangs. Find the density of the bound charges on the surface of the liquid at a distance of $d = 5cm$ from the source.

Answer. $\sigma = 2 \times 10^{-6}\, C/m^2$.

Problem C

Problem description. Inside the dielectric ball, the permittivity of which is $\varepsilon = 5$, a uniform electric field with a intensity of $E = 100 V/m$ was created. Calculate the maximum surface density of bound charges.

Answer. $\sigma_{\max} = 3.54 \times 10^{-9}\, C/m^2$.

Problem D

Problem description. On opposite sides of the boundary of air and kerosene ($\varepsilon = 2$) on the same perpendicular to it are two point charges. The charge in kerosene is four times more than the charge in the air. The position of these charges is such that at the point on the border between them there are no polarization charges. How many times do the distances differ from the charge to the boundary in the air and in kerosene?

Answer. $N = 0.5$.

Problem E

Problem description. The vector of electric field intensity in water ($\varepsilon_1 = 81$) near the border with glass is directed at an angle of $\alpha = 60°$ to the normal. Calculate the angle β between the normal and the direction of the electric field in the glass, if the permittivity in it is $\varepsilon_2 = 7$.

Answer. $\beta = 0.149 rad$.

CHAPTER 3. CONDUCTORS IN ELECTRIC FIELD

3.1. Charge Distribution in Conductors

Let us consider the case of charged conductor or conductor in an external electric field. The electric field will act on the charges of the conductor and lead to their displacement. The charge displacement (electric current) will continue until an equilibrium distribution of charges is established, at which the electrostatic field inside the conductor vanishes. The movement of electric charges occurs in a very short time. In fact, if the field were not equal to zero, then an ordered movement of charges would arise in the conductor without consuming energy from an external source, which contradicts the law of conservation of energy. So, the electric field intensity at all points inside the conductor is zero $E = 0$.

The absence of a field inside the conductor means that the potential at all points inside the conductor is constant $(\varphi = const)$, i.e. the conductor surface in the electrostatic field is equipotential. It follows from this that the field intensity vector on the outer surface of the conductor is directed along the normal to each point of its surface. Otherwise, under the action of the tangential component $\vec{E}$, charges would begin to move along the surface of the conductor, which, in turn, would contradict the equilibrium distribution of charges.

Uncompensated charges are located only on the surface of the conductor. This follows directly from Gauss's theorem, according to which the charge Q inside the conductor is equal to

$$Q = \oint_S \vec{D} d\vec{S} = \oint_S D_n dS = 0 \tag{3.1}$$

since at all points inside the surface $D = 0$.

Let us find the relationship between the electric field intensity E near the surface of a charged conductor and the surface density of charges on its surface. We apply the Gauss's theorem to an infinitesimal cylinder with bases ΔS, which crosses the conductor-dielectric boundary. The axis of the cylinder is oriented along the vector $\vec{E}$. The flux of the electric displacement vector through the inner part of the cylindrical surface is zero, since inside the conductor the value of $\vec{E}_1$ (and consequently $\vec{D}_1$) is equal to zero. It follows that the flux of vector $\vec{D}$ through a closed cylindrical surface is determined only by flow through the outer base of the cylinder. According to the Gauss's theorem, this flux ($D\Delta S$) is equal to the sum of the charges ($Q = \sigma \Delta S$) covered by the surface

$$D\Delta S = \sigma \Delta S \tag{3.2}$$

i.e.

$$D = \sigma \tag{3.3}$$

or

$$E = \frac{\sigma}{\varepsilon_0 \varepsilon}, \qquad (3.4)$$

where ε is the dielectric permeability of the medium surrounding the conductor.

Thus, the intensity of the electrostatic field at the surface of the conductor is determined by the surface density of charges. If a neutral conductor is placed in an external electrostatic field, then free charges (electrons, ions) will shift. Positive charges will shift in the direction of the field, and negative charges will shift in the direction opposite to the field (Figure 3.1, A). At one end of the conductor, an excess of positive charge will accumulate, at the other end of the conductor there is an excess of negative charge.

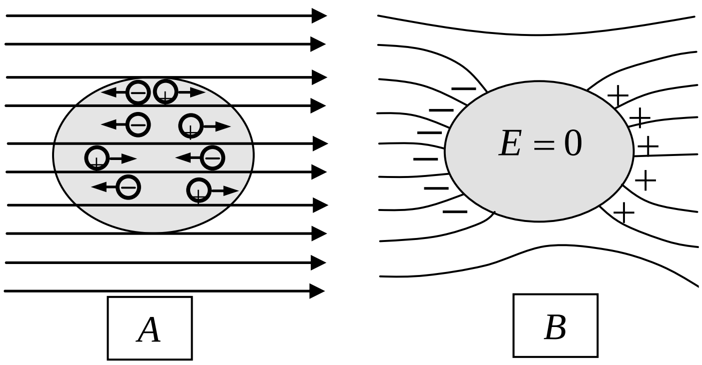


Figure 3.1. Distribution of electrical charges inside the conductor.

These charges are called <u>*induced charges*</u>. This process will occur until the electric field intensity inside the conductor becomes zero, and field lines outside the conductor become perpendicular to its surface (Figure 3.1, B). Thus, a neutral conductor located in the electrostatic field breaks a portion of the field lines; they end on negative induced charges and again start on positive ones. Induced charges are distributed on the outer surface of the conductor. The phenomenon of redistribution of surface charges on a conductor in an external electrostatic field is called <u>*electrostatic induction*</u>.

It follows from Figure 3.1, B that the induced charges appear on the conductor due to their displacement under the action of the field. The value of σ is the surface density of the displaced charges. The electrical displacement D near the conductor is numerically equal to the surface density of the displaced charges. Therefore, the vector $\vec{D}$ is called the <u>*electric displacement vector*</u>.

Since there are no charges in the state of equilibrium inside the conductor, the appearance of a cavity inside the conductor will not affect the arrangement of the charges and thereby the electrostatic field. Consequently, the field will be absent inside the cavity. If now this conductor is grounded, then the potential at all points of the cavity will be zero, that is, the cavity is completely isolated from the influence of external electrostatic fields. This property is the cause of electrostatic protection, i.e. screening of bodies, for example measuring instruments, from the influence of external electrostatic fields. A dense metal grid can be used to protect instead of a solid conductor. Such a grid, by the way, is effective in the presence of not only permanent, but also variable electric fields.

3.2. Electrical Capacitance of a Solitary Conductor

Consider a conductor that is remote from other conductors, bodies and charges. Such a conductor is called a solitary conductor. Its electric potential is directly proportional to the charge of the conductor. It follows from experience that different conductors, being equally charged, have different potentials. Therefore, for a solitary conductor, we can write

$$Q = C\varphi . \tag{3.5}$$

The quantity

$$C = \frac{Q}{\varphi} \tag{3.6}$$

is called the *electrical capacity* (or simply the capacity) of a solitary conductor.

The conductor capacity depends on its size and shape, but it does not depend on the material, the aggregate state, the shape and dimensions of the cavities inside the conductor. This is due to the fact that excess charges are distributed on the outer surface of the conductor. The capacity does not depend either on the charge of the conductor or on its potential. The potential of a solitary sphere with a radius R in a homogeneous medium with a permittivity of ε is

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\varepsilon R} . \tag{3.7}$$

From the above formulas it can be obtained that the capacity of the sphere is

$$C = 4\pi\varepsilon_0 \varepsilon R . \tag{3.8}$$

3.3. Capacitors

Devices that are capable of accumulating significant charges with small dimensions and small potentials with respect to surrounding bodies are often needed for practical applications. In other words, such devices must have a large capacity. These devices are called *capacitors*. Reducing the distance between the charged body and other bodies leads to the appearance of induced (on a conductor) or connected (on dielectric) charges. Charges with opposite sign are placed near charged conductor. These charges weaken the field created by the charged conductor, i.e. lower the potential of the conductor, which leads to an increase in its electrical capacitance.

The capacitor consists of two conductors (plates) separated by a dielectric. The surrounding bodies should not affect the capacitance of the capacitor. Therefore, the conductors are shaped so that the field created by the accumulated charges is concentrated in the narrow gap between the capacitor plates. This condition is satisfied by: 1) two flat plates; 2) two coaxial cylinders; 3) two concentric spheres. The shape of the capacitor determines their type. The technology uses flat, cylindrical and spherical capacitors.

Since the field is concentrated inside the capacitor, the field lines begin on one plate and end on the other, so the free charges on different plates are equal in modulus with opposite charges. The physical quantity, which is equal to the ratio of the charge Q , accumulated in the capacitor, to the potential difference $(\varphi_1 - \varphi_2)$ between its plates is called the *capacitance* of the capacitor

$$C = \frac{Q}{\varphi_1 - \varphi_2}. \tag{3.9}$$

We calculate the capacitance of a flat capacitor consisting of two parallel metal plates of area S each, located at a distance d from each other and having charges of $+Q$ and $-Q$. The edge effects can be neglected and the field between the plates considered homogeneous in the case then the distance between the plates is small in comparison with their linear dimensions. Let's consider the presence of a dielectric between the plates. In this case the potential difference between plates is

$$\varphi_1 - \varphi_2 = \frac{\sigma d}{\varepsilon_0 \varepsilon}, \tag{3.10}$$

where ε is the permittivity.

Then from the last formula, replacing

$$Q = \sigma S, \tag{3.11}$$

we obtain an expression for the capacitance of a flat capacitor

$$C = \frac{\varepsilon_0 \varepsilon S}{d}. \tag{3.12}$$

We calculate the capacitance of a cylindrical capacitor consisting of two hollow coaxial cylinders with radii r_1 and r_2 $(r_2 > r_1)$ inserted one into the other. Neglecting edge effects, we assume the field to be radially symmetric and concentrated between the cylindrical plates. The potential difference between the plates is calculated from the formula for the field of a uniformly charged infinite cylinder with a linear charge density

$$\tau = \frac{Q}{l}, \tag{3.13}$$

where l is the length of the plates.

The potential difference for the case then dielectric is placed between the plates is

$$\varphi_1 - \varphi_2 = \frac{\tau}{2\pi\varepsilon_0\varepsilon}\ln\frac{r_2}{r_1}. \tag{3.14}$$

Consequently, the expression for the capacitance of a cylindrical capacitor can be written in the following form

$$C = \frac{2\pi\varepsilon_0\varepsilon l}{\ln\dfrac{r_2}{r_1}}. \tag{3.15}$$

Let us determine the capacitance of a spherical capacitor consisting of two concentric plates separated by a spherical dielectric layer. We use the formula for the potential difference between two points located at distances r_1 and r_2 ($r_2 > r_1$) from the centre of a charged spherical surface. Taking into account the presence of a dielectric between the plates, the potential difference is

$$\varphi_1 - \varphi_2 = \frac{Q}{4\pi\varepsilon_0\varepsilon}\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \tag{3.16}$$

Then we obtain for the capacitance of the spherical capacitor

$$C = 4\pi\varepsilon_0\varepsilon\frac{r_1 r_2}{r_2 - r_1}. \tag{3.17}$$

It follows from the above formulas that the capacitance of capacitors of any shape is directly proportional to the dielectric constant of the dielectric filling the space between the plates.

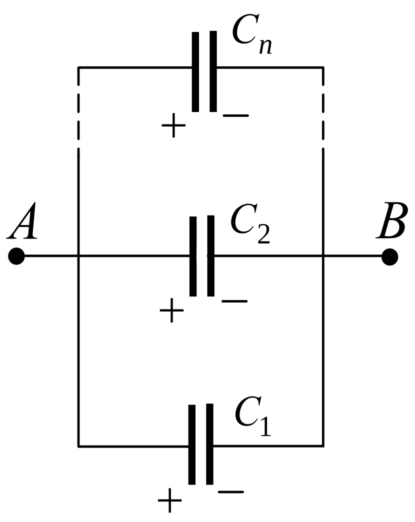


Figure 3.2. Parallel connection of capacitors.

Therefore, the use of ferroelectrics as a layer significantly increases the capacitance of capacitors. Capacitors are characterized by breakdown voltage, i.e. the potential difference between the capacitor plates, at which breakdown occurs. An electric discharge through a dielectric layer in a capacitor is called a breakdown. The breakdown voltage depends on the shape of the plates, the properties of the dielectric and its thickness.

To increase capacitance and vary its possible values, capacitors are connected to batteries, using their parallel and serial connection.

1. *Parallel connection of capacitors* (Figure 3.2). Parallel connected capacitors are characterized by the same potential difference $\varphi_A - \varphi_B$ between the capacitor plates. We denote the capacities of the capacitors as follows $C_1, C_2, \ldots, C_n$. Capacities charges are

$$\begin{aligned} Q_1 &= C_1(\varphi_A - \varphi_B), \\ Q_2 &= C_2(\varphi_A - \varphi_B), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ Q_n &= C_n(\varphi_A - \varphi_B) \end{aligned} \qquad (3.18)$$

The charge of the capacitor bank is

$$Q = \sum_{i=1}^{n} Q_i = (C_1 + C_2 + \ldots + C_n)(\varphi_A - \varphi_B). \qquad (3.19)$$

The total battery capacity is

$$C = \frac{Q}{\varphi_A - \varphi_B} = C_1 + C_2 + \ldots + C_n = \sum_{i=1}^{n} C_i\,, \qquad (3.20)$$

that is, when the capacitors are connected in parallel, it is equal to the sum of the capacitances of the individual capacitors.

2. *Serial connection of capacitors* (Figure 3.3). A battery of series-connected capacitors is characterized by equal charges on all plates.

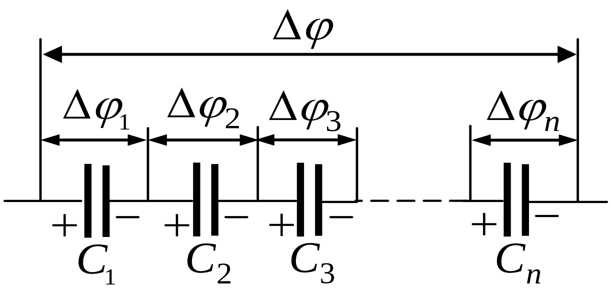


Figure 3.3. Serial connection of capacitors.

The potential difference in this case is

$$\Delta\varphi = \sum_{i=1}^{n} \Delta\varphi_i \,. \tag{3.21}$$

The potential difference for any of the capacitors is

$$\Delta\varphi_i = \frac{Q}{C_i} \,. \tag{3.22}$$

On the other hand,

$$\Delta\varphi = \frac{Q}{C} = Q\sum_{i=1}^{n}\left(\frac{1}{C_i}\right), \tag{3.33}$$

whence

$$\frac{1}{C} = \sum_{i=1}^{n}\left(\frac{1}{C_i}\right), \tag{3.34}$$

that is, when the capacitors are connected in series, the quantities inverse to the electric capacitances are added up. Thus, with a series connection of capacitors, the resultant capacitance C is always less than the smallest capacitance used in the battery.

3.4. Electrostatic Field Energy

A. The energy of electric field generated by a system of fixed point charges. Electrostatic forces of interaction are conservative forces; therefore, the system of charges has potential energy. Let us find the potential energy of a system of two fixed point charges Q_1, and Q_2, which are at a distance of r from each other. Each of these charges has a potential energy

$$W_1 = Q_1\varphi_{12},$$
$$W_2 = Q_2\varphi_{21} \tag{3.35}$$

where φ_{12} and φ_{21} are, respectively, the potentials created by the charge Q_2 at the point of location of the charge Q_1 and the charge Q_1 at the location of the charge Q_2. For potential φ_{12} formulas

$$\varphi_{12} = \frac{1}{4\pi\varepsilon_0}\frac{Q_2}{r} \tag{3.36}$$

and

$$\varphi_{21} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r} \tag{3.37}$$

are valid.

That's why

$$W_1 = W_2 = W \tag{3.38}$$

and

$$W = Q_1\varphi_{12} = Q_2\varphi_{21} = \frac{1}{2}\left(Q_1\varphi_{12} + Q_2\varphi_{21}\right). \tag{3.39}$$

Adding charges Q_3, Q_4,... to the system of two charges, we can verify that in the case of n stationary charges, the interaction energy of the system of point charges is

$$W = \frac{1}{2}\sum_{i=1}^{n} Q_i\varphi_i , \tag{3.40}$$

where φ_i is the potential created by all charges except the Q_i at the point where the charge Q_i is located.

B. The energy of electric field generated by a charged solitary conductor. Consider a solitary conductor whose charge, capacity and potential, respectively, are Q, C, φ. Let us increase the charge of this conductor by dQ. To do this, it is necessary to transfer the charge dQ from infinity to a solitary conductor, performing a work equal to

$$dA = \varphi dQ = C\varphi d\varphi . \tag{3.41}$$

The work that needs to be done to increase the potential from 0 to φ is

$$A = \int_0^{\varphi} C\varphi d\varphi = \frac{C\varphi^2}{2} . \tag{3.42}$$

The energy of a charged conductor is equal to the work that must be done to charge this conductor

$$W = \frac{C\varphi^2}{2} = \frac{Q\varphi}{2} = \frac{Q^2}{2C} . \tag{3.43}$$

C. The energy of electric field generated by a charged capacitor. Like any charged conductor, the capacitor has an energy that is

$$W = \frac{C(\Delta\varphi)^2}{2} = \frac{Q\Delta\varphi}{2} = \frac{Q^2}{2C} , \tag{3.44}$$

where Q is the charge of the capacitor,

C is its capacitance,

$\Delta\varphi$ is the potential difference between the plates of the capacitor.

We transform the formula expressing the energy of a flat capacitor, using the expression for the capacitance of a flat capacitor $C=\frac{\varepsilon_0\varepsilon}{d}$ and the potential difference between its plates $\Delta\varphi = Ed$. Then we get

$$W=\frac{\varepsilon_0\varepsilon E^2}{2}Sd=\frac{\varepsilon_0\varepsilon E^2}{2}V\ , \tag{3.45}$$

where $V = Sd$ is the volume of the capacitor.

The formula shows that the energy of the capacitor is expressed in terms of the value characterizing the electrostatic field, namely through the intensity of the electrostatic field. The volume energy density of the electrostatic field (energy per unit volume) is

$$\omega=\frac{W}{V}=\frac{\varepsilon_0\varepsilon E^2}{2}=\frac{ED}{2}. \tag{3.46}$$

Test questions

1. Explain the absence of an electric field inside the conductors using energy balance.
2. Can the equipotential surface and the conductor surface intersect?
3. Set the value of the electric displacement inside the conductor.
4. What is the value of the electrical displacement inside the conductor?
5. What factors determine the value of the modulus of the electric field intensity vector on the surface of the conductor?
6. Give the definition of induced charges.
7. Describe the phenomenon of electrostatic induction.
8. Does the cavity inside the conductor affect on the arrangement of the charges?
9. Explain the principle of electrostatic protection.
10. What character has the dependence of the potential of a solitary capacitor on the charge located on it?
11. Does the potential of a solitary capacitor affect its electrical intensity?
12. Write the defining formula for the potential of the solitary sphere.
13. Describe the capacitor device.
14. Why can we neglect edge effects on capacitor plates?
15. Build a graph of the electric capacitance of a flat capacitor on the distance between its plates.
16. Calculate the relative change in potential difference on the plates of a cylindrical capacitor with a decrease in the relative dielectric constant of the substance between its plates by 2 times.
17. Write the formula for the electrical capacitance of a spherical capacitor.
18. What are the reasons for using serial and parallel capacitor connections?
19. Calculate the capacity of four identical capacitors connected in series.

20. Write the formula for the energy of a charged conductor, using the values of its electric charge and potential.

Problem-solving examples

Problem 3.1

<u>*Problem description*</u>. The capacitance of the two capacitors are respectively $C_1 = 3\,\mu F$, $C_2 = 7\,\mu F$. These capacitors are charged to voltage $U_1 = 100\,V$ and $U_2 = 150\,V$. Determine the voltage on the capacitor plates after their connection.

<u>*Known quantities*</u>: $C_1 = 3\,\mu F$, $C_2 = 7\,\mu F$, $U_1 = 100\,V$, $U_2 = 150\,V$.

<u>*Quantities to be calculated*</u>: U_2.

<u>*Problem solution*</u>. The charges of the capacitors before they were connected to each other were equal:

$$Q_1 = C_1 U_1,\ Q_2 = C_2 U_2, \tag{3.1.1}$$

where Q is the electric charge on the capacitor plate,
U is the voltage on the capacitor plates,
indices 1 and 2 correspond to the states of capacitors before and after their connection.

For the case when the capacitors are connected in parallel with oppositely charged plates, the total charge on the capacitor plates will be equal to:

$$Q = Q_2 - Q_1 = C_2 U_2 - C_1 U_1. \tag{3.1.2}$$

Since, when they are connected in parallel, the capacitance of capacitors is

$$C = C_1 + C_2, \tag{3.1.3}$$

the voltage on the plates of capacitors after their connection is

$$U_2 = \frac{Q}{C} = \frac{C_2 U_2 - C_1 U_1}{C_1 + C_2} = 75\,V. \tag{3.1.4}$$

<u>*Answer*</u>. the voltage on the capacitor plates after their connection is $U_2 = 75\ V$.

Problem 3.2

<u>*Problem description*</u>. Flat air capacitor with a capacity of $C_1 = 15\ pF$ is charged to a voltage of $U_1 = 500\ V$. After disconnecting the capacitor from the voltage source, the distance between the plates of the capacitor was increased by 3 times. Determine the following values: 1) potential difference on the capacitor plates after increasing the distance between them; 2) the work of external forces that must be done to increase the distance between the plates.

<u>*Known quantities*</u>: $C_1 = 15\ pF$, $U_1 = 500\ V$, $d_2 = 3d_1$.

<u>*Quantities to be calculated*</u>: U_2, A.

<u>*Problem solution*</u>. Due to the fact that the capacitor was disconnected from the source, the charges Q on its plates before and after the connection are equal

$$Q_1 = Q_2. \qquad (3.2.1)$$

Therefore, for the voltage U on the plates of the capacitor and the electric capacity C we get

$$Q = UC,$$
$$U_1C_1 = U_2C_2,$$
$$d_2 = 3d_1,$$
$$C_2 = \frac{\varepsilon\varepsilon_0 S}{d_2} = \frac{\varepsilon\varepsilon_0 S}{3d_1} = \frac{C_1}{3}, \qquad (3.2.2)$$

where ε is the dielectric constant of the substance between the plates,
ε_0 is the electric constant,
S is the capacitor plate area,
d is the distance between the capacitor plates, indices 1 and 2 correspond to the states of the capacitor before and after increasing the distance between the plates.

The voltage on the capacitor plates after increasing the distance between them is equal to

$$U_2 = \frac{U_1C_1}{C_2} = \frac{U_1C_1}{(C_1/3)} = 3U_1 = 1500\ V. \qquad (3.2.3)$$

The work of external forces, which must be done in order to push the plates of the capacitor to a distance of d_2, is equal to:

$$A = \Delta W_p = \Delta W_{p2} - \Delta W_{p1} = \frac{C_2 {U_2}^2}{2} - \frac{C_1 {U_1}^2}{2} = 3.75 \times 10^{-6} \ J \,, \quad (3.2.4)$$

where W_p is the potential energy of the electric field between the capacitor plates.

Answer. The potential difference on the capacitor plates after increasing the distance between them is $U_2 = 1500 \ V$. The work of external forces that must be done to increase the distance between the plates is $A = 3.75 \times 10^{-6} \ J$.

Problem 3.3

Problem description. A metal sphere with an electrical capacity of $C = 4.5 \, pF$ is charged up to a potential of $\varphi = 1.2 \, kV$. Determine the electric field energy enclosed in the spherical layer between the sphere and the spherical surface concentric with it, whose radius is 4 times larger than the radius of the metal sphere.

Known quantities: $C = 4.5 \ pF$, $\varphi = 1.2 \, kV$.

Quantities to be calculated: W.

Problem solution. The electric field strength in a spherical layer is equal to

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \,, \quad (3.3.1)$$

where Q is the electric charge of the metal sphere,
ε_0 is the electrical constant,
r is the distance from the charge to the center of the metal sphere.

Then the volume density of the electric field energy in a spherical layer is equal to

$$\omega = \frac{\varepsilon_0 E^2}{2} \,. \quad (3.3.2)$$

Substitute the expression for the electric field strength E in the formula for the bulk energy density of the electric field

$$\omega = \frac{\varepsilon_0 \left[Q / \left(4\pi\varepsilon_0 r^2 \right) \right]^2}{2} = \frac{Q^2}{32\pi^2 \varepsilon_0 r^4} \,. \quad (3.3.3)$$

The elementary volume of the spherical layer is equal to

$$dV = 4\pi r^2 dr\,. \tag{3.3.4}$$

The energy of a spherical layer with an inner radius of $r_1 = R$ and an outer radius of $r_2 = 4R$ is

$$W = \int_{r_1}^{r_2} \omega\, dV = \int_{r_1}^{r_2} \frac{Q^2}{32\pi^2 \varepsilon_0 r^4} 4\pi r^2 dr = \int_{r_1}^{r_2} \frac{Q^2}{8\pi\varepsilon_0 r^2} dr = -\frac{Q^2}{8\pi\varepsilon_0 r}\Bigg|_{r_1}^{r_2} =$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{R} - \frac{1}{4R}\right) = \frac{3Q^2}{32\pi\varepsilon_0 R}\,. \tag{3.3.5}$$

Electrical intensity of the metal sphere equals

$$C = 4\pi\varepsilon_0 R\,. \tag{3.3.6}$$

Then the relationship between the energy of the electric field and the electric capacity is

$$W = \frac{3Q^2}{8C} \tag{3.3.7}$$

In addition, for electric charge and potential we get

$$Q = C\varphi\,. \tag{3.3.8}$$

Consequently, the energy of the electric field, which is in the spherical layer between the sphere and the spherical surface concentric with it, is equal to

$$W = \frac{3C^2\varphi^2}{8C} = \frac{3C\varphi^2}{8} = 2.43 \times 10^{-6}\ J\,. \tag{3.3.9}$$

<u>Answer</u>. The electric field energy enclosed in the spherical layer between the sphere and the spherical surface concentric with it is $W = 2.43 \times 10^{-6}\ J$.

Problems

Problem A

<u>Problem description</u>. A capacitor filled with a substance with a dielectric constant of $\varepsilon = 2$ was charged to voltage $U_1 = 220\,V$ and disconnected from the source. The

dielectric was removed from the capacitor and doubled the distance between the plates. Calculate the voltage U_2 on the capacitor plates.

Answer. $U_2 = 8.8 \times 10^2 V$.

Problem B

Problem description. The radius of the inner plate of a spherical air condenser is $R_1 = 2 cm$, and the radius of the outer plate is $R_2 = 6 cm$. A potential difference of $U = 400 V$ is applied between the spheres. Determine the energy of this capacitor after filling the space between the plates with paraffin with a dielectric constant of $\varepsilon = 2$.

Answer. $W = 5.3 \times 10^{-7} J$.

Problem C

Problem description. In parallel to three series-connected capacitors $C_1 = C_2 = C_3 = 0.36 \mu F$, two series-connected capacitors $C_4 = 0.2 \mu F$ and $C_5 = 0.3 \mu F$ are connected. Calculate the electrical capacity of this capacitor bank.

Answer. $C = 1.24 \times 10^{-7} F$.

Problem D

Problem description. A constant voltage of $U = 113 V$ is applied to the flat square-faced air condenser. The side of the square is $a = 10 cm$. The width of the gap is $d = 0.8 mm$. The capacitor is lowered into the water ($\varepsilon = 81$) at a constant speed of $\upsilon = 1\ m/s$. What current flows in this circuit?

Answer. $I = 8.85 \times 10^{-8} A$.

Problem E

Problem description. A voltage of $U = 20V$ is applied to the plates of a cylindrical capacitor. The radii of the coaxial cylinders forming the capacitor are $R_1 = 4\ mm$ and $R_2 = 16\ mm$. Calculate the surface charge densities on each plate.

Answer. $\sigma_1 = 3.192 \times 10^{-8} C/m^2$, $\sigma_2 = 7.98 \times 10^{-9} C/m^2$.

CHAPTER 4. LAWS OF DIRECT CURRENT

4.1. Quantitative Characteristics of Electric Current

Section, which deals with phenomena and processes caused by the movement of electric charges or macroscopic charged bodies, is called *electrodynamics*. The most important concept in electrodynamics is the concept of electric current. Any ordered (directed) motion of electric charges is called an *electric current*. Free electrical charges of the conductor under the action of the applied electric field move both along the field (for positive charges) and against the field (for negative charges). Consequently, an electric current appears in the conductor. This current is called the *conduction current*. The ordered motion of electric charges can be cased by the displacement of a charged macroscopic body in the space. An orderly movement of this type is called convection current [2].

For the appearance and existence of an electric current, it is necessary to fulfil two conditions. 1. The presence of free carriers i.e. charged particles, capable of moving in an orderly manner. 2. The presence of electric field which energy is spent on ordered movement of free carriers. The direction of the ordered motion of positive charges coincides with the direction of the current.

The current strength (current) is a quantitative measure of the electric current. The scalar physical quantity determined by the electric charge passing through the cross section of the conductor per unit time is called the current strength (current)

$$I = \frac{dQ}{dt}. \tag{4.1}$$

Direct current (DC) is the unidirectional flow of electric charge. Direct current is determined by formula

$$I = \frac{Q}{t}, \tag{4.2}$$

where Q is the electric charge passing through the fixed cross section of the conductor.

The physical quantity determined by the strength of the current passing through the unit of the cross-sectional area of the conductor perpendicular to the direction of the current is called the *current density*

$$j = \frac{dI}{dS_\perp}. \tag{4.3}$$

We express the force and current density through the velocity $<\upsilon>$ of the ordered motion of charges in the conductor. Suppose that a carriers concentration is n and each carrier has an elementary charge e (which is not necessary for ions). During time dt, the charge

$$dQ = ne<\upsilon>Sdt \tag{4.4}$$

is passed through the conductor cross-section S. The current strength is

$$I = \frac{dQ}{dt} = ne < \upsilon > S\,, \tag{4.5}$$

and the current density is

$$\vec{j} = ne < \vec{\upsilon} >. \tag{4.6}$$

The current density is a vector that is oriented along the direction of the current, that is, the direction of the vector coincides with the direction of the ordered motion of the positive charges.

The current through an arbitrary surface is defined as the flux of the vector $\vec{j}$, i.e.

$$I = \oint_S \vec{j}d\vec{S}\,, \tag{4.7}$$

where $d\vec{S} = \vec{n}dS$ ($\vec{n}$ is the unit normal vector to the area dS);

α is the angle between the normal $\vec{n}$ and the vector $\vec{j}$.

4.2. Extraneous Forces. Electromotive Force and Tension

The presence only the forces of the electrostatic field in the electric circuit leads to the carriers transport (they are assumed to be positive) from points with a large potential to points with a smaller potential. This will lead to equalization of the potentials at all points of the circuit and to the disappearance of the electric field. Therefore, for the existence of a direct current, it is necessary to have in the circuit a device capable of creating and maintaining a potential difference due to the operation of forces of non-electrostatic origin. Such devices are called *current sources*. Forces of non-electrostatic origin, acting on charges from current sources, are called *extraneous forces*.

The nature of extraneous forces can be different. For example, in galvanic cells these forces arise due to the energy of chemical reactions between electrodes and electrolytes; in the generator, extraneous forces arise due to the mechanical energy of rotation of the generator of the generator, and so on. The role of the current source in the electrical circuit is the same as, for example, the role of the pump, which is necessary for pumping fluid in the hydraulic system. Under the action of extraneous forces, the electric charges move inside the current source against the forces of the electrostatic field, due to which the potential difference is maintained at the ends of the circuit and a constant electric current flows in the circuit.

The work on moving electric charges is performed by extraneous forces. The physical quantity, which is determined by the work done by extraneous forces when moving a single positive charge, is called the *electromotive force* (EMF)

$$\varepsilon = \frac{A}{Q_0}. \tag{4.8}$$

The value ε can also be called the electromotive force of the current source included in the circuit. The term "electromotive force" is used as a characteristic of extraneous forces. EMF, as well as potential, is expressed in volts. The extraneous force $\vec{F}_e$ acting on the charge Q_0 can be expressed as

$$\vec{F}_e = \vec{E}_e Q_0, \tag{4.9}$$

where $\vec{E}_e$ is the intensity of the electric field of the extraneous forces. The work of third-party forces on charge Q_0 transfer on a closed circuit segment is equal to

$$A = \oint \vec{F}_e d\vec{l} = Q_0 \oint \vec{E} d\vec{l} . \tag{4.10}$$

Dividing the equation by Q_0, we obtain the expression for the EMF acting in the circuit

$$\varepsilon = \int_1^2 \vec{E}_e d\vec{l} . \tag{4.11}$$

The EMF acting in a closed circuit can be defined as the circulation of the field vector of extraneous forces. The EMF acting on section 1 – 2 of the electrical circuit is

$$\varepsilon_{12} = \int_1^2 \vec{E}_e d\vec{l} . \tag{4.12}$$

Electrostatic field forces

$$\vec{F}_c = Q_0 \vec{E} \tag{4.13}$$

also act on the charge Q_0 in addition to extraneous forces. Thus, the resultant force acting on the charge Q_0 in the electrical circuit is

$$\vec{F} = \vec{F}_e + \vec{F}_c = Q_0 \left(\vec{E}_e + \vec{E} \right). \tag{4.14}$$

The work done by the resultant force over charge Q_0 in section 1-2 is

$$A_{12} = Q_0 \int_1^2 \vec{E}_e d\vec{l} + Q_0 \int_1^2 \vec{E} d\vec{l} . \tag{4.15}$$

In this case, the formula

$$A_{12} = Q_0 \varepsilon_{12} + Q_0 (\varphi_1 - \varphi_2) \tag{4.16}$$

is valid. The work of electrostatic forces for a closed circuit is zero, so in this case

$$A_{12} = Q_0 \varepsilon_{12}. \quad (4.17)$$

The physical quantity determined by the work done by the total field of electrostatic (Coulomb) and extraneous forces when a unit positive charge moves on a given section of the circuit is called the *voltage* U in this section. Thus, for the voltage we get

$$U_{12} = \varphi_1 - \varphi_2 + \varepsilon_{12}. \quad (4.18)$$

The concept of voltage is a generalization of the concept of potential difference. The voltage at the ends of the section of the circuit is equal to the potential difference if there is no EMF on this section, that is, there are no extraneous forces.

4.3. Ohm's Law. Conductor Resistance

German physicist Georg Simon Ohm (1789 – 1854) experimentally found that the current I passing through a homogeneous metallic conductor (i.e., a conductor in which the extraneous forces do not act) is proportional to the voltage U at the ends of the conductor

$$I = \frac{U}{R}, \quad (4.19)$$

where R is the *electrical resistance* of the conductor. The equation expresses the *Ohm's law* for the circuit section (which does not contain the source of the electromotive force): the current in the conductor is directly proportional to the applied voltage and inversely proportional to the resistance of the conductor. A homogeneous linear conductor is characterized by a resistance R that is directly proportional to its length l and inversely proportional to its cross-sectional area S:

$$R = \frac{\rho l}{S}, \quad (4.20)$$

where ρ is the coefficient of proportionality characterizing the material of the wire and is called the *resistivity*.

Ohm's law can be represented in a differential form

$$\frac{I}{S} = \left(\frac{1}{\rho}\right)\left(\frac{U}{l}\right) = \gamma \frac{U}{l}, \quad (4.21)$$

where the value of γ, the reciprocal of the resistivity, is caused by the specific electrical conductivity of the conductor substance.

Considering that

$$\frac{U}{l} = E \quad (4.22)$$

is the intensity of the electric field in the conductor, and

$$\frac{I}{S} = j \tag{4.23}$$

is the current density, Ohm's law can be written in the form

$$j = \gamma E . \tag{4.24}$$

Since in an isotropic conductor the current carriers at each point move in the direction of the vector $\vec{E}$, the directions $\vec{j}$ and $\vec{E}$ coincide. Therefore, the following formula holds

$$\vec{j} = \gamma \vec{E} . \tag{4.25}$$

The last expression is Ohm's law in differential form, connecting the current density at any point inside the conductor with the electric field intensity at the same point. This relation is also valid for variable fields. Experience shows that, in the first approximation, the change in resistivity and, consequently, in resistance, with temperature is described by a linear law

$$\rho = \rho_0 (1 + \alpha t) \tag{4.26}$$

or

$$R = R_0 (1 + \alpha t), \tag{4.27}$$

where ρ and ρ_0, R and R_0 are respectively the resistances and resistivity of the conductor at t and $0^\circ C$, α is the temperature coefficient of resistance, for pure metals (at not very low temperatures) close to 1/273 K^{-1}. The temperature dependence of the resistance can be represented in the form

$$R = \alpha R_0 T , \tag{4.28}$$

where T is the *thermodynamic temperature*. The qualitative temperature dependence of the resistance of the metal is shown in Figure 4.1 (curve 1). Subsequently, it was found that the resistance of many metals (for example, Al, Pb, Zn etc.) and their alloys at very low temperatures $T_k (0{,}14 - 20 K)$, called *critical temperatures*, which are characteristic of each substance, abruptly decreases to zero (Figure 4.1, curve 2), i.e. metal becomes an absolute conductor.

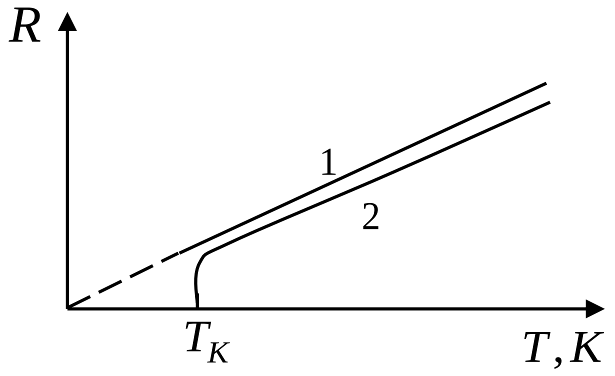


Figure 4.1. Temperature dependence of resistance for metals.

For the first time this phenomenon, called *superconductivity*, was discovered by Dutch physicist Heike Kamerlingh Onnes (1853 – 1926) for mercury. The phenomenon of superconductivity is explained on the basis of quantum theory. The principle of action of *resistance thermometers* is based on the dependence of the electrical resistance of metals on temperature. Resistance thermometers allow to measure the temperature with accuracy up to 0,003 K according to the graded relationship of resistance to temperature. The use of semiconductors, prepared according to a special technology as the working substance of resistance thermometers, makes it possible to measure temperatures in millionths of kelvin. Such resistance thermometers are called *thermistors*. Thermistors are used to measure temperatures in the case of small dimensions of semiconductors.

4.4. Joule–Lenz Law

Let us consider a homogeneous conductor. Constant voltage U is applied to the ends of this conductor. Through the cross section of the conductor, a charge

$$dq = Idt \tag{4.29}$$

is transferred during a time of dt. Since the current represents the charge dq transfer under the action of the electric field the work done by current is equal to

$$dA = Udq = IUdt. \tag{4.30}$$

In the assumption that the resistance of the conductor is equal to R we get

$$dA = I^2 Rdt = \left(\frac{U^2}{r}\right)dt. \tag{4.31}$$

In this case the power of the current is

$$P = \frac{dA}{dt} = UI = I^2 R = \frac{U^2}{R}\,. \qquad (4.32)$$

If the current passes through a fixed metal conductor, then all the work is performed to heat this conductor. According to the law of conservation of energy we get

$$dQ = dA \qquad (4.33)$$

or, using the previously mentioned relationships

$$dQ = IUdt = I^2 Rdt = \frac{U^2}{R} dt\,. \qquad (4.34)$$

The last expression represents *Joule-Lenz law*. This law appeared thanks to the studies of English physicist James Prescott Joule (1818 – 1889).

We select an elementary cylindrical volume $dV = dSdl$ in the conductor (the axis of the cylinder coincides with the direction of the current), whose resistance is

$$R = \rho\left(\frac{dl}{dS}\right). \qquad (4.35)$$

According to the Joule-Lenz law we obtain

$$dQ = I^2 Rdt = \frac{\rho dl}{dS}\left(jdS\right)^2 dt = \rho j^2 dVdt \qquad (4.36)$$

Therefore, the heat will be released in this volume during the time dt.

The amount of heat that is released per unit of volume per unit time is called the *specific thermal power*. It is equal to

$$\omega = \rho j^2. \qquad (4.37)$$

Using the differential form of Ohm's law

$$j = \gamma E \qquad (4.38)$$

and the relation

$$\rho = 1/\gamma\,, \qquad (4.39)$$

we obtain

$$\omega = jE = \gamma E^2. \qquad (4.40)$$

These formulas are a generalized expression of the Joule-Lenz law in differential form, they are suitable for any conductor.

4.5. Ohm's Law for the Inhomogeneous Section of the Circuit

We consider an inhomogeneous section of the circuit where the actual EMF in section 1-2 is denoted by ε_{12}, and the potential difference applied at the ends of the section will be denoted by $\varphi_1 - \varphi_2$.

If the current passes through the fixed conductors forming section 1-2, then the work A_{12} of all forces (extraneous and electrostatic) performed over the current carriers, according to the law of conservation and transformation of energy, is equal to the heat released in this section. The work performed when the charge Q_0 is moved in section 1-2, is

$$A_{12} = Q_0\varepsilon_{12} + Q_0(\varphi_1 - \varphi_2). \qquad (4.41)$$

EMF ε_{12} as well as the current I is a scalar quantity. EMF must be taken either with a positive or a negative sign, depending on the sign of the work performed by external forces. If the EMF facilitates the movement of positive charges in the chosen direction (direction 1-2), then $\varepsilon_{12} > 0$. The amount of heat that is released in the conductor during time t is

$$Q = I^2Rt = IR(It) = IRQ_0. \qquad (4.42)$$

For the quantity IR we obtain

$$IR = (\varphi_1 - \varphi_2) + \varepsilon_{12}, \qquad (4.43)$$

then

$$I = \frac{\varphi_1 - \varphi_2 + \varepsilon_{12}}{R}. \qquad (4.44)$$

The expression for I is the Ohm's law for the inhomogeneous section of a circuit in integral form, which is a generalized Ohm's law. For the case when there is no current source in this section of the circuit ($\varepsilon_{12} = 0$), we obtain Ohm's law for a homogeneous part of the circuit

$$I = \frac{(\varphi_1 - \varphi_2)}{R} = \frac{U}{R} \qquad (4.45)$$

Voltage U at the ends of the section is equal to the potential difference in the absence of extraneous forces. For the case when the electrical circuit is closed, i.e. the selected points 1 and 2 are the same ($\varphi_1 = \varphi_2$) we obtain Ohm's law for a closed chain

$$I = \frac{\varepsilon}{R}, \qquad (4.46)$$

where ε is the EMF acting in the circuit,

R is the total resistance of the entire circuit.

In the general case the total resistance is

$$R = r + R_1, \qquad (4.47)$$

where r is the internal resistance of the EMF source,

R_1 is the resistance of the external circuit.

Therefore, Ohm's law for a closed chain will have the form

$$I = \frac{\varepsilon}{r + R_1}. \tag{4.48}$$

If the circuit is open and, consequently, there is no current in it ($I = 0$), then from Ohm's law we get that

$$\varepsilon_{12} = \varphi_2 - \varphi_1, \tag{4.49}$$

i.e., the EMF acting in an open circuit is equal to the potential difference at its ends. Therefore, in order to find the EMF of the current source, it is necessary to measure the potential difference when the circuit is open.

4.6. Kirchhoff's Laws

The generalized Ohm's law allows us to calculate almost any complex chain. However, the direct calculation of branched chains containing several closed contours (contours can have common areas, each of the circuits can have several sources of EMF, etc.) is rather complicated. This problem is solved more simply with the help of two Kirchhoff's laws. These laws were established by German physicist Gustav Robert Kirchhoff (1824 – 1887).

The *Kirchhoff's first law*: the algebraic sum of currents converging at a node is zero

$$\sum_k I_k = 0. \tag{4.50}$$

Figure 4.2 presents the scheme for the Kirchhoff's first law. According to this scheme we can write

$$I_1 - I_2 + I_3 - I_4 - I_5 = 0. \tag{4.51}$$

The Kirchhoff's first law is based on the law of conservation of electric charge. Indeed, in the case of steady-state direct current, no electrical charges should accumulate at any point in the conductor and in any of its sections. Otherwise, the currents could not remain constant.

The *Kirchhoff' second law* is based on the generalized Ohm's law for branched chains. Consider a contour consisting of three sections (Figure 4.3). The direction of the clockwise rotation will be taken as positive, noting that the choice of this direction is completely arbitrary. All currents that coincide in direction with the direction of contour traversal are considered positive, and currents that do not coincide with the direction of traversal are considered negative.

The sources of EMF are considered positive if they create a current directed towards the contour traversal. Applying Ohm's law to the sections, we can write

$$I_1R_1 = \varphi_A - \varphi_B + \varepsilon_1$$
$$-I_2R_2 = \varphi_B - \varphi_C - \varepsilon_2$$
$$I_3R_3 = \varphi_C - \varphi_A + \varepsilon_3 \qquad (4.52)$$

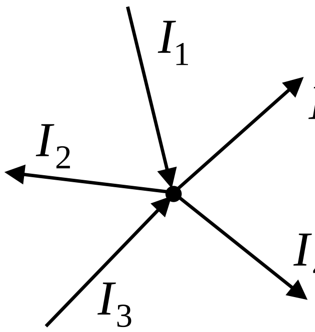


Figure 4.2. Scheme for the Kirchhoff's first law.

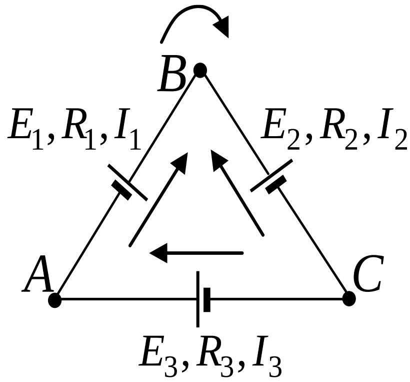


Figure 4.3. Scheme for the Kirchhoff's second law.

Adding these equations, we obtain

$$I_1R_1 - I_2R_2 + I_3R_3 = \varepsilon_1 - \varepsilon_2 + \varepsilon_3. \qquad (4.53)$$

The resulting equation expresses Kirchhoff's second law: in any closed contour arbitrarily chosen in a branched electrical circuit, the algebraic sum of the products of the currents by the resistances of the corresponding sections of this circuit is equal to the algebraic sum of the EMF encountered in this contour

$$\sum_i I_i R_i = \sum_k \varepsilon_k \, . \qquad (4.54)$$

Additional rules should be taken into account.

1. It is necessary to select an arbitrary direction of currents on all sections of the circuit. The actual direction of the currents is determined when solving the problem. If the desired current is positive, then its direction was chosen correctly, and if the current value is negative, then its true direction is opposite to the chosen one.
2. It is necessary to choose the direction of circumvention and strictly adhere to it. The product IR is positive if the current in this section coincides with the direction of the circumvention, and vice versa. The EMF values acting in the selected circumvention direction are considered positive, and those acting against the bypass direction are considered negative.
3. The number of equations that are compiled according to Kirchhoff's laws must be equal to the number of unknown quantities. The system of equations should include all the resistance and EMF of the electrical circuit. Each contour must

contain at least one element that is not contained in the previous outlines. Otherwise, we obtain equations that are a simple combination of already compiled ones.

Test questions

1. Can any movement of electric charges be called an electric current?
2. List the conditions under which a convection current occurs.
3. What conditions are necessary for the appearance and maintenance of any electrical current?
4. Does the current depend on the cross-section area of the conductor?
5. Give the units of current density.
6. Specify the direction of current density.
7. Describe the main task for which the current sources are intended.
8. Give some examples of extraneous forces.
9. Write the formula by which the electromotive force is determined.
10. Indicate the functional relationship between the electromotive force and the electric field intensity.
11. Specify the components of the resultant force that acts on the charge in the electrical circuit.
12. Determine the voltage in the electrical circuit.
13. In which case the voltage is equal to the potential difference at the ends of the circuit?
14. Formulate Ohm's law.
15. Calculate the relative change in resistance of the conductor if its cross section decreases four times.
16. Explain the phenomenon of superconductivity.
17. What is the difference between thermistors and ordinary thermometers?
18. Formulate the Joule law.
19. Specify the conditions under which instead of the generalized Ohm's law, we can apply Ohm's law for a homogeneous section of the circuit.
20. List the laws on which the first and second Kirchhoff rules are based.

Problem-solving examples

Problem 4.1

Problem description. Resistance, voltmeter and current source are connected in parallel. The amount of resistance is $R_1 = 6\ \Omega$. The voltmeter voltage is $U_1 = 12\ V$. If one increase the resistance to $R_2 = 15\,\Omega$, then the voltmeter will show the voltage $U_2 = 14\ V$. Determine the EMF and internal resistance of the current source. The current through the voltmeter is neglected.

Known quantities: $R_1 = 6\ \Omega$, $U_1 = 12\ V$, $R_2 = 15\,\Omega$, $U_2 = 14\ V$.

Quantities to be calculated: ε, R_0.

Problem solution. The current through the voltmeter can be neglected. Consequently, the current through the resistance is equal to the current through the current source. Ohm's law for a complete chain is

$$I = \frac{\varepsilon}{R + R_0}, \tag{4.1.1}$$

where R_0 is the internal source resistance,
R is load resistance,
I is a current.

We write the formulas for the forces of currents with different resistances R_1 and R_2

$$I_1 = \frac{\varepsilon}{R_1 + R_0}, \tag{4.1.2}$$

$$I_2 = \frac{\varepsilon}{R_2 + R_0}. \tag{4.1.3}$$

These resistances provide voltages U_1 and U_2

$$U_1 = I_1 R_1,$$
$$U_2 = I_2 R_2. \tag{4.1.4}$$

Use Ohm's law and write down the ratio for the currents

$$\frac{I_1}{I_2} = \frac{R_2 + R_0}{R_1 + R_0}. \tag{4.1.5}$$

Rewrite the last equation

$$I_1 R_1 + I_1 R_0 = I_2 R_2 + I_2 R_0, \tag{4.1.6}$$

then

$$R_0 = \frac{I_2 R_2 - I_1 R_1}{I_1 - I_2} = (U_2 - U_1)\left(\frac{U_1}{R_1} - \frac{U_2}{R_2}\right)^{-1}. \tag{4.1.7}$$

Using the expression for Ohm's law, we find the EMF

$$\varepsilon = I_1 (R_1 + R_0) = \frac{U_1}{R_1}(R_1 + R_0). \tag{4.1.8}$$

Substitute the numerical values in the formulas for resistance and EMF

$$R_0 = 1.87\,\Omega,\ \varepsilon = 15.75V. \qquad (4.1.9)$$

<u>*Answer*</u>. The EMF is $\varepsilon = 15.75V$. The internal resistance of the current source is $R_0 = 1.87\,\Omega$.

Problem 4.2

<u>*Problem description*</u>. The current in the conductor increases uniformly from $I_1 = 0$ to $I_2 = 6\,A$ during the time interval $t_1 = 0$, $t_2 = 1.5\,s$. Conductor resistance equals $R = 20\ \Omega$. Determine the amount of heat that will be released in the conductor during the time interval $t_1 = 0$, $t_3 = 1.2\,s$. The temperature dependence of the conductor resistance is neglected.

<u>*Known quantities*</u>: $I_1 = 0$, $I_2 = 6\,A$, $t_1 = 0$, $t_2 = 1.5\,s$, $R = 20\ \Omega$, $t_3 = 1.2\,s$.

<u>*Quantities to be calculated*</u>: ΔQ.

<u>*Problem solution*</u>. We write the Joule-Lenz law for thermal power that will stand out on the resistance

$$P = I^2 R, \qquad (4.2.1)$$

where I is the current that passes through the resistance,
R is the resistance value.

The amount of heat dQ, which is formed during the time interval $(t, t+dt)$ equals

$$dQ = Pdt = I^2 Rdt. \qquad (4.2.2)$$

The current increases evenly, i.e. is a linear function of time

$$I = at + b. \qquad (4.2.3)$$

At the initial moment of time $t_1 = 0$, the current strength is zero $I_1 = 0$, therefore $b = 0$, thus

$$I = at. \qquad (4.2.4)$$

The coefficient "a" we find from the condition that $I_2 = 4\,A$ at $t_2 = 2\,s$:

$$I_2 = at_2. \qquad (4.2.5)$$

Then we get

$$a = \frac{I_2}{t_2} = 4\,A/s. \qquad (4.2.6)$$

Now we can find the amount of heat that will stand out in the conductor

$$\Delta Q = \int_{t_1}^{t_3} I^2 R dt = a^2 R \int_{t_1}^{t_3} t^2 dt = \frac{a^2 R}{3}\left(t_3^3 - t_1^3\right). \qquad (4.2.7)$$

Substitute the numerical values

$$\Delta Q = 184.3\,J\,. \qquad (4.2.8)$$

Answer. The amount of heat that will be released in the conductor is $\Delta Q = 184.3\,J$.

Problem 4.3

Problem description. Determine the average speed of the ordered movement of electrons in a copper conductor with a current of $I = 12\,A$ and a conductor cross section of $S = 1mm^2$. Assume that for each copper atom there are two conduction electrons.

Known quantities: $I = 12\,A$, $S = 1mm^2$.

Quantities to be calculated: υ.

Problem solution. The current density j in a conductor is by definition equal to

$$j = \frac{I}{S}. \qquad (4.3.1)$$

On the other hand, the relationship for current density can be obtained through the average speed of charge carriers in a conductor (electrons) υ and the carrier concentration (number of carriers per unit volume of the conductor) n using the expression

$$j = en\upsilon, \qquad (4.3.2)$$

Where e is an elementary charge ($e = 1.6 \times 10^{-19}\,C$).

Equating the right parts of the obtained formulas, we obtain the expression for the average speed

$$\upsilon = \frac{I}{enS}. \qquad (4.3.3)$$

The electron concentration is found from the following considerations.. First, from the periodic table we find the molar mass of copper: $M = 64 \times 10^{-3}\ kg/mol$. One mole of any substance contains $N_A = 6.02 \times 10^{23}$ atoms (Avogadro number).

The volume of one mole of copper is equal to

$$V = M/\rho, \qquad (4.3.4)$$

Where ρ is the density of copper ($\rho = 8.93 \times 10^3\ kg/m^3$).

Therefore, the number of copper atoms per unit volume will be equal to

$$n_0 = \frac{N_A}{V} = \frac{N_A \rho}{M}. \qquad (4.3.5)$$

Since for each copper atom there are two conduction electrons, the concentration of conduction electrons will be equal to $n = 2n_0$. As a result, the average electron velocity is:

$$\upsilon = \frac{M}{2e\rho N_A}\frac{I}{S}. \qquad (4.3.6)$$

Substituting numerical values into this formula, we get $\upsilon = 4.46 \times 10^{-4}\ m/s$.

<u>Answer</u>. The average speed of the ordered movement of electrons in a copper conductor $\upsilon = 4.46 \times 10^{-4}\ m/s$.

Problems

Problem A

<u>Problem description</u>. The voltage on the tires of the power station is $U = 6.6\ kV$. The consumer is at a distance of $L = 10\ km$. Determine the cross-sectional area S of copper wire, which should be taken for the device of a two-wire transmission line, if the current in the line is $I = 20\,A$ and the voltage loss in the wires should not exceed 3%.

<u>Answer</u>. $S = 3.42 \times 10^{-5}\ m^2$.

Problem B

Problem description. A coil with a resistance of $R = 0.1\,\Omega$ was connected to a current source with an EMF of $E = 1.5\,V$. The ammeter showed a current strength of $I_1 = 0.5\,A$. When another current source with the same EMF was connected in series to a current source, the current strength in the same coil was $I_2 = 0.4\,A$. Calculate the internal resistances of the first r_1 and second r_2 current sources.

Answer. $r_1 = 2.9\,\Omega$, $r_2 = 4.5\,\Omega$.

Problem C

Problem description. Three batteries with EMF $E_1 = 12\,V$, $E_2 = 5\,V$, $E_3 = 7\,V$ and the same internal resistance $r = 1\,\Omega$, interconnected by the same poles. The resistance of the connecting wires is negligible. Determine the strength of the currents flowing through each battery.

Answer. $I_1 = 3\,A$, $I_2 = 4\,A$, $I_3 = 1\,A$.

Problem D

Problem description. A battery heater is attached to the battery terminals. The EMF of the battery is $E = 24\,V$, and the internal resistance is $r = 1\,\Omega$. The heater included in the circuit consumes $P = 80\,W$ power. Calculate the current I in the circuit and the efficiency η of the heater.

Answer. $I = 4\,A$, $\eta = 0.83$.

Problem E

Problem description. A conductor, whose resistance is equal to $R = 3\,\Omega$, passes a current, the strength of which increases. The amount of heat released in the conductor during time $\tau = 8\,s$ is $Q = 200\,J$. Determine the amount of electricity that has flowed through the conductor during this time. At the time taken as the initial, the current in the conductor is zero.

Answer. $q = 20\,C$.

CHAPTER 5. MAGNETIC FIELD IN VACUUM

5.1. Quantitative Description of Magnetic Field

Numerous experiments indicate the presence of a magnetic field both around permanent magnets and in the vicinity of moving charges and electric currents. The presence of a magnetic field is detected by force acting on conductors with current or permanent magnets.

The electric field acts on both static and electric charges moving in it. The magnetic field acts only on moving electric charges and this is its most important feature. The influence of the magnetic field on the electric current depends on the shape of the conductor and its location in space, as well as on the direction of the current. Therefore, in order to characterize the magnetic field, we must consider its effect on a certain current. Just as point charges were used in the investigation of an electrostatic field, a closed planar circuit with a current is used to study the magnetic field. The size of this circuit should be small compared to the distance to circuit.

The orientation of the contour with current in space is characterized by the direction of the normal to the contour. The direction associated with the current by the rule of the right screw is adopted as the positive direction of the normal. The positive direction of the normal assumes the direction of the translational motion of the screw, whose head rotates in the direction of the current flowing in the contour. The magnetic field exerts an orienting action on the frame with the current, rotating it in a certain way. This result is associated with a certain direction of the magnetic field.

The direction of magnetic field at a given point is taken along the positive normal to the contour (Figure 5.1).

The direction of the magnetic field can also be related to the direction of the force that acts on the north pole of the magnetic needle placed at a given point.

Since both poles of the magnetic needle lie at close points of the field, the forces acting on both poles are equal to each other.

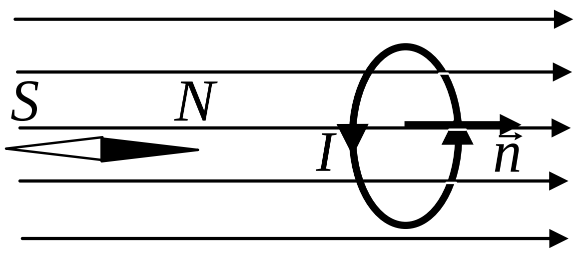


Figure 5.1. Magnetic field of circular current.

Consequently, a pair of forces acts on the magnetic needle, turning it so that the axis of the arrow that connects the south pole with the north pole coincides with the direction of the field.

A contour with current can also be used for quantitative description of the magnetic field. Since the contour experiences the orienting action of the field, a pair of forces acts on it in a magnetic field. The torque of the forces depends both on the properties of the field at a given point, and on the properties of the contour

$$\vec{M} = \left[\vec{p}_m \vec{B}\right], \tag{5.1}$$

where $\vec{B}$ is the vector of magnetic induction, which is a quantitative characteristic of the magnetic field;

$\vec{p}_m$ is the magnetic moment of the contour with current.

The magnetic moment of a plane circuit with a current is

$$\vec{p}_m = IS\vec{n}, \tag{5.2}$$

where S is the surface area of the contour,

I is the current,

$\vec{n}$ is the unit vector of the normal to the surface of the contour.

The $\vec{p}_m$ direction coincides, thus, with the direction of the positive normal. If we place contours with different magnetic moments at a given point of the magnetic field, then different torque acts on them, but the relation $\frac{M_{\max}}{p_m}$ ($M_{\max}$ is the maximum torque) for all contours is the same and therefore can serve as a characteristic of the magnetic field. This characteristic

$$B = \frac{M_{\max}}{p_m} \tag{5.3}$$

is called *magnetic induction*.

The magnetic induction at a given point of a homogeneous magnetic field is determined by the maximum torque acting on the contour with current. The rotational moment is maximum one, when the normal to the contour is perpendicular to the direction of the field. Since the magnetic field is a force field, it is, by analogy with the electric field, represented by lines of magnetic induction. The lines tangent to which at each point coincide with the direction of the vector $\vec{B}$ are called *magnetic field lines*. Their direction is given by the rule of the right screw: the screw head screwed in the direction of the current rotates in the direction of the magnetic field lines.

Magnetic field lines are always closed and cover conductors with current. Magnetic field lines differ from the electrostatic field lines, which are open (start on positive charges and end on negative ones).

The vector of magnetic induction $\vec{B}$ characterizes the resulting magnetic field created by all macro- and micro currents. In the case of the same current and other equal conditions the vector $\vec{B}$ in different media will have different values.

The magnetic field of macro-currents is described by the magnetic field intensity vector $\vec{H}$. For the case of isotropic medium, the vector of magnetic induction is

$$\vec{B} = \mu_0 \mu \vec{H}, \tag{5.4}$$

where μ_0 is the magnetic constant,

μ is a dimensionless quantity, called the *magnetic permeability*.

The magnetic permeability shows how many times the magnetic field H of macro-currents is amplified by the field of micro currents in the medium. Comparing the vector characteristics of the electrostatic ($\vec{E}$, and $\vec{D}$) and magnetic ($\vec{B}$, and $\vec{H}$) fields, we will point out that the vector $\vec{B}$ of magnetic field induction is analogous to the vector $\vec{E}$ of the intensity of the electrostatic field, since the vectors $\vec{E}$, and $\vec{B}$ determine the force actions of these fields and determine the properties of the medium. The magnetic field intensity vector $\vec{H}$ is an analogy of the electric displacement vector $\vec{D}$.

5.2. Biot-Savart Law

French physicist Jean-Baptiste Biot (1774 – 1862) and French physicist Félix Savart (1971 – 1841) established the basic experimental law relating the magnetic induction $\vec{B}$. The *Biot-Savart law* for a conductor with current I, whose element dl creates at some point A the induction of the field $d\vec{B}$, is written in the form

$$d\vec{B} = \frac{\mu_0 \mu}{4\pi} \frac{I\left[d\vec{l}, \vec{r}\right]}{r^3}, \tag{5.5}$$

where $d\vec{l}$ is a vector whose modulus is equal to the length dl of the element of the conductor and coincides in direction with the current,

$\vec{r}$ is the radius vector directed from the conductor element dl to the point A of the field,

r is the module of the radius vector $\vec{r}$.

Assuming that linear superposition holds, the Biot-Savart law can be integrated to determine the magnetic-flux density due to various configurations of current-carrying wires [9]. The direction of vector $d\vec{B}$ is perpendicular to $d\vec{l}$ and $\vec{r}$, and coincides with the tangent to magnetic field lines. This direction can be found by the rule of determining the magnetic field lines direction (rule of the right screw): the direction of the screw head rotation gives the direction of the vector $d\vec{B}$ if the translatory movement of the screw corresponds to the direction of the current in the element.

The modulus of vector $d\vec{B}$ is given by

$$dB = \frac{\mu_0 \mu}{4\pi} \frac{Idl \sin\alpha}{r^2}, \tag{5.6}$$

where α is the angle between vectors $d\vec{l}$ and $\vec{r}$.

The superposition principle is valid for both the magnetic field and the electric field: magnetic induction of the resultant field created by several currents or (and) moving charges is equal to the vector sum of the magnetic inductions of fields created by each current or (and) each moving charge

$$\vec{B} = \sum_{i=1}^{n} \vec{B}_i . \tag{5.7}$$

Determining the characteristics of the magnetic field ($\vec{B}$ and $\vec{H}$) according to the above formulas is quite complicated in general case. However, if the current distribution has a certain symmetry, then the application of the Biot-Savart law together with the superposition principle allows us to calculate the specific fields quite simply. Consider two examples.

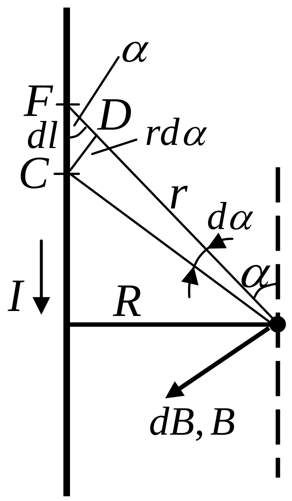


Figure 5.2. Magnetic field of straight current-carrying wire.

1. The magnetic field of straight current current-carrying wire, i.e. current passing through a thin straight wire of infinite length (Figure 5.2). At an arbitrary point A, which is located at a distance R from the axis of the conductor, the vectors $d\vec{B}$ from all elements of the current have the same direction perpendicular to the plane of the drawing ("to us"). Therefore, the addition of vectors $d\vec{B}$ can be replaced by adding their moduli. As the integration constant, we choose the angle α (the angle between vectors $d\vec{l}$ and $\vec{r}$). We express all other quantities through this angle. Using the construction in Figure 5.2, we get

$$r = \frac{R}{\sin\alpha}, dl = \frac{rd\alpha}{\sin\alpha}. \tag{5.8}$$

The radius of the arc CD due to the smallness of dl is r. The angle FDC for the same reason can be considered straight. Taking into account the obtained expressions, we find that the magnetic induction created by element dl of the conductor is

$$dB = \frac{\mu_0 \mu I}{4\pi R} \sin \alpha d\alpha . \quad (5.9)$$

Since the angle α for all elements of the straight current varies from 0 to π, then the formula

$$B = \int dB = \frac{\mu_0 \mu}{4\pi} \frac{I}{R} \int_0^{\pi} \sin \alpha d\alpha = \frac{\mu_0 \mu}{4\pi} \frac{2I}{R} \quad (5.10)$$

is valid. Consequently, the magnetic induction of the forward current field is

$$B = \frac{\mu_0 \mu}{4\pi} \frac{2I}{R} . \quad (5.11)$$

2. The magnetic field in the centre of a circular current loop (Figure 5.3). All elements of a circular conductor with a current create in the centre a magnetic fields of same direction along the normal from the current loop.

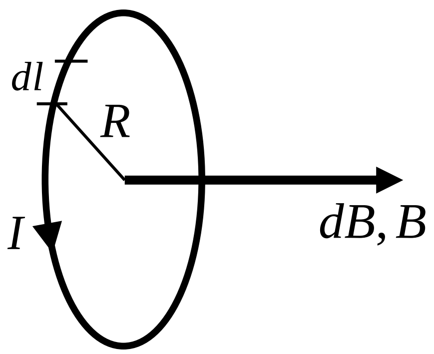


Figure 5.3. Magnetic field in the centre of a circular current loop.

Therefore, the addition of vectors $d\vec{B}$ can be replaced by the addition of their modules. Since all the elements of the conductor are perpendicular to the radius vector ($\sin \alpha = 1$) and the distance of all the elements of the conductor to the centre of the circular current is the same and equal R, then

$$dB = \frac{\mu_0 \mu}{4\pi} \frac{I}{R^2} dl . \quad (5.12)$$

Hence

$$B = \int dB = \frac{\mu_0 \mu}{4\pi} \frac{I}{R^2} \int dl = \frac{\mu_0 \mu I}{4\pi R^2} 2\pi R = \mu_0 \mu \frac{I}{2R} . \quad (5.13)$$

Consequently, the magnetic induction of the field at the center of the circular conductor with current is equal to

$$B = \mu_0 \mu \frac{I}{2R}. \tag{5.14}$$

The magnetic field exerts an orienting action on the contour with current. Consequently, the torque that is experienced by the contour is the result of the action of forces on its individual elements.

Summarizing the results of studying the effect of a magnetic field on various conductors with current, French physicist André-Marie Ampère (1775 – 1836) established that the force $d\vec{F}$ with which a magnetic field acts on an element of a conductor dl with a current is directly proportional to the current I and the vector product of element $d\vec{l}$ of the conductor per magnetic induction $\vec{B}$ (<u>*Ampère's force*</u> law):

$$d\vec{F} = I\left[d\vec{l}, \vec{B}\right]. \tag{5.15}$$

5.3. Magnetic Field of Moving Charge

Each conductor with a current creates a magnetic field in the surrounding space. The electric current is an ordered movement of electric charges. Therefore, we can say that any charge moving in a vacuum or medium creates a magnetic field around itself. The motion of a charge with a constant velocity is called free motion. The law determining the field $\vec{B}$ of the point charge Q moving freely with a nonrelativistic velocity $\vec{\upsilon}$ was established as a result of generalization of the experimental data. This law is expressed by the formula

$$\vec{B} = \frac{\mu_0 \mu}{4\pi} \frac{Q[\vec{\upsilon}\,\vec{r}]}{r^3}, \tag{5.16}$$

where $\vec{r}$ is the radius vector directed from the charge Q to the observation point M.

The vector $\vec{B}$ is directed perpendicular to the plane in which the vectors $\vec{\upsilon}$ and $\vec{r}$ are located, namely: its direction coincides with the direction of the translational motion of the right screw as it rotates from $\vec{\upsilon}$ to $\vec{r}$. The magnetic induction module is calculated by the formula

$$B = \frac{\mu_0 \mu}{4\pi} \frac{Q\upsilon}{r^2} \sin\alpha, \tag{5.17}$$

where α is the angle between vectors $\vec{\upsilon}$ and $\vec{r}$.

5.4. Lorentz Force

Experience shows that the magnetic field acts not only on conductors with current, but also on individual charges moving in a magnetic field. The force acting on the electric charge Q, which moves in a magnetic field with a velocity of $\vec{\upsilon}$, is called the *Lorentz force* and is expressed by the formula

$$\vec{F} = Q\left[\vec{\upsilon}\,\vec{B}\right], \qquad (5.18)$$

where B is the induction of the magnetic field in which the charge moves.

The frmula (5.18) is named after the Dutch physicist Hendrik Antoon Lorenz (1853 – 1928). The direction of the Lorentz force is determined by the rule of the left hand: if the palm of the left hand is positioned so that the vector $\vec{B}$ enters it, and four straight fingers are directed along the vector $\vec{\upsilon}$ (for $Q > 0$ directions I and $\vec{\upsilon}$ coincide , for $Q < 0$ directions are opposite), then the bent thumb will show the direction of the force acting on the positive charge. The Lorentz force is always perpendicular to the velocity of the charged particle, so it changes only the direction of this velocity, without changing its modulus. Consequently, the Lorentz force does not perform the work. For the case when an electric field with intensity of $\vec{E}$ acts on a moving electric charge in addition to a magnetic field with induction $\vec{B}$, the resultant force $\vec{F}$ applied to the charge is equal to

$$\vec{F} = Q\vec{E} + Q\left[\vec{\upsilon}\vec{B}\right]. \qquad (5.19)$$

This expression is called the Lorentz formula. The speed υ in this formula is the charge velocity with respect to the magnetic field.

5.5. Circulation of Magnetic Field Induction

Similarly to the circulation of the vector $\vec{E}$, we introduce the circulation of the vector $\vec{B}$. Circulation of vector $\vec{B}$ along a given closed contour is the integral

$$\oint_L \vec{B}d\vec{l} = \oint_L B_l dl, \qquad (5.20)$$

where $d\vec{l}$ is the vector of the elementary length of the contour directed along the contour;

$B_1 = B\cos\alpha$ is the component of the vector $\vec{B}$ in the direction of the tangent to the contour (taking into account the selected direction of traversal);

α is the angle between the vectors $\vec{B}$ and $d\vec{l}$.

The circulation theorem for vector $\vec{B}$: the circulation of the vector $\vec{B}$ over an arbitrary closed circuit is equal to the product of the magnetic constant μ_0 by the algebraic sum of the currents covered by this circuit

$$\oint_L \vec{B}d\vec{l} = \oint_L B_l dl = \mu_0 \sum_{k=1}^{n} I_k, \qquad (5.21)$$

where n is the number of conductors with currents covered by a contour L of arbitrary shape.

Each current is counted as many times as it is covered by the circuit. The current whose direction is connected with the direction of traversing the contour by the rule of the right screw is considered positive; the current of the opposite direction is considered negative. The circulation of the electrostatic field vector $\vec{E}$ is always equal to zero, that is, the electrostatic field is potential. The circulation of the magnetic field vector $\vec{B}$ is not zero. Such a field is called a *vortex field*.

The circulation theorem for the vector $\vec{B}$ has the same value in the study of a magnetic field as the Gauss's theorem in electrostatics, since it allows one to find the magnetic induction of the field without applying the Biot-Savart law.

5.6. Magnetic Field of Solenoid and Toroid

We calculate, by applying the circulation theorem, the induction of the magnetic field inside the solenoid. Consider a solenoid with length l, having N turns, along which current I passes.

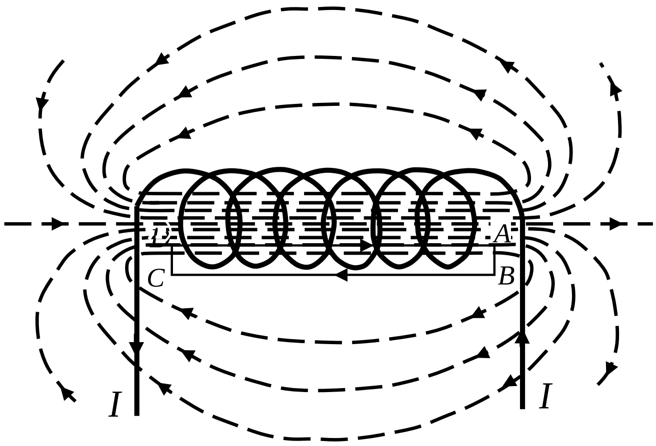


Figure 5.4. Magnetic field of solenoid.

The length of the solenoid is assumed to be many times larger than the diameter of its turns, that is, the solenoid being considered is infinitely long. Experimental study of the magnetic field of the solenoid shows that the field is homogeneous inside the solenoid, and the field outside the solenoid is inhomogeneous and very weak. Figure 5.4 presented the magnetic field lines inside and outside the solenoid. Increasing the length of the solenoid leads to a decrease in the magnetic induction in the surrounding space. Therefore, one can assume that the magnetic field of an infinitely long solenoid is concentrated entirely within it, and the magnetic field outside the solenoid can be neglected. In order to find the magnetic induction $\vec{B}$, we select the closed rectangular contour $ABCDA$, as shown in Figure 5.4. Circulation of vector $\vec{B}$ along the closed contour $ABCDA$, covering all N turns, is equal to

$$\oint_{ABCDA} B_l dl = \mu_0 NI . \tag{5.22}$$

The contour integral $ABCDA$ can be represented as the sum of four integrals over the segments AB, BC, CD and DA. In sections AB and CD the contour is perpendicular to the lines of magnetic induction and $B_1 = 0$. Outside the solenoid, the induction of the magnetic field is $B = 0$. The circulation of vector $\vec{B}$ is Bl in section DA (the contour coincides with the line of magnetic induction), hence

$$\int_{DA} B_l dl = Bl = \mu_0 NI . \tag{5.23}$$

Then the formula

$$B = \frac{\mu_0 NI}{l} \tag{5.24}$$

holds for the vacuum. Consequently, the magnetic field inside the solenoid is uniform (the edge effects in the regions adjacent to the ends of the solenoid are neglected in the calculations).

5.7. Gauss's Law for Magnetism

The flux of the magnetic induction vector (magnetic flux) through the surface dS is a scalar physical quantity equal to

$$d\Phi_B = \vec{B} d\vec{S} = B_n dS , \tag{5.25}$$

where $B_n = B\cos\alpha$ is the projection of vector $\vec{B}$ on the direction of the normal to the surface dS (α is the angle between vectors $\vec{n}$ and $\vec{B}$);

$d\vec{S}$ is a vector whose modulus is dS, and the direction coincides with the direction of the normal $\vec{n}$ to the surface.

The flux of vector $\vec{B}$ can be either positive or negative depending on the sign of $\cos\alpha$ (determined by the choice of the positive direction of the normal $\vec{n}$). Typically, the flux of vector $\vec{B}$ is associated with a specific circuit, over which current passes. In this case, we have already determined the positive direction of the normal to the contour: it is connected with the current by the rule of the right screw. Thus, the magnetic flux created by the contour through a surface limited by itself is always positive.

The flux Φ_B of the vector of magnetic induction through an arbitrary surface S is

$$\Phi_B = \int_S \vec{B} d\vec{S} = \int_S B_n dS . \tag{5.26}$$

For a homogeneous field and a plane surface perpendicular to the vector $\vec{B}$, formulas $B_n = B = const$ and $\Phi_B = BS$ are valid.

Gauss's law for the field $\vec{B}$: the flux of the vector of magnetic induction through any closed surface is equal to zero

$$\oint_S \vec{B} d\vec{S} = \oint_S B_n dS = 0 . \tag{5.27}$$

This theorem reflects the fact that there are no magnetic charges, as a result of which the magnetic field lines are closed.

5.8. Contour with a Current in Magnetic Field

The forces determined by the Ampère's force act on a contour with current in a magnetic field. Consider a non-fixed conductor, for example, one of the sides of the contour in the form of a movable bridge. Such a conductor will move in a magnetic field under the action of Ampère's force. Consequently, the magnetic field performs work to move the conductor with a current. To determine this work, consider a conductor with length l and current I placed in a uniform external magnetic field perpendicular to the plane of the contour. This conductor is influenced by the Ampère's force $F = IBl$.

Under the action of this force, the conductor will move parallel to itself to the segment dx. The work done by the magnetic field is

$$dA = Fdx = IBldx = IBdS = Id\Phi , \tag{5.28}$$

where $dS = ldx$ is the surface, crossed by the conductor when it is moved in a magnetic field;
$BdS = d\Phi$ is the flux of the vector of magnetic induction, which crosses this surface.

Thus, the work that must be performed to move a conductor with a current in a magnetic field is equal to the product of the current by the change in the magnetic flux during the motion of the conductor

$$dA = Id\Phi . \tag{5.29}$$

This formula is also valid for an arbitrary direction of the vector $\vec{B}$.

Test questions

1. Does the magnetic field act on the static electric charges?
2. What device is used to study the local characteristics of the magnetic field?
3. Give the formula for the moment of forces acting on the frame with a current in a magnetic field.
4. Indicate the direction of the magnetic moment vector.
5. Specify a formula by which the magnetic induction can be determined.

6. Give the units of measurement of the magnetic field.
7. Specify the rule for constructing the magnetic induction lines.
8. Specify for a fixed point the relative position of the magnetic field induction vector and line of the magnetic field.
9. Describe the functional relationship between induction and intensity of the magnetic field.
10. Specify the analogy in electricity for the intensity of the magnetic field.
11. Formulate the Biot-Savart law.
12. Specify the rule for determining the direction of the elementary induction vector of a magnetic field.
13. Formulate the superposition principle for the magnetic field induction vector.
14. Derive the formula for the induction of a magnetic field created by direct current.
15. Calculate the magnetic field in the centre of the circular conductor with current.
16. Formulate the Ampère's force law.
17. Specify the physical fields that create a moving charge.
18. Calculate the moving charge magnetic field induction.
19. Does the Lorentz force perform the work?
20. Formulate the Gauss's law for magnetism.

Problem-solving examples

Problem 5.1

Problem description. Determine the magnetic induction of the field created by a segment of length $L = 20\,cm$ of an infinitely long straight wire at a point remote from the ends of the segment at distances $L_1 = 25\,cm$ and $L_2 = 15\,cm$. The current I flowing through the wire is 20 A. The wire is in a vacuum.

Known quantities: $L = 20\,cm$, $L_1 = 25\,cm$, $L_2 = 15\,cm$, $I = 20\,A$.

Quantities to be calculated: B.

Problem solution. According to the Biot-Savart law, the induction of the magnetic field dB, created by a piece of wire with a current I and length of dL at a point located at a distance of r from the middle of the segment dL, is determined by the expression

$$dB = \frac{\mu_0 I \left[d\vec{L}, \vec{r}\right]}{4\pi r^3}, \qquad (5.1.1)$$

where $d\vec{L}$ is vector equal in magnitude to the length of the segment dL and coinciding in direction with the current,

$\vec{r}$ is the radius vector drawn from the middle of the conductor element to the point where the magnetic induction is determined,

μ_0 is magnetic constant.

For the modulus of the magnetic induction vector, we get

$$dB = \frac{\mu_0 I \sin\alpha}{4\pi r^2} dL, \qquad (5.1.2)$$

where α is angle between vectors $d\vec{L}$ and $\vec{r}$.

Let the element of conductor dL is visible from point A at an angle of $d\alpha$, and the distance from point A to the wire is r_0. Then for these values we can write

$$dL = \frac{r d\alpha}{\sin\alpha}, \quad r = \frac{r_0}{\sin\alpha}. \qquad (5.1.3)$$

In this case, for the elementary induction of the magnetic field we get

$$dB = \frac{\mu_0 I \sin\alpha\, d\alpha}{4\pi r_0}. \qquad (5.1.4)$$

We determine the induction of magnetic field generated by the segment of conductor. To do this, we will integrate the resulting expression over an angle ranging from α_1 to α_2:

$$B = \int_{\alpha_1}^{\alpha_2} \frac{\mu_0 I \sin\alpha}{4\pi r_0} d\alpha = \frac{\mu_0 I}{4\pi r_0} \int_{\alpha_1}^{\alpha_2} \sin\alpha\, d\alpha. \qquad (5.1.5)$$

Calculating the integral, we get

$$B = \frac{\mu_0 I}{4\pi r_0} (\cos\alpha_1 - \cos\alpha_2). \qquad (5.1.6)$$

From the condition of the problem it follows that $L_1^2 = L^2 + L_2^2$, i.e. $\alpha_2 = 90^0$, $\cos\alpha_2 = 0$, $r_0 = L_2$, $\cos\alpha_1 = 0.8$.

Substituting the numerical values, we get $B = 10.7\ \mu T$.

Answer. The magnetic induction is $B = 10.7\ \mu T$.

Problem 5.2

<u>Problem description</u>. Electric current $I = 7\,A$ flows along a square frame with side $a = 1.5\,cm$. The frame is in a non-uniform magnetic field that varies in space according to the law $B_z = kx$, where $k = 1.5\;T/m$, $B_y = B_x = 0$. Frame plane perpendicular to field induction lines. One of the sides of the frame coincides with the y-axis, the second coincides with the x-axis. The top of the frame is at the origin of the coordinate system. Calculate the work that needs to be done to slowly rotate the frame around the y-axis so that the lines of force of the field lie in the plane of the frame.

<u>Known quantities</u>: $I = 7\,A$, $a = 1.5\,cm$, $B_z = kx$, $k = 1.5\;T/m$, $B_y = B_x = 0$.

<u>Quantities to be calculated</u>: A.

<u>Problem solution</u>. If the frame is rotated slowly in a magnetic field, the induction currents can be neglected and the current in the circuit can be considered constant. The work that must be done to move the frame with the current in a magnetic field is

$$A = I\Delta\varPhi, \tag{5.2.1}$$

where $\Delta\varPhi$ is magnetic flux change. Since according to the condition of the problem in the final position, the plane of the frame is parallel to the power lines of the field, the magnetic flux in the final position of the frame is zero. Consequently, the change in the magnetic flux will be equal to its original value, at which the orientation of the frame is perpendicular to the field force lines, that is, $\Delta\varPhi = \Delta\varPhi_0$.

To calculate the magnetic flux $\varPhi_0$, we divide the plane of the frame into narrow strips of width dx parallel to y-axis. The area of each strip will be equal $ds = adx$. The magnetic flux through one of these strips, located at a distance of x from the y-axis, will be equal to

$$d\varPhi = B_z(x)ds = kxadx. \tag{5.2.2}$$

As a result of integration, we find the total flux of magnetic induction through the area of the frame:

$$\varPhi_0 = \int_0^a kxadx = \frac{kx^2}{2}. \tag{5.2.3}$$

We finally have:

$$A = I\Delta\varPhi = I\varPhi_0 = \frac{Ika^3}{2}. \tag{5.2.4}$$

We substitute numerically and calculate the work $A = 17.7\,\mu J$.

Answer. The work is $A = 17.7\,\mu J$.

Problem 5.3

Problem description. The electron moves in a uniform magnetic field with induction $B = 8\,mT$. The trajectory of movement is a spiral with a radius of $R = 1\,cm$ and a step of $h = 6.5\,cm$. Determine the period T of electron rotation and its speed υ.

Known quantities: $B = 8\,mT$, $R = 1\,cm$, $h = 6.5\,cm$.

Quantities to be calculated: T, υ.

Problem solution. The trajectory of the electron is a result of two movements: rotation around a circle under the action of a Lorentz force in a plane perpendicular to the magnetic field, and uniform movement along the direction of the field. Newton's second law, describing the rotational motion of an electron, is written as

$$e\upsilon_{\perp}B = \frac{m\upsilon_{\perp}^{2}}{R}, \tag{5.3.1}$$

where e is an electron charge,
m is the electron mass,
$\upsilon_{\perp}$ is a component of velocity that is perpendicular to the magnetic field,
R is the radius of the circle,
B is magnetic field induction.

Then the component $\upsilon_{\perp}$ of the electron rotational speed is equal to

$$\upsilon_{\perp} = \frac{eBR}{m}. \tag{5.3.2}$$

Consequently, the period of rotation of the electron can be found by the formula

$$T = \frac{2\pi R}{\upsilon_{\perp}} = 2\pi\frac{m}{eB}. \tag{5.3.3}$$

The velocity of the electron along the magnetic field is found as

$$\upsilon_{\parallel} = \frac{h}{T} = \frac{heB}{2\pi m}. \tag{5.3.4}$$

Taking into account the expressions obtained above, for full speed we get the following formula

$$\upsilon = \sqrt{\upsilon_\perp^{\ 2} + \upsilon_\parallel^{\ 2}} = \frac{eB}{m}\sqrt{R^2 + \frac{h^2}{4\pi^2}}\,. \qquad (5.3.5)$$

We substitute numerically: $T = 4.46\,ns$, $\upsilon = 20.2\,Mm/s$.

Answer. The period is $T = 4.46\,ns$. The velocity is $\upsilon = 20.2\,Mm/s$.

Problems

Problem A

Problem description. Two long parallel wires are at a distance of $r = 5\ cm$ from each other. Equal currents of $I = 10\ A$ each flow along wires in opposite directions. Find the magnetic field at a point located at a distance of $r_1 = 2\ cm$ from the first wire and $r_2 = 3\ cm$ from the second wire.

Answer. $H = 132\ A/m$.

Problem B

Problem description. The thin wire is curved into a regular hexagon. The length of the side of the hexagon is equal to $d = 10\,cm$. Determine the magnetic induction at the centre of the hexagon if a current of $I = 25\ A$ is flowing along the wire.

Answer. $B = 1.73 \times 10^{-4}\ T$.

Problem C

Problem description. An electron in an unexcited hydrogen atom moves around a nucleus along a circle with a radius of $r = 53\ pm$. Calculate the force of the equivalent circular current and the magnetic field at the center of the circle.

Answer. $I = 1.1 \times 10^{-3}\ A$, $H = 10^7\ A/m$.

Problem D

Problem description. Determine the maximum induction $B_{\max}$ of the magnetic field generated by an electron moving in a straight line at a speed of $\upsilon = 10\ Mm/s$ at a point away from the trajectory at a distance of $d = 1nm$.

<u>Answer</u>. $B_{max} = 1.6 \times 10^{-2}\ T$.

Problem E

<u>*Problem description*</u>. In a uniform magnetic field with induction $B = 0.4\ T$ in a plane perpendicular to the magnetic induction lines, a rod $L = 10\ cm$ length rotates. The axis of rotation passes through one of the ends of the rod. Determine the potential difference U at the ends of the rod at a frequency of $n = 16\ s^{-1}$.

<u>Answer</u>. $U = 2.01 \times 10^{-1}\ V$.

CHAPTER 6. MAGNETIC PROPERTIES OF MATTER

6.1. Magnetic Moments of Electrons and Atoms

In order to understand the magnetic properties of media and their effect on magnetic induction, it is necessary to consider the effect of a magnetic field on the atoms and molecules of matter. The properties of the medium were taken into account formally with the help of magnetic permeability μ.

All substances placed in a magnetic field are magnetized. Consider the cause of this phenomenon in terms of the structure of atoms and molecules, based on the Ampère hypothesis. According to the Ampère hypothesis microscopic currents, caused by the motion of electrons in atoms and molecules are present in any substance.

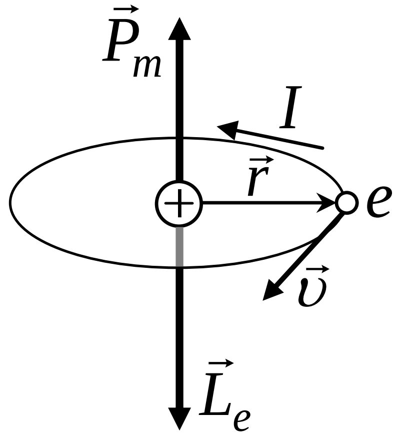


Figure 6.1. Electron orbital magnetic moment.

For a qualitative explanation of magnetic phenomena with sufficient approximation, we can assume that the electron moves in an atom along circular orbits. Electron moving along one of these orbits is equivalent to a circular current, so it has an orbital magnetic moment

$$\vec{p}_m = \vec{I}S\vec{n} . \qquad (6.1)$$

The modulus of this moment is equal to

$$p_m = IS = e\nu S , \qquad (6.2)$$

where $I = e\nu$ is the current;

ν is the frequency of the electron rotation in the orbit;

S is the area of the orbit.

If the electron moves clockwise (Figure 6.1), then the current is directed counter clockwise and the vector $\vec{p}_m$ in accordance with the rule of the right screw is directed perpendicular to the electron orbit plane. On the other hand, the electron moving along the orbit has a mechanical moment $\vec{L}_e$, whose modulus is

$$L_e = m\upsilon r = 2m\upsilon S\,, \tag{6.3}$$

where $\upsilon = 2\pi\nu r$, and $\pi r^2 = S$.

Vector $\vec{L}_e$ (its direction can also be determined with the right-handed screw rule) is called the orbital mechanical moment of the electron. Directions $\vec{p}_m$ and $\vec{L}_e$ are opposite, therefore

$$\vec{p}_m = -(e/2m)\vec{L}_e = g\vec{L}_e, \tag{6.4}$$

where the quantity $g = -e/(2m)$ is called the *gyromagnetic ratio* of the orbital mechanical moment (it is customary to write the relation for $\vec{p}_m$ with the sign "-", indicating that the directions of the moments are opposite).

This ratio, defined by universal constants, is the same for any orbit. Values υ and r are different for different orbits. In addition to orbital moments, the electron has its own mechanical moment $\vec{L}_{eS}$, called *spin*. It has now been established that spin is an inherent property of the electron, like its charge and mass. Magnetic moment $\vec{p}_{mS}$ is proportional to $\vec{L}_{eS}$ and directed to the opposite side

$$\vec{p}_{mS} = g_S\vec{L}_{eS} \tag{6.5}$$

with respect to the spin of the electron. The quantity g_S is called the gyromagnetic ratio of the spin moments. The projection of the intrinsic magnetic moment on the direction of the vector $\vec{B}$ can take only one of the following two values:

$$\vec{p}_{mSB} = \pm\frac{e\hbar}{2m} = \pm\mu_B, \tag{6.6}$$

where $\hbar = h/(2\pi)$ (h is the *Planck constant*),

μ_B is the *Bohr magneton*, which is the unit of the magnetic moment of the electron. Values h and μ_B are named after German physicist Max Karl Ernst Ludvig Planck (1858 – 1947) and Danish physicist Niels Henrik David Bohr (1885 – 1962), correspondingly.

In the general case, the magnetic moment of an electron is consists of the orbital and spin magnetic moments. The magnetic moment of the atom, therefore, is the sum of the magnetic moments of the electrons entering into the atom and the magnetic moment of the nucleus (due to the magnetic moments of the protons and neutrons). However, the magnetic moments of the nuclei are thousands of times smaller than the magnetic moments of the electrons, and therefore they are neglected. Thus, the total magnetic moment of the atom (molecule) $\vec{p}_a$ is equal to the vector sum of the magnetic moments (orbital and spin) of the electrons entering into the atom (molecule)

$$\vec{p}_a = \sum\vec{p}_m + \sum\vec{p}_{mS}\,. \tag{6.7}$$

When we considered the magnetic moments of electrons and atoms, we used the classical theory, not taking into account the limitations imposed on the motion of electrons by the laws of quantum mechanics. However, this does not contradict the obtained results, since for further explanation of the magnetization of substances, only the fact that the atoms have magnetic moments is essential.

6.2. Diamagnetic and Paramagnetic Phenomena

Any substance is a magnet, that is, it is capable of acquiring a magnetic moment (magnetize) under the action of a magnetic field. To understand the mechanism of this phenomenon, it is necessary to consider the effect of a magnetic field on electrons moving in an atom. For simplicity, we assume that the electron in the atom moves in a circular orbit. Suppose that the orbit of an electron is oriented with respect to the vector $\vec{B}$ in an arbitrary way and forms an angle α with it (Figure 6.2). In this case, the orbit plane moves around $\vec{B}$ and the vector of the magnetic moment $\vec{p}_m$ rotates around the direction $\vec{B}$ with some angular velocity. The angle α remains constant. This movement in mechanics is called *precession*.

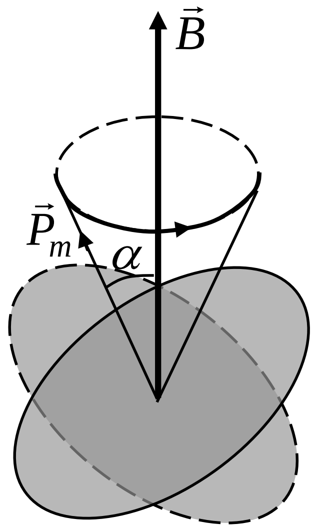


Figure 6.2. Mutual arrangement of the electron orbit and vector $\vec{B}$.

Thus, the electron orbits of an atom under the action of an external magnetic field perform a precession motion, which is equivalent to a circular current. This micro current is induced by an external magnetic field. Then, according to the Lenz rule, a component of the magnetic field appears and this component has a direction opposite to the external field. The induced components of the magnetic fields of atoms (molecules) add up and form an intrinsic magnetic field of matter, weakening the external magnetic field. This effect is called the *diamagnetic effect*, and substances that are magnetized in an external magnetic field against the direction of the field are called diamagnetic substances.

In the absence of an external magnetic field, the diamagnetic is non magnetic, since in this case the magnetic moments of the electrons are mutually compensated, and the total magnetic moment of the atom (it is equal to the vector sum of the magnetic moments (orbital and spin) constituents of the electron atom) is zero. Metals (for example, Bi, Ag, Au, Cu) and most organic compounds, resins, carbon are diamagnetic.

Since the diamagnetic effect is due to the action of an external magnetic field on the electrons of atoms, diamagnetism is manifested in all substances. However, along with diamagnetic substances, there are also paramagnetic substances. Substances that are magnetized in an external magnetic field along the direction of the field are called *paramagnetic*. In the absence of an external magnetic field, the magnetic moments of electrons of paramagnetic substances do not compensate each other, and the atoms (molecules) of paramagnetic always have a magnetic moment. However, due to the thermal motion of molecules, their magnetic moments are randomly oriented; therefore, paramagnetic substances do not possess magnetic properties.

The placement of a paramagnetic into an external magnetic field leads to an advantageous orientation of the magnetic moments of the atoms over the field. The thermal motion of atoms prevents the complete orientation of the magnetic moments. Thus, the paramagnetic is magnetized by creating its own magnetic field, co-incident with the external field and amplifying it. This effect is called *paramagnetic effect*. The thermal motion breaks the orientation of the magnetic moments and leads to the demagnetization of the paramagnet in the case when the external magnetic field decreases to zero. Rare-earth elements, Pt, Al, refer to paramagnetic substances. The diamagnetic effect is also observed in paramagnets, but it is much weaker than the paramagnetic effect, and therefore remains invisible. The explanation of the phenomenon of paramagnetic effect coincides with the explanation of the orientation (dipole) polarization of dielectrics with polar molecules. The electric moment of the atoms in the case of polarization must be replaced by the magnetic moment of the atoms in the case of magnetization. Summarizing the qualitative examination of diamagnetic and paramagnetic effects, we note once again that the atoms of all substances are carriers of diamagnetic properties.

The large magnetic moment of atoms leads to the fact that the paramagnetic properties predominate over the diamagnetic ones and the substance is paramagnetic. If the magnetic moment of the atoms is small, diamagnetic properties predominate and the material is diamagnetic.

6.3. Magnetization

Just as the polarization was introduced for the quantitative description of the properties of dielectrics, a magnetization is introduced for a quantitative description of the properties of magnets. The magnetization is determined by the magnetic moment of a unit volume of the magnet

$$\vec{J} = \vec{p}_m / V = \sum \vec{p}_a / V, \qquad (6.8)$$

where $\vec{p}_m = \sum \vec{p}_a$ is the magnetic moment of the magnet, which is the vector sum of the magnetic moments of the individual molecules.

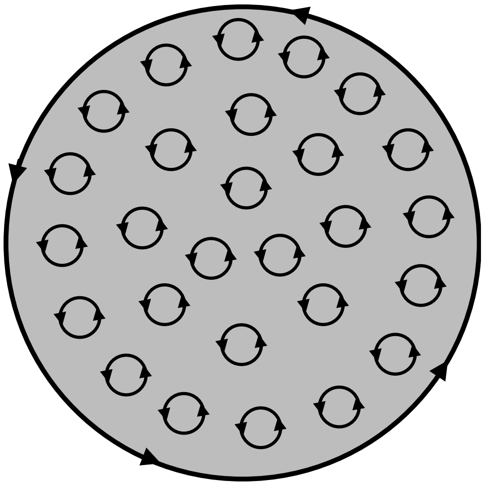


Figure 6.3. Molecular currents.

Considering the characteristics of the magnetic field, we introduced a vector of magnetic induction $\vec{B}$ characterizing the resulting magnetic field created by all macro- and micro-currents, and a vector of intensity $\vec{H}$ characterizing the magnetic field of macro-currents (Figure 6.3). Consequently, the magnetic field in the substance is the sum of their two fields: the external field produced by the current, and the field created by the magnetized substance. Then the vector of magnetic induction of the resulting magnetic field in the magnet is equal to the vector sum of the magnetic inductions of the external field $\vec{B}_0$ (field created by the magnetizing current in vacuum) and the field of micro-currents $\vec{B}'$ (the field created by the molar currents):

$$\vec{B} = \vec{B}_0 + \vec{B}', \tag{6.9}$$

where $\vec{B}_0 = \mu_0 \vec{H}$.

To describe the field produced by molecular currents, let us consider a magnet in the form of a circular cylinder of section S and length l, placed into a homogeneous external magnetic field with induction $\vec{B}_0$. The magnetic field of molecular currents arising in a magnetic field will be directed opposite to the external field for diamagnetic substances and coincide with it in the direction for paramagnetic substances. The planes of all molecular currents will be located perpendicular to vector $\vec{B}_0$, since vectors of their magnetic moments $\vec{p}_m$ are anti parallel to vector $\vec{B}_0$ (for diamagnetic substances) and are parallel to $\vec{B}_0$ (for paramagnetic substances). We consider any section of the cylinder perpendicular to its axis.

The molecular currents of neighbouring atoms in the internal sections of the cross section are directed opposite to each other and mutually compensated (Figure 6.3). Only the molecular currents emerging on the side surface of the cylinder will be uncompensated. The current passing along the side surface of the cylinder is similar to the current in the solenoid and creates a field inside it. The magnetic induction B' of this field can be calculated, taking into account that $N = 1$

$$B' = \mu_0 I'/l, \tag{6.10}$$

where I' is the molecular current,

l is the length of the cylinder, and the magnetic permeability μ is assumed to be unity.

On the other hand, the value I'/l is the current per unit length of the cylinder, or its linear density, so the magnetic moment of this current is

$$p = I'lS/l = I'V/l, \tag{6.11}$$

where V is the volume of the magnet.

If the value of P is the magnetic moment of a magnet of volume V, then P/V is the magnetization of the magnet. Thus,

$$J = I'/l. \tag{6.12}$$

Comparing the formulas, we get

$$B' = \mu_0 J, \tag{6.13}$$

or in the vector form

$$\vec{B}' = \mu_0 \vec{J}. \tag{6.14}$$

Substituting the expressions for $\vec{B}_0$ and $\vec{B}'$, we get

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{J} \tag{6.15}$$

or

$$\vec{B}/\mu_0 = \vec{H} + \vec{J}. \tag{6.16}$$

The magnetization in non-strong fields is directly proportional to the field intensity $\vec{H}$ causing magnetization, i.e.

$$\vec{J} = \chi \vec{H}, \tag{6.17}$$

where χ is a dimensionless quantity called the *magnetic susceptibility* of matter.

For diamagnetic substances χ is negative (the field of molecular currents is opposite to external), for paramagnetic substances χ is positive (the field of molecular currents coincides with the external field).

We rewrite the expression for $\vec{B}$:

$$\vec{B} = \mu_0 (1 + \chi) \vec{H}. \tag{6.18}$$

Hence

$$\vec{H} = \vec{B}/[\mu_0(1+\chi)]. \tag{6.19}$$

The dimensionless quantity

$$\mu = 1+\chi \tag{6.20}$$

is the magnetic permeability of the substance.

Since the absolute value of the magnetic susceptibility for diamagnetic and paramagnetic substances is very small (of the order of 10^{-4} –10^{-6}), the value of μ is insignificantly different from the unit. This is easy to understand, since the magnetic field of molecular currents is much weaker than the magnetizing field. Thus, for diamagnetic substances $\chi < 0$, and $\mu < 1$, for paramagnetic substances $\chi > 0$, and $\mu > 1$.

The circulation theorem for vector $\vec{B}$ in matter:

$$\oint_L \vec{B}d\vec{l} = \oint_L B_l dl = \mu_0(I + I'), \tag{6.21}$$

where I and I' are the algebraic sums of macro-currents (currents of conductivity) and micro-currents (molecular currents), covered by an arbitrary closed circuits.

Thus, the circulation of magnetic induction $\vec{B}$ along an arbitrary closed circuit is equal to the algebraic sum of the conduction currents and the molecular currents covered by this circuit, multiplied by the magnetic constant. The vector $\vec{B}$ thus characterizes the resulting field created both by macroscopic currents in conductors (conduction currents) and by microscopic currents in magnets, so the magnetic field lines have no sources and are closed.

It can be shown that the circulation of magnetization $\vec{J}$ over an arbitrary closed circuit L is equal to the algebraic sum of the molecular currents covered by this circuit

$$\oint_L \vec{J}d\vec{l} = I'. \tag{6.22}$$

Then the circulation of magnetic induction $\vec{B}$ in matter can also be written in the form

$$\oint_L \left(\frac{\vec{B}}{\mu_0} - \vec{J}\right) dl = I, \tag{6.23}$$

where I is the algebraic sum of conduction currents.

The expression in parentheses is equal to the previously introduced vector of magnetic field intensity $\vec{H}$. Thus, the circulation of vector $\vec{H}$ along an arbitrary closed circuit L is equal to the algebraic sum of the conduction currents enclosed by this contour:

$$\oint_L \vec{H}dl = I \qquad (6.24)$$

(circulation theorem for vector $\vec{H}$).

6.4. Magnetic Field Boundary Conditions

Consider the conditions for the vectors $\vec{B}$ and $\vec{H}$ at the interface of two homogeneous magnets (magnetic permeabilities μ_1 and μ_2) in the absence of conduction current at the boundary. Let us construct a straight cylinder near the interface between magnets 1 and 2 of a negligible height, one base of which is placed in the first magnetic, and the other is placed in the second magnetic. The bases of ΔS are so small that within each of them the vector $\vec{B}$ is the same. According to the Gauss's theorem,

$$B_{2n}\Delta S - B_{1n}\Delta S = 0 \qquad (6.25)$$

(normals $\vec{n}$ and $\vec{n}'$ to the bases of the cylinder are directed opposite). Therefore $B_{1n} = B_{2n}$. Taking into account that $\vec{B} = \mu\mu_0\vec{H}$, we obtain

$$H_{n1} / H_{n2} = \mu_2 / \mu_1. \qquad (6.26)$$

Near the interface of the two magnets 1 and 2, we construct a small closed rectangular contour ABCDA with a length l, placing it as shown in Figure 6.4.

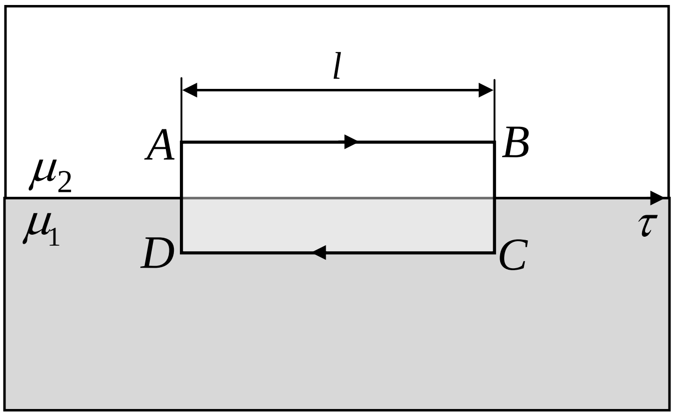


Figure 6.4. Scheme for determining the tangential component of the magnetic field.

According to the circulation theorem for vector $\vec{H}$,

$$\oint_{ABCDA} \vec{H}d\vec{l} = 0 \qquad (6.27)$$

(there are no conduction currents at the interface), whence

$$H_{2\tau}l - H_{1\tau}l = 0 \quad (6.28)$$

(the signs of the integrals over AB and CD are different, since the paths of integration are opposite, and the integrals along the sections of BC and DA are negligibly small). Therefore

$$H_{1\tau} = H_{2\tau}. \quad (6.29)$$

Replacing, according to formula $\vec{B} = \mu\mu_0\vec{H}$, the projections of the vector $\vec{H}$ by the projections of the vector $\vec{B}$, divided by $\mu\mu_0$, we obtain

$$B_{1\tau} / B_{2\tau} = \mu_1 / \mu_2. \quad (6.30)$$

Thus, when passing through the interface of two magnets, the normal component of the vector $\vec{B}$ (B_n) and the tangential component of the vector $\vec{H}$ (H_τ) change continuously (do not undergo a jump), and the tangential component of the vector $\vec{B}$ (B_τ) and the normal component of the vector $\vec{H}$ (H_n) are undergoing a jump.

The conditions for vectors $\vec{B}$ and $\vec{H}$ indicate that the lines of these vectors undergo a break (refracted). As in the case of dielectrics, one can find the law of refraction of lines $\vec{B}$ (and, in other words, of lines $\vec{H}$):

$$tg\alpha_2 / tg\alpha_1 = \mu_2 / \mu_1. \quad (6.31)$$

6.5. Ferromagnetic Substances

In addition to two classes of substances, namely diamagnetic substances and paramagnets substances, called weakly magnetic substances, there are still strongly magnetic substances (ferromagnetic substances). Substances with spontaneous magnetization which are magnetized even in the absence of an external magnetic field are called *ferromagnetic substances*. Iron, cobalt, nickel, gadolinium, and their alloys and compounds belong to ferromagnetic substances.

In addition to the ability to strongly magnetize, ferromagnetic substances also have other properties, which substantially distinguish them from diamagnetic substances and paramagnetic substances. Magnetization $\vec{J}$ of ferromagnetic substances first increases rapidly with increasing $\vec{H}$, then slower increases, and finally reaches the so-called *magnetic saturation* J_S, which no longer depends on the field intensity. A similar character of the dependence of J on H can be explained by the fact that an increase in the magnetizing field leads to an increase in the degree of orientation of the molecular magnetic moments over the field. However, this process will begin to slow down when there are less and less moments with random orientation, and, finally, when all the moments are oriented along the field. A further degree of orientation increase process stops and a magnetic saturation begins. The magnetic induction

$$B = \mu_0 (H + J) \quad (6.32)$$

in weak fields increases rapidly with growth of H as a result of the magnetization J increase. The value B grows according to a linear law in strong fields.

The dependence of J on H (and hence B on H) is determined by the prehistory of the magnetization of the ferromagnetic substances. This phenomenon was called the *magnetic hysteresis*.

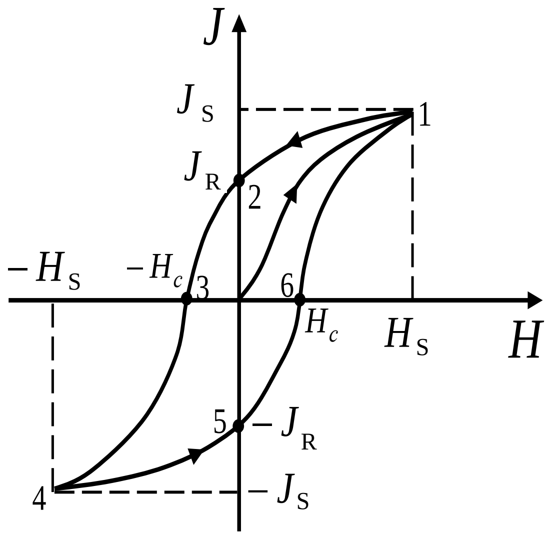


Figure 6.5. Magnetic hysteresis.

If we magnetize the ferromagnetic substance before saturation (Figure 6.5, point 1), and then begin to reduce the intensity H of the magnetizing field, then the decrease in J is described by curve 1–2 lying above the curve 1–0. In the case when $H = 0$ the quantity J differs from zero, that is, a residual magnetization J_R is observed in the ferromagnetic substance. The existence of permanent magnets is associated with residual magnetization. The magnetization vanishes under the action of the field H_C, which has a direction opposite to the field that caused the magnetization. The intensity H_C of the magnetic field is called *coercive force*.

Thus, when an alternating magnetic field acts on a ferromagnetic substance, the magnetization J varies in accordance with the curve 1-2-3-4-5-6-1, which is called the *hysteresis loop*. Hysteresis leads to the fact that the magnetization of a ferromagnetic substance is not a single-valued function of H, i.e., several values of H correspond to the same value J.

Different ferromagnetic substances correspond to different hysteresis loops. Ferromagnetic substances with a small (up to 1–2 A/cm) coercive force and with a narrow hysteresis loop are called soft ferromagnetic substances. Ferromagnetic substances with a large (up to 1000 A/cm) coercive force and a wide hysteresis loop are called rigid ferromagnetic substances. Quantities of H_C, I_R, μ_{MAX} determine the applicability of ferromagnetic substances for various practical purposes. Rigid ferromagnetic substances (for example, carbon and tungsten steels) are used to make permanent magnets. Soft ferromagnetic substances (for example, soft iron, iron alloy with nickel) are used for the production of converter cores.

Each ferromagnetic substance can be associated with the Curie temperature. The ferromagnetic substance turns into an ordinary paramagnetic substance when the sample is heated above the *Curie point*. This temperature is named after French physicist Pierre Curie (1859 – 1906).

The process of magnetization of ferromagnetic substances is accompanied by a change in its linear dimensions and volume. This phenomenon was called *magnetostriction*. The magnitude and sign of this effect depend on the intensity of the magnetizing field, on the nature of the ferromagnetic substance, and on the orientation of the crystallographic axes relative to the magnetic field.

6.6. Nature of Ferromagnetism

According to French physicist Pierre-Ernest Weiss (1865 – 1940) ideas, ferromagnetic substances at temperatures below the Curie point possess spontaneous magnetization, regardless of the presence of an external magnetizing field. Spontaneous magnetization, however, is in apparent contradiction with the fact that many ferromagnetic materials, even at temperatures below the Curie point, are not magnetized. To eliminate this contradiction, Weiss introduced the hypothesis that a ferromagnetic substances below the Curie point is broken down into a large number of small macroscopic regions, self-magnetically magnetized to saturation. These regions are called *domains*. The magnetic moments of individual domains are randomly oriented and compensate each other in the absence of an external magnetic field. Therefore, the resulting magnetic moment of the ferromagnetic substance is zero and it is not magnetized. The external magnetic field orientates the entire regions of spontaneous magnetization. Therefore, the magnetization J and the magnetic induction B grow very rapidly in rather weak fields with increasing magnetic intensity.

In the case when the external magnetic field decreases to zero, the ferromagnetic substances have the residual magnetization, since the heat movement is not able to quickly disorient the magnetic moments of such large formations as the domains. Therefore, the phenomenon of magnetic hysteresis is observed. In order to demagnetize the ferromagnetic substances, it is necessary to apply a coercive force. Shaking and heating also leads to demagnetization of the ferromagnetic substances.

Test questions

1. Formulate the Ampère hypothesis.
2. What is the orbital mechanical moment of the electron that moves in orbit around the nucleus of an atom?
3. Specify the relationship between the magnetic moment and the orbital mechanical moment of the electron.
4. Give the definition of the electron's orbital mechanical moment.

5. Explain the choice of sign for the gyromagnetic ratio.
6. What physical quantity of an electron can be compared with its spin?
7. List the values that the projection of the intrinsic magnetic moment can take on the direction of the magnetic induction vector.
8. List the values of the projection of the intrinsic magnetic moment on the direction of the magnetic induction vector.
9. Compare the values of the magnetic moments of the electron and the nucleus.
10. Describe the precession phenomenon.
11. Is the diamagnetism manifested in all substances?
12. Describe the paramagnetic effect.
13. What types of magnets are rare earth metals?
14. Give the definition of the magnetization vector.
15. Specify the relationship between the magnetic induction and magnetization vectors.
16. Indicate the magnetic susceptibility signs for the main types of magnets.
17. Write down the determining equation for magnetic permeability.
18. Formulate the theorem on the circulation of the magnetic field strength vector.
19. Give formulas that describe the conditions for the vectors of induction and strength of magnetic field at the interface of two homogeneous magnets.
20. What physical phenomenon causes the existence of permanent magnets?

Problem-solving examples

Problem 6.1

Problem description. An electron in a hydrogen atom rotates in a circular orbit of radius $r = 5.3 \times 10^{-11}\, m$ at a speed of $\upsilon = 2.2 \times 10^{6}\, m/s$. Calculate the orbital magnetic moment of the electron.

Known quantities: $r = 5.3 \times 10^{-11}\, m$, $\upsilon = 2.2 \times 10^{6}\, m/s$.

Quantities to be calculated: p_m.

Problem solution. The motion of an electron in an orbit can be considered as a circular current I, which has a magnetic moment

$$p_m = IS\,, \qquad (6.1.1)$$

where $S = \pi r^2$ is the area bounded by the electron orbit.

For the case when the frequency of rotation of the electron is n, the current is equal to

$$I = en, \qquad (6.1.2)$$

where e is an electron charge.

Then for the electron speed, frequency of turns and current we get

$$\upsilon = 2\pi rn,\ n = \frac{\upsilon}{2\pi r},\ I = \frac{e\upsilon}{2\pi r}. \qquad (6.1.3)$$

The magnetic moment of the electron is

$$p_m = IS = \frac{e\upsilon}{2\pi r}\pi r^2. \qquad (6.1.4)$$

And finally, with numbers: $p_m = 9.3\times10^{-24}\ A\cdot m^2$.

Answer. The orbital magnetic moment of the electron is $p_m = 9.3\times10^{-24}\ A\cdot m^2$.

Problem 6.2

Problem description. A solenoid of length $l = 25\,cm$, with a cross-sectional area of $S = 12\,cm^2$ and a total number of coils $N = 410$ is in a diamagnetic environment. Determine the strength of the current in the coil of the solenoid, if its inductance is $L = 1\,mH$ and the magnetization inside the solenoid is $J = 22\,A/m$.

Known quantities: $l = 25\,cm$, $S = 12\,cm^2$, $N = 410$, $L = 1\,mH$, $J = 22\,A/m$.

Quantities to be calculated: I.

Problem solution. The magnetization inside the solenoid is

$$J = \chi H, \qquad (6.2.1)$$

where χ is the magnetic susceptibility of matter,
H is the magnetic field strength.

We can write the following expressions for magnetic permeability and magnetization of a substance

$$\mu = 1+\chi,\ J = (\mu-1)H. \qquad (6.2.2)$$

The circulation of the magnetic field strength vector is equal to the algebraic sum of the currents that are covered by the contour l:

$$\oint_l \vec{H}d\vec{l} = \oint_l H_L dl = \sum_k I_k. \qquad (6.2.3)$$

The relationship of the magnetic field strength and the number of turns of the solenoid is

$$Hl = NI, \tag{6.2.4}$$

thus, the magnetic field is

$$H = \frac{NI}{l}. \tag{6.2.5}$$

The inductance of the solenoid is

$$L = \mu_0 \mu N^2 S / l, \tag{6.2.6}$$

where μ_0 is magnetic constant,

μ is the relative magnetic permeability of the diamagnetic environment.

Then for μ we get

$$\mu = \frac{Ll}{\mu_0 N^2 S}. \tag{6.2.7}$$

We use the formulas for μ and H, and then we write the expression for magnetization

$$J = \left(\frac{Ll}{\mu_0 N^2 S} - 1 \right) \frac{NI}{l}, \tag{6.2.8}$$

consequently, current is

$$I = \frac{Jl}{N\left(\frac{Ll}{\mu_0 N^2 S} - 1 \right)}. \tag{6.2.9}$$

We substitute numerically: $I = 1.01 A$.

<u>Answer</u>. The strength of the current in the coil of the solenoid $I = 1.01 A$.

Problem 6.3

<u>Problem description</u>. The solenoid is in a diamagnetic environment. The solenoid has a length of $l = 50\,cm$, a cross-sectional area of $S = 10\,cm^2$ and the number of turns $N = 1200$. The inductance of the solenoid is $L = 36\,mH$, and the current flowing through the solenoid is $I = 0.6\,A$. Determine the magnetic induction and magnetization inside the solenoid.

<u>Known quantities</u>: $l = 50\,cm$, $S = 10\,cm^2$, $N = 1200$, $L = 36\,mH$, $I = 0.6\,A$.

<u>*Quantities to be calculated*</u>: J, B.

<u>*Problem solution*</u>. The inductance of the solenoid is related to the number of turns N, the cross-sectional area is S and the length is l by a ratio

$$L = \frac{\mu_0 \mu N^2 S}{l}, \tag{6.3.1}$$

where μ is the magnetic permeability of the medium.

Due to the fact that the medium is diamagnetic, the magnitude μ does not depend on the characteristics of the magnetic field that is created by the solenoid.

We apply the theorem on the circulation of the magnetic field strength vector to the solenoid

$$\oint_l \vec{H} d\vec{l} = NI, \tag{6.3.2}$$

then

$$H = \frac{NI}{l}. \tag{6.3.3}$$

Contour l covers the coils of the solenoid, partially passing through it. In this case, we will take into account only that part of the contour that is located inside the solenoid, i.e. where the field is approximately uniform.

The magnetization of the substance inside the solenoid is

$$J = (\mu - 1)H \tag{6.3.4}$$

or

$$J = \left(\frac{Ll}{\mu_0 N^2 S} - 1 \right) \frac{NI}{l}. \tag{6.3.5}$$

The relationship of induction and magnetic field strength is

$$B = \mu \mu_0 H = \frac{LI}{NS}. \tag{6.3.6}$$

where μ_0 is magnetic constant.

We substitute numerically and calculate the magnetization and magnetic induction: $J = 7.6\, A/m$, $B = 0.52\, T$.

<u>*Answer*</u>. The magnetization and magnetic induction are $J = 7.6\, A/m$, $B = 0.52\, T$.

Problems

Problem A

Problem description. On the iron ring is wound in a single layer $N=500$ turns of wire. The average ring diameter is $d=25\ cm$. Determine the magnetic induction in iron and the magnetic permeability of iron, if the current in the winding is $I=0.5\ A$.

Answer. $B=1\,T$, $\mu=2.5\times10^{3}$.

Problem B

Problem description. A closed steel core toroid has $n=10$ turns for every centimetre of length. The current that flows through the solenoid is $I=2\ A$. Calculate the magnetic flux Φ in the core if its cross section is $S=4\ cm^{2}$.

Answer. $\Phi=5.2\times10^{-4}\ Wb$.

Problem C

Problem description. The electromagnet is made in the form of a toroid. The core of the toroid with an average diameter of $d=51\,cm$ has a vacuum gap of length $L_0=2\ mm$. The toroid winding is evenly distributed over its entire length. How many times will the induction of the magnetic field decrease in the gap, if, without changing the current in the winding, the gap is increased three times? The scattering of the magnetic field near the gap is neglected. The magnetic permeability of the core is considered constant and taken equal to 800.

Answer. $N=2$.

Problem D

Problem description. In the iron core of the solenoid, the magnetic field induction is $B=1.3\,T$. The iron core was replaced with a steel one. Determine how many times the current in the coil of the solenoid should be changed so that the induction in the core remains unchanged.

Answer. $N=2.4$.

Problem E

Problem description. The length of the cast-iron torus along the midline is $L = 1.2\ m$, and the cross section is $S = 20\ cm^2$. A current flows in the toroid winding, creating a magnetic flux in a narrow vacuum gap equal to $\Phi = 0.5\ mWb$. The gap length is $L_0 = 8\ mm$. What should be the length of the gap so that did the current strength doubled?

Answer. $L = 1.8 \times 10^{-3}\ mm$.

CHAPTER 7. ELECTROMAGNETIC INDUCTION

7.1. Faraday's Law of Induction

British scientist Michael Faraday (1791–1867) came to the quantitative law of electromagnetic induction by generalizing the results of experiments. Faraday showed that a change in the flux of magnetic induction coupled to the circuit results in the generation of an induction current in the circuit. The appearance of an induction current indicates the occurrence of an electromotive force (EMF) in a circuit, called the electromotive force of *electromagnetic induction*. The value of the induction current and, consequently, the EMF ε_i of electromagnetic induction is determined only by the rate of change of the magnetic flux, i.e.

$$\varepsilon_i \approx \frac{d\Phi}{dt}. \tag{7.1}$$

The sign of the magnetic flux depends on the choice of the positive normal to the contour. In turn, the positive direction of the normal is connected with the current by the rule of the right screw. Consequently, choosing a certain positive direction of the normal, we define both the sign of the flux of magnetic induction, and the direction of the current and EMF in the circuit. Using these ideas and conclusions, one can accordingly come to the formulation of Faraday's law of electromagnetic induction: whatever the reason for the change in flux of magnetic induction, enclosed by a closed conducting circuit, the EMF generated in the contour is

$$\varepsilon_i = -\frac{d\Phi}{dt}. \tag{7.2}$$

The minus sign shows that increasing the magnetic flux ($\frac{d\Phi}{dt} > 0$) causes the EMF $\varepsilon_i < 0$, that is, the field of the induction current is directed towards the flux; the decrease in magnetic flux ($\frac{d\Phi}{dt} < 0$) causes EMF $\varepsilon_i > 0$. The minus sign in the formula for the law of electromagnetic induction is a mathematical expression of the *Lenz's law*: the induction current in the circuit always has such a direction that the magnetic field created by it prevents the change of the magnetic flux that caused this induction current. This law is named after Russian physicist Heinrich Friedrich Emil Lenz (1804 – 1865).

Faraday's law can also be formulated in this way: the EMF of the electromagnetic induction in the contour with circuit is numerically equal and opposite in sign in compare with the change rate of the magnetic flux through the surface bounded by this contour. This law is universal: EMF does not depend on the method of magnetic flux changing.

7.2. Rotating Frame in Magnetic Field

The phenomenon of electromagnetic induction is used to convert mechanical energy into electric current. Generators are used for this purpose. Let us consider the operation of generators using a model of a plane frame rotating in a homogeneous magnetic field (Figure 7.1).

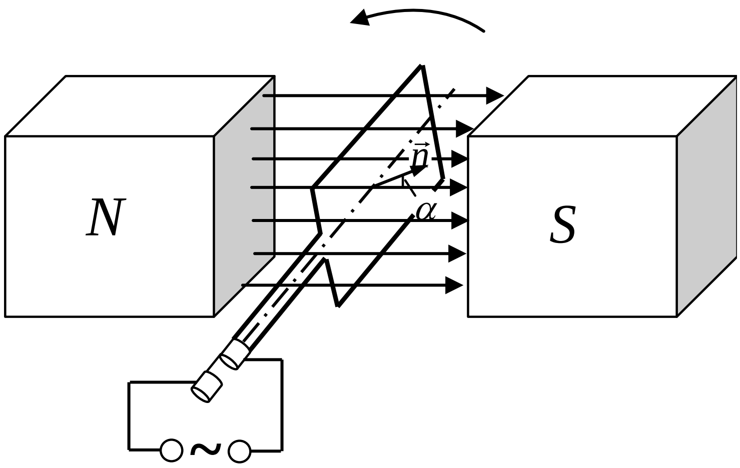


Figure 7.1. Electric generator.

Suppose that the frame rotates in a homogeneous magnetic field ($B = const$) uniformly with angular velocity $\omega = const$. The magnetic flux coupled to the frame with an area of S at any time t is

$$\Phi = B_n S = BS\cos\alpha = BS\cos\omega t, \tag{7.3}$$

where $\alpha = \omega t$ is the angle of rotation of the frame at time t ($\alpha = 0$ at $t = 0$).

When the frame rotates, a variable induction EMF will appear in it

$$\varepsilon_i = -\frac{d\Phi}{dt} = BS\omega\sin\omega t. \tag{7.4}$$

The EMF reaches its maximum value

$$\varepsilon_{\max} = BS\omega \ (\sin\omega t = 1). \tag{7.5}$$

The EMF changes according to the harmonic law. It follows from

$$\varepsilon_{max} = BS\omega \tag{7.6}$$

that ε_{max} (and hence also the EMF of induction) is directly dependent on the magnitudes of ω, B and S.

7.3. Inductance of the Circuit

The electric current passing in a closed circuit creates around itself a magnetic field. The induction of magnetic field, according to the Biot-Savart law, is proportional to the current. The magnetic flux Φ connected to the circuit is proportional to the current I in the circuit:

$$\Phi = LI, \tag{7.7}$$

where the proportionality coefficient L is called the *inductance* of the circuit.

The change in the current in the circuit results in a change in the magnetic flux coupled to the circuit. Consequently, the EMF will be induced in the circuit. The phenomenon of self-induction consists in the occurrence of EMF induction in a conducting circuit under the condition that the current in it changes.

We calculate the inductance of an infinitely long solenoid. The total magnetic flux through the solenoid is $\mu_0 \mu S N^2 I / l$. By inserting this expression into formula $\Phi = LI$, we obtain

$$L = \mu_0 \mu S N^2 / l, \tag{7.8}$$

that is, the inductance of the solenoid depends on the number of turns of the solenoid N, its length l, the surface S and the magnetic permeability μ of the substance from which the core of the solenoid is made.

It can be shown that the inductance of the contour in the general case depends only on the geometric shape of the contour, its dimensions and the magnetic permeability of the medium in which it is located. In this sense, the inductance of the circuit is an analogy of the electrical capacity of a solitary conductor, which also depends only on the shape of the conductor, its dimensions, and dielectric permeability of the medium. Applying Faraday's law to the self-induction phenomenon, we find that the self-induction EMF is

$$\varepsilon_S = -\frac{d\Phi}{dt} = -\frac{d}{dt}(LI) = -\left(L\frac{dI}{dt} + I\frac{dL}{dt}\right). \tag{7.9}$$

In the case when the contour is not deformed and the magnetic permeability of the medium does not change (it will be shown later that the last condition is not always satisfied), then $L = const$ and

$$\varepsilon_S = -L\frac{dI}{dt}. \tag{7.10}$$

The minus sign in Lenz’s law shows that the presence of inductance in the circuit leads to a slowing of the current change. The increase in current leads to $dI/dt > 0$ and $\varepsilon_S < 0$, that is, the current of self-induction is opposite directed towards the current caused by the external source, and slows its increase. The decrease in current leads to $dI/dt < 0$ and $\varepsilon_S > 0$, that is, the induction current has the same direction as the decreasing current in the circuit, and slows its decrease. Thus, the circuit, having a certain inductance, acquires an electrical inertia, which is included in the fact that any change in the current is inhibited the stronger, the larger the inductance of the circuit.

7.4. Magnetic Field Energy

The conductor with the electric current is always surrounded by a magnetic field. A magnetic field appears and disappears together with the appearance and disappearance of current. A magnetic field, like the electric field, is the carrier of energy. It is natural to assume that the energy of the magnetic field is equal to the work that is performed by the current on the creation of this field.

Consider current loop with inductance L. The magnetic flux $\Phi = LI$ corresponds to this circuit. A change in current dI is the cause of a change in magnetic flux

$$d\Phi = LdI. \tag{7.11}$$

However, to change the magnetic flux by a value of $d\Phi$, it is necessary to perform the work

$$dA = Id\Phi = LIdI. \tag{7.12}$$

Then the work on creating the magnetic flux Φ will be

$$A = \int_0^I LIdI = \frac{LI^2}{2}. \tag{7.13}$$

Consequently, the energy of the magnetic field associated with the circuit is

$$W = \frac{LI^2}{2}. \tag{7.14}$$

The energy of the magnetic field can be represented as a function of the quantities characterizing this field in the surrounding space. Consider a particular case, namely, a homogeneous magnetic field inside a long solenoid. Substituting the expression for L into the formula for the work A on creating a magnetic field, we obtain

$$W = \frac{1}{2}\mu_0\mu\frac{N^2I^2}{l}S. \tag{7.15}$$

Since

$$I = \frac{Bl}{\mu_0\mu N} \tag{7.16}$$

and

$$B = \mu_0 \mu H \,, \tag{7.17}$$

then

$$W = \frac{B^2}{2\mu_0 \mu} V = \frac{BH}{2} V \,, \tag{7.18}$$

where $Sl = V$ is the volume of the solenoid.

The magnetic field of the solenoid is uniform and concentrated inside it; therefore the energy is enclosed in the volume of the solenoid and is distributed in it with a constant bulk density

$$\omega = \frac{W}{V} = \frac{B^2}{2\mu_0 \mu} = \frac{\mu_0 \mu H^2}{2} = \frac{BH}{2} \,. \tag{7.19}$$

The formula is derived for a homogeneous field, but it is also valid for inhomogeneous fields. However, this expression is valid only for media for which the dependence of B on H is linear, that is, it refers only to paramagnetic and diamagnetic substances.

Test questions

1. Explain what the physical factor causes the appearance of an electromotive force.
2. Write the expression for the electromotive force module.
3. Formulate Faraday's law for electromotive force.
4. What is the essence of the Lenz’s law?
5. Does the EMF depend on the method of changing the magnetic flux?
6. What physical phenomenon underlies the conversion of mechanical energy into electric current?
7. Write down the law of change with time of the magnetic flux, which is intersected by a flat frame, rotating in a uniform magnetic field.
8. Specify the physical quantities whose values affect the maximum value of the EMF of a flat frame rotating in a uniform magnetic field.
9. Write a formula for the magnetic flux associated with a closed circuit through which an electric current passes.
10. What is the phenomenon of self-induction?
11. How does the total magnetic flux through a solenoid depend on its length?
12. What determines the inductance of the circuit?
13. Write down the formula for the electromotive force of self-induction.
14. Calculate the work of creating a magnetic flux in the circuit with a constant inductance.
15. Write the formula for the energy of the magnetic field associated with the circuit?

16. Make a quantitative and qualitative description of the magnetic field energy of the solenoid.
17. Where is the solenoid magnetic field energy concentrated?
18. Is it possible to say that the magnetic field energy outside the solenoid is exactly zero?
19. Is the energy distribution inside the solenoid uniform?
20. Write the formula for the bulk density of energy inside the solenoid.

Problem-solving examples

Problem 7.1

Problem description. A charged particle moves around a circle of radius $R = 2.5 cm$ in a uniform magnetic field with induction $B = 0.3\,T$. The electric field is switched parallel to the magnetic field. The electric field depends on time according to the law $E = \alpha t^2$, where $\alpha = 6\,V/s^2$. Determine the time after switching on the field, which is necessary for the kinetic energy of the particle to double.

Known quantities: $R = 2.5 cm$, $B = 0.3\,T$, $E = \alpha t^2$, $\alpha = 6\,V/s^2$.

Quantities to be calculated: t_0.

Problem solution. In the absence of an electric field, the particle was moving in a circle under the action of the Lorentz force. Particle acceleration is

$$a_1 = \frac{F_L}{m} = \frac{B\upsilon_1 Q}{m}, \qquad (7.1.1)$$

where m is a particle mass,
Q is a particle charge,
υ_1 is the speed of the particle before the electric field is turned on.

In this case, the force F_L acts perpendicular to the speed. Therefore, the particle acceleration is centripetal

$$a_1 = \frac{\upsilon_1^2}{R}, \qquad (7.1.2)$$

We find the rate, which had particle before switching the electric field.

$$\upsilon_1 = \frac{RQB}{m}. \qquad (7.1.3)$$

The kinetic energy of a particle depends on its mass and speed.

$$E_1 = \frac{m\upsilon_1^2}{2}. \quad (7.1.4)$$

After turning on the electric field at time $t = 0$, the particle begins to accelerate in the direction perpendicular to the direction of the vector $\vec{E}$. Acceleration of a particle in this direction is determined from Newton's second law

$$a(t) = \frac{QE(t)}{m}. \quad (7.1.5)$$

After time t_0, the particle speed will be equal to

$$\upsilon_2 = \int_0^{t_0} a(t)dt = \frac{Q\alpha}{3m} t_0^3. \quad (7.1.6)$$

The resulting velocity square will take the value

$$\upsilon = \upsilon_1^2 + \upsilon_2^2. \quad (7.1.7)$$

Kinetic energy will be equal, respectively

$$E_2 = \frac{m}{2}\left(\upsilon_1^2 + \upsilon_2^2\right). \quad (7.1.8)$$

Since this energy is twice the initial energy, then $\upsilon_1 = \upsilon_2$.

Rewrite the expression for the initial and final speeds

$$\upsilon_1 = \upsilon_2 \text{ или } \frac{RQB}{m} = \frac{Q\alpha}{3m} t_0^3. \quad (7.1.9)$$

Express time from the last relation

$$t_0 = \left(\frac{3RB}{\alpha}\right)^{1/3}. \quad (7.1.10)$$

We substitute numerically $t_0 = 0.155\ s$.

Answer. The time is $t_0 = 0.155\ s$.

Problem 7.2

Problem description. A flat wire frame, the area of which is equal to $S = 3 \times 10^{-2}\ m^2$, and the resistance $R = 1.5\ \Omega$, is located in a uniform magnetic field with induction $B = 0.2\ T$ initially, the coil plane is perpendicular to the magnetic induction lines. The frame is connected to the galvanometer. The total charge flowing through the galvanometer is equal to $Q = 8.5 \times 10^{-4}\ C$. Determine the angle of rotation of the frame.

Known quantities: $S = 3 \times 10^{-2}\ m^2$, $R = 1.5\ \Omega$, $B = 0.2\ T$, $Q = 8.5 \times 10^{-4}\ C$.

Quantities to be calculated: α.

Problem solution. Let the normal to the coil plane coincide with the direction of the magnetic induction vector. The initial magnetic flux through the area bounded by a coil is equal to

$$\Phi_1 = BS\cos 0^0 = BS, \tag{7.2.1}$$

where B is magnetic field induction,
S is a coil area.

The rotation of the plane of the frame at an angle of α causes the rotation of the normal to the frame at an angle of α, so the magnetic flux becomes 44

$$\Phi_2 = BS\cos\alpha. \tag{7.2.2}$$

Since the magnetic flux Φ has changed, an EMF of induction has appeared in the frame. However, the law of change in the magnetic flux in time is not specified. The magnetic flux Φ may vary unevenly over time. Therefore, to calculate the EMF induction, we use the formula

$$\varepsilon_i = -\frac{d\Phi(t)}{dt}. \tag{7.2.3}$$

Induction current flows through the frame

$$I(t) = \frac{\varepsilon_i}{R} = -\frac{1}{R}\frac{d\Phi(t)}{dt}. \tag{7.2.4}$$

The electric charge that flows through the coil is

$$Q = \int_{t_1}^{t_2} I(t)dt, \tag{7.2.5}$$

where t_1 and t_2 are the initial and final times, respectively.

Rewrite the previous equality

$$Q=\int_{t_1}^{t_2}\left(-\frac{1}{R}\frac{d\Phi(t)}{dt}\right)dt=-\frac{1}{R}\int_{t_1}^{t_2}\frac{d\Phi(t)}{dt}dt=-\frac{1}{R}[\Phi(t_2)-\Phi(t_1)]=-\frac{1}{R}\Delta\Phi. \quad (7.2.6)$$

Thus, regardless of how the turn turns, the charge flowing through the closed loop is equal to

$$Q=-\frac{\Delta\Phi}{R}. \quad (7.2.7)$$

The formula for the charge is obtained under the assumption that the inductance of the circuit is negligible.

Rewrite the formula for charge

$$Q=-\frac{BS(\cos\alpha-1)}{R}, \quad (7.2.8)$$

then

$$\cos\alpha=1-\frac{QR}{BS}, \quad (7.2.9)$$

therefore

$$\alpha=\arccos\left(1-\frac{QR}{BS}\right). \quad (7.2.10)$$

Numerically $\alpha=38^0$.

Answer. The angle of rotation of the frame $\alpha=38^0$.

Problem 7.3

Problem description. The solenoid with a diameter of $d=8cm$ is in a uniform magnetic field with induction $B=0.6\,T$. The solenoid has $N=80$ turns of copper wire with a cross-sectional area of $\sigma=1mm^2$. The solenoid is turned at an angle of $\alpha=180^0$ over time $\Delta t=0.2\,s$ so that its axis remains directed along the field. Determine the average value of the EMF that occurs in the solenoid, as well as the induction charge. The resistivity of copper is $\rho=1.7\times10^{-8}\,\Omega\times m$.

Known quantities: $d=8cm$, $B=0.6\,T$, $N=80$, $\sigma=1mm^2$, $\alpha=180^0$, $\Delta t=0.2\,s$, $\rho=1.7\times10^{-8}\,\Omega\times m$.

Quantities to be calculated: ε_i, Q.

Problem solution. The change in the magnetic flux $\Delta\Phi$, which penetrates the solenoid, leads to the appearance of an EMF

$$\varepsilon_i = -N\frac{\Delta\Phi}{\Delta t}, \qquad (7.3.1)$$

where N is the number of turns of the solenoid.

When the axis of the solenoid is rotated from angle α_1 to angle α_2, the magnetic flux penetrating the solenoid changes by

$$\Delta\Phi = \Phi_2 - \Phi_1 = BS\cos(\alpha_1 + \alpha_2) - BS\cos\alpha_1, \qquad (7.3.2)$$

where S is the cross section of the solenoid,
B is magnetic field induction.

According to the condition of the problem, the axis of the coil in the initial position coincided with the direction of the field ($\alpha_1 = 0$), and the angle of rotation is $\alpha_2 = 180^0$. The change in magnetic flux in this case will be equal to

$$\Delta\Phi = -2BS. \qquad (7.3.3)$$

The cross-sectional area of the solenoid is equal to

$$S = \frac{\pi d^2}{4}, \qquad (7.3.4)$$

where d is the diameter of the solenoid.

In this case, the EMF is equal to

$$\varepsilon_i = \frac{\pi d^2 NB}{2\Delta t}. \qquad (7.3.5)$$

For the given numerical values we get: $\varepsilon_i = 2.4\,V$.

A change in the magnetic flux in the solenoid leads to the appearance of a charge

$$Q = \frac{\Delta\Phi}{R}. \qquad (7.3.6)$$

The coil resistance of the solenoid is equal to

$$R = \frac{N\pi\rho d}{\sigma}, \qquad (7.3.7)$$

where σ is the cross-sectional area of copper wire.

Finally, we get the expression for the charge

$$Q = \frac{\sigma d B}{2\rho}. \tag{7.3.8}$$

Numerically: $Q = 1.4C$.

Answer. The average value of the EMF is $\varepsilon_i = 2.4\,V$. The induction charge is $Q = 1.4C$.

Problems

Problem A

Problem description. Between the poles of the electromagnet is placed a coil connected to a ballistic galvanometer. The axis of the coil is parallel to the magnetic induction lines. The coil has a resistance of $R_1 = 4\ \Omega$ and a cross-sectional area of $S = 2\ cm^2$. The number of turns of the wire that is wound on the coil is $N = 15$. The resistance of the galvanometer is $R_2 = 46\ \Omega$. When the current in the electromagnet winding was turned off, a quantity of electricity leaked along the galvanometer circuit equal to $Q = 9 \times 10^{-5}\ C$. Calculate the magnetic induction generated by the electromagnet.

Answer. $B = 1.5\ T$.

Problem B

Problem description. At a distance of $a = 1\ m$ from the long straight wire with a current of $I = 1\ kA$, there is a ring with a radius of $r = 1\ cm$. The ring is located so that the flow penetrating it is maximum. Determine the amount of electricity Q that will flow through the ring when the current in the conductor is turned off. The ring resistance is $R = 10\ \Omega$.

Answer. $Q = 6.28 \times 10^{-5}\ C$.

Problem C

Problem description. The inductance of the coil is equal to $L = 2\ mH$. The current frequency of $\nu = 50\ Hz$, flowing through the coil, varies according to a sinusoidal law. Determine the average value of the self-induced EMF over a period of time during which the current in the coil varies from minimum to maximum. The amplitude value of the current is $I_0 = 10\ A$.

Answer. $E = 4\ V$.

Problem D

Problem description. The inductance of a solenoid with a length of $a = 1\ m$ wound in a single layer on a nonmagnetic frame is $L = 1.6\ mH$. The solenoid cross section is $S = 20\ cm^2$. Determine the number of turns on each centimeter of the length of the solenoid.
Answer. $n = 8\ cm^{-1}$.

Problem E

Problem description. The solenoid contains $N = 1000$ turns. The current in its winding is $I = 1\ A$. The magnetic flux Φ through the cross section of the solenoid is $1\times10^{-4}\ Wb$. Calculate the energy of the magnetic field.

Answer. $W = 5\times10^{-2}\ J$.

CHAPTER 8. MAXWELL'S EQUATIONS

8.1. Vortex Electric Field

The theoretical analysis carried out by Scottish scientists James Clerk Maxwell (1831 – 1879) showed that a time-varying magnetic field generates an electric field $\vec{E}_B$ whose circulation is

$$\oint_L \vec{E}_B d\vec{l} = \oint_L \vec{E}_{Bl} dl = -\frac{d\Phi}{dt} , \tag{8.1}$$

where $\vec{E}_{Bl}$ is the projection of the vector $\vec{E}_B$ on the direction $d\vec{l}$. Substituting expression

$$\Phi = \int_S \vec{B} d\vec{S} \tag{8.2}$$

into formula for the electric field circulation, we obtain

$$\oint_L \vec{E}_B d\vec{l} = -\frac{d}{dt}\int_S \vec{B} d\vec{S} . \tag{8.3}$$

If the surface and the contour are fixed, then the operations of differentiation and integration can be interchanged. Consequently

$$\oint_L \vec{E}_B d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} d\vec{S} , \tag{8.4}$$

where the symbol of a partial derivative emphasizes the fact that the integral $\int_S \vec{B} d\vec{S}$ is a function only of time. It is known that the circulation of the vector of the intensity of the electrostatic field (we denote it $\vec{E}_Q$) along any closed circuit is equal to zero

$$\oint_L \vec{E}_Q d\vec{l} = \oint_L E_{Ql} dl = 0 . \tag{8.5}$$

Between these fields ($\vec{E}_B$ and $\vec{E}_Q$) there is a fundamental difference: the circulation of vector $\vec{E}_B$, i.e. $\vec{E}_B$ in contrast to the circulation of vector $\vec{E}_Q$ is not zero. Consequently, the electric field generated by the magnetic field, like the magnetic field itself, is vortex.

8.2. Displacement Current

Maxwell argued that if every variable magnetic field generates a vortex electric field in the surrounding space, then there must be another phenomenon: any change in the electric field should cause the appearance of a vortex magnetic field in the surrounding space. To establish the quantitative relationships between the changing electric field and the magnetic field caused by it, Maxwell introduced the so-called *displacement current* into consideration.

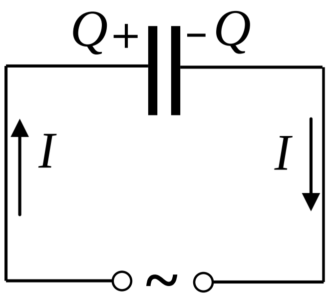


Figure 8.1. AC circuit containing a capacitor.

Consider an AC circuit containing a capacitor (Figure 8.1). Between the plates of the charging and discharging capacitor there is an alternating electric field, therefore, displacement currents flow through the capacitor, and in those areas where there are no conductors. Let us find a quantitative relationship between the varying electric and the magnetic fields that are caused by them. The alternating electric field in the capacitor at each instant of time produces such a magnetic field as if there was conduction current between the capacitor plates equal to the current in the lead wires. Then it can be argued that the conduction (I) and displacement (I_d) currents are equal: $I_d = I$. The conduction current near the capacitor plates is

$$I = \frac{dQ}{dt} = \frac{d}{dt}\int_S \sigma\, dS = \int_S \frac{\partial \sigma}{\partial t} dS = \int_S \frac{\partial D}{\partial t} dS \tag{8.6}$$

(the surface charge density σ on the plates is equal to the electric displacement D in the capacitor). The integrand can be viewed as a special case of the scalar product $\left(\frac{\partial \vec{D}}{\partial t}\right) d\vec{S}$ when $\frac{\partial \vec{D}}{\partial t}$ and $d\vec{S}$ are mutually parallel. Therefore, for the general case, we can write

$$I = \int_S \frac{\partial D}{\partial t} dS. \tag{8.7}$$

Comparing this expression with

$$I = I_d = \int_S j_d\, d\vec{S}, \tag{8.8}$$

we have

$$j_d = \frac{\partial \vec{D}}{\partial t}. \qquad (8.9)$$

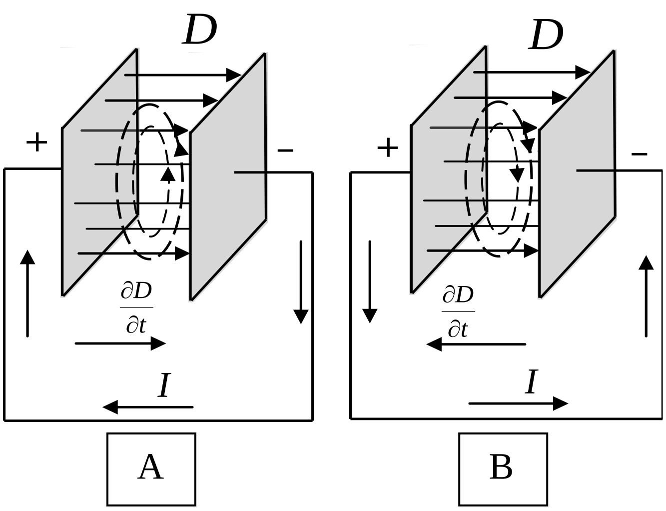


Figure 8.2. Displacement current.

The expression for j_d is called the *displacement current density*.

Let us consider the direction of the density vectors of the conduction and displacement currents $\vec{j}$ and $\vec{j}_{\text{d}}$. Charging the capacitor (Figure 8.2) through the conductor connecting the electrodes causes the current passing from the right side to the left. The field in the capacitor is amplified, the vector $\vec{D}$ grows with time, hence $\frac{\partial \vec{D}}{\partial t} > 0$, i.e. vector $\frac{\partial \vec{D}}{\partial t}$ is directed in the same direction as $\vec{D}$. It can be seen from the figure that the directions of the vectors $\frac{\partial \vec{D}}{\partial t}$ and $\vec{j}$ coincide. When the capacitor is discharged (Figure 8.2) the current passes from the left to the right; the field in the capacitor is weakened, the vector $\vec{D}$ decreases with time; consequently, $\frac{\partial \vec{D}}{\partial t} < 0$, that is, the vector $\frac{\partial \vec{D}}{\partial t}$ is directed opposite to the vector $\vec{D}$.

However, the directions of vectors $\frac{\partial \vec{D}}{\partial t}$ and $\vec{j}$ coincide. From the analyzed examples it follows that the direction of the vector $\vec{j}$, and hence of the vector $\vec{j}_d$, coincides with the direction of the vector $\frac{\partial \vec{D}}{\partial t}$.

Thus, the displacement current (in a vacuum or substance) creates a magnetic field in the surrounding space (lines of induction of the magnetic fields of displacement currents when charging and discharging the capacitor are shown in Figure 8.2 by a dashed lines).

The displacement current in dielectrics consists of two parts. Since

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}, \tag{8.10}$$

where $\vec{E}$ is the strength of the electrostatic field,

$\vec{P}$ is the polarization.

The displacement current density is

$$\vec{j}_{см} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t}, \tag{8.11}$$

where $\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ is the displacement current density in vacuum,

$\frac{\partial \vec{P}}{\partial t}$ is the polarization current density.

The current due to the orderly motion of electrical charges in the dielectric (the displacement of charges in non polar molecules or the rotation of dipoles in polar molecules) is called the *polarization current*. The excitation of the magnetic field by the polarization currents is correct, since the polarization currents are by their nature no different from the conduction currents. However, the fact that the other part of the displacement current density ($\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$), which is not connected with charge motion, but is caused only by the change of the electric field in time, also excites the magnetic field, is a fundamentally new statement. Even in vacuum, any change in the time of the electric field leads to the appearance of a magnetic field in the surrounding space. It should be noted that the name "displacement current" is conditional, and more precisely historically developed, since the bias current in its essence is a changing electric field with time. The displacement current therefore exists not only in vacuum or dielectrics, but also inside conductors, over which an alternating current passes. However, in this case it is negligibly small in comparison with the conductivity current.

Maxwell introduced the concept of a total current equal to the sum of the conduction currents (and also the convection currents) and the displacement current. The density of the total current is

$$j_t = j + \frac{\partial \vec{D}}{\partial t}. \tag{8.12}$$

Introducing the concepts of the displacement current and the total current, Maxwell explored the closed nature of AC circuits in a new way. The total current in AC circuits is always closed, that is, only the conduction current breaks off at the ends of the conductor, and in the dielectric (vacuum) between the ends of the conductor there is a displacement current that closes the conduction current. Maxwell generalized the theorem of the vector $\vec{H}$ circulation by introducing into the right-hand side of formula the total current

$$I_t = \int_S j_t d\vec{S} \tag{8.13}$$

through the surface S stretched over the closed contour L. Then the generalized theorem of the vector $\vec{H}$ circulation is written in the form

$$\oint_L \vec{H}d\vec{l} = \int_S \left(\vec{j} + \frac{\partial \vec{D}}{\partial t} \right) d\vec{S} . \tag{8.14}$$

8.3. Maxwell's Equations for the Electromagnetic Field

The concept of displacement current allowed Maxwell to complete the macroscopic theory of the electromagnetic field. This theory explained not only electrical and magnetic phenomena, but also predicted new phenomena, the existence of which was subsequently confirmed.

Maxwell's theory is based on the four equations considered above.

1. The electric field can be both potential ($\vec{E}_Q$) and vortex ($\vec{E}_B$), so the intensity of the total field is

$$\vec{E} = \vec{E}_Q + \vec{E}_B. \tag{8.15}$$

Since the circulation of the vector $\vec{E}_Q$ is zero, the circulation of the vector of the intensity $\vec{E}$ of the total field is

$$\oint_L \vec{E}d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} dS . \tag{8.16}$$

This equation shows that the sources of the electric field can be not only electric charges, but also changing magnetic fields in time.

2. The generalized theorem of the vector $\vec{H}$ circulation has the form

$$\oint_L \vec{H}d\vec{l} = \int_S \left(\vec{j} + \frac{\partial \vec{D}}{\partial t} \right) d\vec{S} . \tag{8.17}$$

This equation shows that magnetic fields can be excited either by moving charges (electric currents) or alternating electric fields.

3. The Gauss's law for vector $\vec{D}$ has the form

$$\oint_S \vec{D}d\vec{S} = Q . \tag{8.18}$$

If the charge is distributed within a closed surface continuously with a volume density ρ, then formula

$$\oint_S \vec{D}d\vec{S} = Q \qquad (8.19)$$

will be written as

$$\oint_S \vec{D}d\vec{S} = \int_V \rho dV . \qquad (8.20)$$

4. The Gauss's law for vector $\vec{B}$ has the form

$$\oint_S \vec{B}d\vec{S} = 0 . \qquad (8.21)$$

Thus, the complete system of Maxwell's equations in integral form includes 4 equations:

$$\oint_L \vec{E}d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} d\vec{S} , \qquad (8.22)$$

$$\oint_S \vec{D}d\vec{S} = \int_V \rho dV ; \qquad (8.23)$$

$$\oint_L \vec{H}d\vec{l} = \int_S \left(\vec{j} + \frac{\partial D}{\partial t} \right) d\vec{S} , \qquad (8.24)$$

$$\oint_S \vec{B}d\vec{S} = 0 . \qquad (8.25)$$

The quantities appearing in the *Maxwell's equations* are not independent and there is a following relationship between them (for isotropic non ferroelectric and non-ferromagnetic substances)

$$\vec{D} = \varepsilon_0 \varepsilon \vec{E} ,$$
$$\vec{B} = \mu_0 \mu \vec{H} ,$$
$$\vec{j} = \gamma \vec{E} , \qquad (8.26)$$

where ε_0 and μ_0 are, respectively, electric and magnetic constants,
ε and μ are, respectively, dielectric and magnetic permeability,
γ is the specific conductivity of the substance.

It follows from Maxwell's equations that an electric field can be created either by electric charges or by time-varying magnetic fields. Magnetic fields can be generated either by moving electric charges (electric currents) or by alternating electric fields. Maxwell's equations are not symmetric with respect to the electric and magnetic fields. This is due to the fact that in nature there are electric charges, but there are no magnetic charges.

For stationary fields ($E = const$ and $B = const$) Maxwell's equations assume the form

$$\oint_L \vec{E}d\vec{l} = 0,$$
$$\oint_S \vec{D}d\vec{S} = Q,$$
$$\oint_L \vec{H}d\vec{l} = I,$$
$$\oint_S \vec{B}d\vec{S} = 0, \qquad (8.27)$$

that is, the electric fields in this case are generated only by electric charges, and the magnetic fields are generated only by conduction currents. In this case, the electric and magnetic fields are independent of each other. This fact allows us to study separately the constant electric and magnetic fields.

Using the *Stokes'* and Gauss's *theorems* known from vector analysis, we can represent the complete system of Maxwell's equations in differential form (characterizing the field at each point of space)

$$rot\vec{E} = -\frac{\partial \vec{B}}{\partial t},$$
$$div\,\vec{D} = \rho,$$
$$rot\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t},$$
$$div\vec{B} = 0. \qquad (8.28)$$

Stokes' theorem is named after Anglo-Irish physicist Sir George Gabriel Stokes, 1st Baronet (1819 – 1903). If the charges and currents are continuously distributed in space, then both forms of the Maxwell's equations (integral and differential) are equivalent. However, when the area under investigation contains discontinuity surfaces, i.e. such surfaces, on which the properties of the medium or fields change in an abrupt manner, then the integral form of the equations is more general.

Maxwell's equations in differentiated form assume that all the magnets in space and time change continuously. In order to achieve the mathematical equivalence of both forms of Maxwell's equations, the differential form is supplemented by the boundary conditions that the electromagnetic field at the interface between the two media must satisfy. The integral form of the Maxwell's equations contains these conditions

$$D_{1n} = D_{2n},$$
$$E_{1\tau} = E_{2\tau},$$
$$B_{1n} = B_{2n},$$
$$H_{1\tau} = H_{2\tau} \qquad (8.29)$$

(the first and last equations correspond to the cases when there are neither free charges nor conductivity currents at the interface).

Maxwell's equations are the most general equations for electric and magnetic fields in stationary media. They play the same role in the doctrine of electromagnetism as Newton's laws in mechanics. English physicist Sir Isaak Newton (1642 – 1726) was one of the founders of classical physics. It follows from Maxwell's equations that an alternating magnetic field is always connected with the electric field generated by it, and an alternating electric field is always associated with the magnetic field generated by it, that is, the electric and magnetic fields are inextricably linked with each other. They form an electromagnetic field. Maxwell's theory, being a generalization of the basic laws of electrical and magnetic phenomena, was able to explain not only the already known experimental facts, which is also an important consequence of it, but also predicted new phenomena.

One of the important conclusions of Maxwell's theory was the existence of a magnetic field of displacement currents, which allowed Maxwell to predict the existence of electromagnetic waves. Variable electromagnetic fields, which propagate in a space with a finite velocity, are called *electromagnetic waves*. The propagation velocity of a free electromagnetic field (not connected with charges and currents) in a vacuum is equal to the speed of light. This conclusion and theoretical study of the properties of electromagnetic waves led Maxwell to create the electromagnetic theory of light, according to which light is also electromagnetic waves.

8.4. Electromagnetic Waves

The existence of electromagnetic waves is one of the most important consequences of Maxwell's equations. Consider a homogeneous and isotropic medium far from charges and currents that create an electromagnetic field. It follows from Maxwell's equations that in such a medium, the intensity vectors $\vec{E}$ and $\vec{H}$ of the alternating electro magnetic field satisfy the wave equation

$$\Delta\vec{E} = \frac{1}{\upsilon^2}\frac{\partial^2\vec{E}}{\partial t^2}, \tag{8.30}$$

$$\Delta\vec{H} = \frac{1}{\upsilon^2}\frac{\partial^2\vec{H}}{\partial t^2}, \tag{8.31}$$

where $\Delta = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}$ is the *Laplace operator* (this operator is named after French physicist Pierre-Simon, marquis de Laplace (1749 – 1827));

υ is the phase velocity.

Any function that satisfies the above equations describes a certain wave. Consequently, electro magnetic fields can exist in the form of electromagnetic waves. The phase velocity of electromagnetic waves is determined by the formula

$$\upsilon = \frac{1}{\sqrt{\varepsilon_0\mu_0}}\frac{1}{\sqrt{\varepsilon\mu}} = \frac{c}{\sqrt{\varepsilon\mu}} \tag{8.32}$$

where $c = 1/\sqrt{\varepsilon_0 \mu_0}$;

ε_0 and μ_0, respectively, electric and magnetic constants;

ε and μ are, respectively, the electrical and magnetic permeabilities of the medium.

In a vacuum ($\varepsilon = 1$, and $\mu = 1$), the propagation velocity of electromagnetic waves coincides with the velocity c. Since $\varepsilon\mu > 1$, the propagation velocity of electromagnetic waves in matter is always less than in a vacuum.

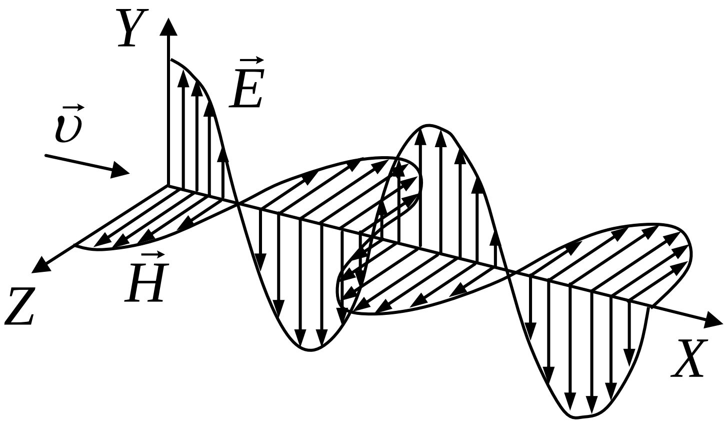


Figure 8.3. Electromagnetic wave.

Vectors $\vec{E}$ and $\vec{H}$ of the electric and magnetic field intensity of the electromagnetic wave are mutually perpendicular (Figure 8.3) and lie in a plane perpendicular to the velocity vector $\vec{\upsilon}$ of the wave propagation. Vectors $\vec{E}$, $\vec{H}$ and $\vec{\upsilon}$ form a right-screw system. It follows also from the Maxwell's equations that in the electromagnetic wave vectors $\vec{E}$ and $\vec{H}$ always oscillate in the same phases. Consequently, the modules of vectors $\vec{E}$ and $\vec{H}$ simultaneously reach a maximum and simultaneously turn to zero. The following equations are a consequence of the wave equations

$$\frac{\partial^2 E_Y}{\partial X^2} = \frac{1}{\upsilon^2}\frac{\partial^2 E_Y}{\partial t^2},$$

$$\frac{\partial^2 H_Z}{\partial X^2} = \frac{1}{\upsilon^2}\frac{\partial^2 H_Z}{\partial t^2}. \qquad (8.33)$$

Plane monochromatic electromagnetic waves (electromagnetic waves of one strictly defined frequency) are described by equations

$$E_Y = E_0 \cos(\omega t - kX + \varphi);$$

$$H_Z = H_0 \cos(\omega t - kX + \varphi), \qquad (8.34)$$

where E_0 and H_0 are, respectively, the amplitude of the intensity of the electric and magnetic fields of the wave;

ω is the circular frequency of the wave;
$k = \omega / \upsilon$ is the wave number;
φ are the initial phases of the oscillations at points with the coordinate $X = 0$.

8.5. Electromagnetic Waves Energy

The possibility of detecting electromagnetic waves indicates that they transfer energy. The volume density ω of the energy of the electromagnetic wave consists of the volume densities ω_E and ω_M of the electric and magnetic fields

$$\omega = \omega_E + \omega_M = \varepsilon\varepsilon_0 E^2/2 + \mu\mu_0{}^2 H^2/2 . \qquad (8.35)$$

The energy density of the electric and magnetic fields at each moment of time is the same, that is, $\omega_E = \omega_M$.

Therefore

$$\omega = 2\omega_E = \varepsilon\varepsilon_0 E^2 = \sqrt{\varepsilon_0\mu_0}\sqrt{\varepsilon\mu} EH . \qquad (8.36)$$

Multiplying the energy density ω by the velocity υ of propagation of the wave in the medium, we obtain the energy flux density modulus

$$S = \omega\upsilon = EH .$$

Since the vectors $\vec{E}$ and $\vec{H}$ are mutually perpendicular and form a right-handed system the direction of the vector $\left[\vec{E}\vec{H}\right]$ coincides with the energy transfer direction, and the modulus of this vector is EH. The flux density vector of electromagnetic energy is called the *Poynting vector*

$$\vec{S} = \left[\vec{E}\vec{H}\right] . \qquad (8.37)$$

Pointing vector is named after English physicist John Henry Poynting (1852 – 1914). The vector $\vec{S}$ is directed towards the electromagnetic wave propagation, and its modulus is equal to the energy transferred by the electromagnetic wave per unit time through a single area perpendicular to the propagation direction of the wave.

Electromagnetic waves, according to Maxwell's theory, should exert pressure on bodies if these bodies absorb or reflect electromagnetic waves. The pressure of electromagnetic waves is explained by the fact that under the action of the electric field of the wave, the charged particles of matter begin to move in an orderly manner. The reason for this motion is the action of the Lorentz force on particles from the side of the magnetic field of the wave. However, the magnitude of this pressure is negligible. It can be estimated that with an average solar radiation power coming to Earth, the pressure for an absolutely absorbing surface is about 5 μPa.

The existence of pressure of electromagnetic waves leads to the conclusion that the mechanical impulse can be associated with a electromagnetic field. The impulse of the electromagnetic field is $P = W / c$, where W is the energy of the electromagnetic field. Expressing the momentum as $P = mc$ (the field in the vacuum propagates at a speed of c), we obtain $P = mc = W / c$, whence $W = mc^2$. This relationship between the mass and energy of a free electromagnetic field is a universal law of nature.

Test questions

1. What physical field is generated by the time-varying magnetic field?
2. Write the integral expression for the electric field intensity generated by an alternating magnetic field.
3. Calculate the circulation of the electrostatic field intensity vector.
4. What is the difference between the intensity of the electrostatic field and the intensity of the electric field generated by an alternating magnetic field?
5. What is the reason for introducing the displacement current into consideration?
6. Write down the differential formula for the displacement current density.
7. What is the relationship between the displacement current density and the polarization current density?
8. Describe the concept of total current proposed by Maxwell.
9. What is the essence of the generalized theorem on the circulation of the magnetic field strength vector?
10. Is it possible to say that the electric field is always potential?
11. Is it possible to say that the source of the electric field can be only changing magnetic field in time?
12. What physical field can generate alternating electric fields?
13. Write down the Gauss’s theorem for the displacement vector.
14. Calculate the flux of the displacement vector in a closed circuit.
15. Write Maxwell's equations in integral form.
16. Write Maxwell's equations in differential form.
17. Are the quantities appearing in Maxwell's equations independent?
18. Explain the reason for the asymmetry of Maxwell's equations.
19. Write down the boundary conditions for Maxwell's equations.
20. What is the significance of a Poynting vector?

Problem-solving examples

Problem 8.1

Problem description. The oscillatory circuit consists of a solenoid and a 12.5 nF capacitor. The current in the solenoid changes by 1.5 A in 0.4 s. In this case, an EMF of 0.3 mV is induced in the solenoid. Calculate the radio wave length radiated by the oscillating circuit.

Known quantities: $C = 12.5\,nF$, $\Delta I = 1.5\,A$, $\Delta t = 0.4\,s$, $\varepsilon = 0.3\,mV$.

Quantities to be calculated: λ .

Problem solution. The wavelength emitted by the generator

$$\lambda = c2\pi\sqrt{LC}\,, \qquad (8.1.1)$$

where c is the speed of propagation of electromagnetic waves in a vacuum;
L is the inductance of the solenoid;
C is the capacitance of the capacitor.
EMF of self-induction arising in the solenoid is equal to

$$\varepsilon = -L\frac{\Delta I}{\Delta t}\,, \qquad (8.1.2)$$

where ΔI is the change in current over time Δt .
Hence the inductance of the solenoid is equal to

$$L = \varepsilon\left|\frac{\Delta t}{\Delta I}\right|. \qquad (8.1.3)$$

Then for the wavelength we get

$$\lambda = c2\pi\sqrt{C\varepsilon\left|\frac{\Delta t}{\Delta I}\right|} = 2450\ m\,. \qquad (8.1.4)$$

Answer. The length of the radio waves that the generator emits is equal to $\lambda = 1884\,m$.

Problem 8.2

Problem description. The area of the capacitor plates is equal to 70 cm^2. The initial distance between the plates of the capacitor is 0.4 cm. The charge on each capacitor plate is 10^{-9} C. Condenser plates began to move apart with speed 3 mm/min. Determine the density of the displacement current in the capacitor 240 s after the beginning of the movement of the plates. Consider two cases: 1) plate charges remain constant; 2) the potential difference between the plates remains constant.

Known quantities: $S = 70\,cm^2$, $d_0 = 0.4\,cm$, $q = 10^{-9}\,C$, $\upsilon = 3\,mm/\min$, $t = 240\,s$.

Quantities to be calculated: j_d .

Problem solution. Since the conduction current lines pass into the bias current lines, the bias current density is equal to

$$j_d = \frac{1}{S}\frac{dq}{dt} = \frac{d}{dt}\left(\frac{CU}{S}\right) = \frac{U}{S}\frac{d}{dt}\left(\frac{\varepsilon\varepsilon_0 S}{d_0 + \upsilon t}\right) = \frac{U\varepsilon\varepsilon_0\upsilon}{(d_0 + \upsilon t)^2} \quad (8.2.1)$$

where S is the area of the capacitor plates;
q is an electric charge located on a plate;
C is the capacitance of the capacitor;
U is the potential difference between the capacitor plates;
ε is the relative dielectric constant of the substance between the capacitor plates;
ε_0 is an electrical constant;
υ is the plate speed;
t is the time for which the plates will move apart.

For the first case we get

$$q = const \Rightarrow \frac{dq}{dt} = 0 \Rightarrow j_d = 0. \quad (8.2.2)$$

According to the condition of the problem, in the second case $U = const$, then

$$U = \frac{q_1}{C} = \frac{qd_0}{\varepsilon\varepsilon_0 S}. \quad (8.2.3)$$

For the displacement current density, the ratio is

$$j_d = \frac{qd_0\upsilon}{S(d_0 + \upsilon t)^2} = 1.1\times10^{-10}\ A/m^2. \quad (8.2.4)$$

Answer. displacement current density is: 1) $j_d = 0$; 2) $j_d = 1.1\times10^{-10}\ A/m^2$.

Problem 8.3

Problem description. A plane electromagnetic wave falls normally on the surface of a plane-parallel layer. The layer has a thickness of L and is made of a non-magnetic material. The dielectric constant of this material falls exponentially from a value of ε_1 on the front surface to ε_2 on the back. Determine the formula for the propagation time of this phase of the wave through this layer.

Known quantities: L, ε_1, ε_2.

Quantities to be calculated: t.

Problem solution. Instantaneous speed is

$$\upsilon = \frac{dx}{dt}. \tag{8.3.1}$$

Dielectric constant changes exponentially

$$\varepsilon = \varepsilon_1 \exp(-kx). \tag{8.3.2}$$

Calculate the derivative of the dielectric constant per coordinate

$$\frac{d\varepsilon}{dx} = -k\varepsilon_1 \exp(-kx) = -k\varepsilon, \tag{8.3.3}$$

then

$$dx = -\frac{1}{k\varepsilon} d\varepsilon. \tag{8.3.4}$$

The speed of propagation of electromagnetic waves in the medium is

$$\upsilon = \frac{c}{\sqrt{\varepsilon}} \tag{8.3.5}$$

or

$$\upsilon = \frac{dx}{dt} = \frac{(-d\varepsilon/(k\varepsilon))}{dt} = \frac{c}{\sqrt{\varepsilon}} \tag{8.3.6}$$

Then

$$dt = -\frac{d\varepsilon}{\sqrt{\varepsilon}kc}. \tag{8.3.7}$$

We integrate in the range from ε_1 to ε_2:

$$t = -\int_{\varepsilon_1}^{\varepsilon_2} \frac{d\varepsilon}{\sqrt{\varepsilon}kc} = \frac{2}{kc}\left(\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}\right). \tag{8.3.8}$$

Value k we find from the condition

$$\varepsilon_2 = \varepsilon_1 \exp(-kL), \tag{8.3.9}$$

Consequently

$$k = \frac{1}{L}\ln\left(\frac{\varepsilon_1}{\varepsilon_2}\right). \tag{8.3.10}$$

For time t, we can write

$$t=\frac{2L}{c\ln\left(\varepsilon_1/\varepsilon_2\right)}\left(\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}\right). \qquad (8.3.11)$$

Answer. the propagation time of the wave phase through this layer is equal to $t=\frac{2L}{c\ln\left(\varepsilon_1/\varepsilon_2\right)}\left(\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}\right)$.

Problems

Problem A

Problem description. The rectangular ring is made of a material with a conductivity of $\lambda=6\times10^7\ (\Omega\times m)^{-1}$. The inner radius of the ring is $R_1=3\ cm$, the outer radius of the ring is $R_2=5\ cm$, and the height of the ring is $h=1\ cm$. The ring is in a non-stationary magnetic field, the induction vector of which is parallel to the axis of symmetry of the ring. The magnetic field varies according to law $B(t)=\left((kt,0\le r\le R)\cup(0,r>R)\right)$, where r is the distance from the axis of the ring, and $R>R_2$. Determine the strength of the current that flows through the ring, if $k=0.1T/s$.

Answer. $I=24A$.

Problem B

Problem description. A plane-parallel diode is an evacuated vessel, in which the anode and cathode are located. As an anode and cathode, two plane-parallel plates can be imagined, the distance between them is d. Determine the potential distribution between the anode and cathode in such an instrument, assuming that electrons emitted by the cathode with a low initial velocity due to the phenomenon of thermionic emission, create a cloud around the cathode and only a fraction of the electrons moves toward the anode. The anode potential is U_0, and the cathode potential is zero. The distance between the anode and the cathode is small compared with the transverse dimensions of the plate.

Answer. $\varphi(x)=U_0(x/d)^{4/3}$.

Problem C

Problem description. Charged and disconnected from the source, a flat capacitor with round plates with a radius of R is punched by an electric spark along its axis. Considering the discharge as a quasi-stationary and neglecting edge effects, calculate the instantaneous value of the magnetic field strength H inside the capacitor as a function of the distance r from its axis if the current in the electric spark is I.

<u>Answer</u>. $H = I\left(1 - r^2 / R^2\right)/(2\pi r)$.

Problem D

<u>Problem description</u>. The plates of a flat capacitor are in the form of disks whose radius is $R = 10\ cm$. The space between the plates is filled with a uniform dielectric with a dielectric and magnetic permeability of ε and μ, respectively. The capacitor is connected to the AC circuit $I = I_0 \cos(\omega t)$. Neglecting the edge effects, calculate the ratio k of maximum magnetic energy to maximum electric energy. When calculating, it should be assumed that $\varepsilon = \mu = 1$, and the frequency of the current is $\nu = \omega/(2\pi) = 100\ Hz$.

<u>Answer</u>. $k = 5 \times 10^{-15}$.

Problem E

<u>Problem description</u>. Determine the displacement current density j_d in a flat capacitor, the plates of which are moved apart at a speed of υ, while remaining parallel to each other. Consider the cases: 1) the charges on the plates of the capacitor remain constant, 2) the potential difference U between the plates remains constant. The distance d between the plates of the capacitor remains all the time small compared with the linear dimensions of the plates.

<u>Answer</u>. 1) $j_d = \partial D / \partial t = 0$, 2) $j_d = -\varepsilon_0 \upsilon U / d^2$.

CHAPTER 9. GEOMETRICAL OPTICS. PHOTOMETRY

9.1. Laws of Geometrical Optics

The following basic laws of optics are known: the law of rectilinear propagation of light in an optically homogeneous medium; the law of independence of light beams (valid only in linear optics); law of light reflection; law of refraction of light.

The law of rectilinear propagation of light: light rays propagate in straight-line paths as they travel in a homogeneous medium. The proof of this law is the presence of a shadow with sharp boundaries from opaque objects when illuminated by point sources of light (sources, whose dimensions are much smaller than the illuminated object). Careful experiments have shown, however, that this law is violated if light passes through very small holes, and the deviation from the straightness of the propagation is the greater, the smaller the aperture.

The law of independence of light rays: the effect produced by a separate ray does not depend on the presence of other rays. By breaking the light flux into separate light rays (for example, using diaphragms), it can be shown that the action of the extracted light rays is independent. If light falls on the interface between two media (two transparent substances), then the incident ray 1 (Figure 9.1) is divided into two – the reflected ray II and the refracted ray III. Directions of these rays are given by the laws of reflection and refraction.

The *law of reflection*: the reflected ray lies in the same plane as the incident ray and the perpendicular drawn to the boundary between the two media at the point of incidence; the angle of reflection i_1' is equal to the angle of incidence i_1:

$$i_1' = i_1. \tag{9.1}$$

Snell’s law describes the resulting deflection of the light ray: the ray incident, the ray refracted and perpendicular drawn to the interface at the point of incidence lie in the same plane; the ratio of the sine of the angle of incidence to the sine of the angle of refraction is a constant value for these media:

$$\frac{\sin i_1}{\sin i_2} = n_{21}, \tag{9.2}$$

where n_{21} is the relative index of refraction of the second medium relative to the first one.

This result was discovered experimentally in 1621 by Dutch scientist Willebrord Snell (1580–1626) and is known as *Snell’s law* or the law of refraction [3]. The indices in the notation of the angles i_1, i_1', i_2 indicate in which medium (first or second) the ray passes. The relative index of refraction of two media is equal to the ratio of their absolute refractive indices

$$n_{21} = \frac{n_2}{n_1}. \tag{9.3}$$

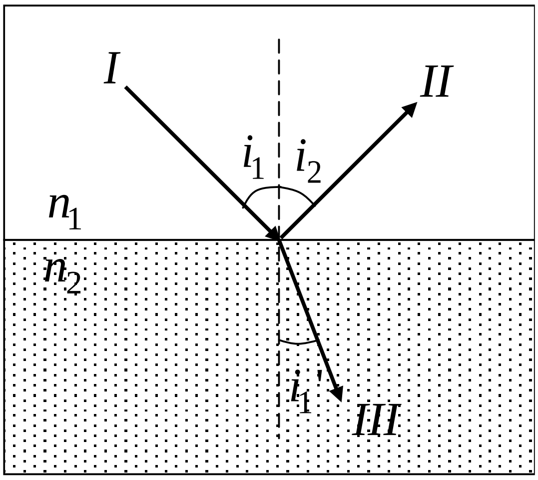


Figure 9.1. Reflection and refraction of light.

The *absolute index of refraction* n is a quantity equal to the ratio of the speed c of electromagnetic waves in a vacuum to their phase velocity υ in a medium

$$n = \frac{c}{\upsilon}. \tag{9.4}$$

On the other hand, the absolute refractive index is equal to

$$n = \sqrt{\varepsilon\mu}, \tag{9.5}$$

where ε and μ are, respectively, the electrical and magnetic permeabilities of the medium.

The law of refraction can be written in the form

$$n_1 \sin i_1 = n_2 \sin i_2 . \tag{9.6}$$

The symmetry of the resulting expression implies the reversibility of light rays. If one turn ray III, causing it to fall to the interface at an angle of i_2, then the refracted ray in the first medium will propagate at an angle of i_1, i.e. pass in the opposite direction along the ray I. If the light propagates from a medium with a high refractive index n_1 (optically denser) to a medium with a smaller refractive index n_2 (optically less dense) $(n_1 > n_2)$, for example glass to water, then, according to the refraction law,

$$\frac{\sin i_2}{\sin i_1} = \frac{n_1}{n_2} > 1 \tag{9.7}$$

and the refracted ray moves away from normal. In this case, the refraction angle i_2 is larger than the angle of incidence i_1. As the angle of incidence increases, the refraction angle increases until, at a certain angle of incidence ($i_1 = i_{la}$), the angle of refraction is $\pi/2$. The angle i_{la} is called the <u>*critical angle*</u>. All incident light is completely reflected at the angles of incidence $i_1 > i_{la}$.

For the case when $i_1 = i_{la}$ the intensity of the refracted ray vanishes and the intensity of the reflected ray is equal to the intensity of the incident ray. Thus, for angles of incidence in the range from i_{la} to $\pi/2$, the ray is not refracted, but completely reflected in the first medium, and the intensities of the incident and reflected rays are the same. This phenomenon is called <u>*total internal reflection*</u>.

The critical angle i_{la} is determined from the refraction law upon substitution $i_2 = \pi/2$ into it. Then

$$\sin i_{la} = \frac{n_2}{n_1} = n_{21}. \quad (9.8)$$

The phenomenon of total internal reflection occurs only when the light propagate in the direction of the optically less dense medium. The phenomenon of total internal reflection is used in prisms of total reflection. The glass refracting index is $n=1.5$, so the critical angle for the glass-air interface is $i_{la} = \arcsin\left(\frac{1}{1.5}\right) = 42^0$.

Therefore, when the light falls on the glass-air boundary, if the condition $i > 42^0$ is satisfied, a complete internal reflection will always take place. Figures 9.2, $A-C$ present the prisms of total reflection, allowing: A) to turn the ray by 90 °; B) rotate the image; C) wrap the rays.

Such prisms are used in optical devices (for example, in binoculars, periscopes), as well as in refractometers that allow one to determine the refractive

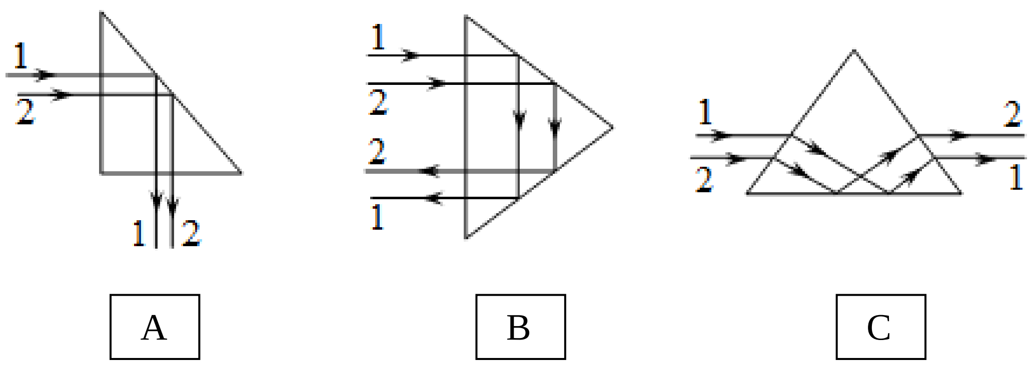


Figure 9.2. Prisms of total reflection.

indices of bodies. Using the law of refraction and measuring angle i_{la}, we can determine the relative refractive index of two media, and also the absolute refractive index of one of the media, if the refractive index of the second medium is known.

The phenomenon of total internal reflection is also used in optical fibres (light guides), which are thin, arbitrarily bent filaments (fibres) made of optically transparent material. In fibre parts, a glass fibre is used, the light-conducting core of which is surrounded by a glass-shell of another glass with a smaller refractive index. The light incident on the end of the light guide at angles greater than the critical angle is completely reflected on the interface between the core and the shell and propagates only along the light-conducting core. Thus, with the help of light guide one can curl the light beam path in any way. The diameter of the light guide cores varies in the range from several micrometers to several millimetres. Light guides are used in electron-beam tubes, in electronic computers, for encoding information, in medicine, as well as for integrated optics.

9.2. Thin Lenses

The optics section, in which the laws of propagation of light are considered on the basis of the concept of light rays, is called *geometrical optics*. Lines normal to the wave surfaces along which the light energy flux propagates are called *light rays*. Geometrical optics, while remaining an approximate method of constructing images in optical systems, makes it possible to study the main phenomena associated with the light propagation through them, and is therefore the basis of the theory of optical instruments.

Lenses are transparent bodies, bounded by two surfaces (one of them is usually spherical, sometimes cylindrical, and the second – spherical or flat). These surfaces can refract light rays and form optical images of objects. Glass, quartz, crystals, plastics are used to make lenses. According to the external form (Figure 9.3), the lenses are divided into: 1) biconvex; 2) plano-convex; 3) biconcave; 4) plano-concave; 5) positive meniscus; 6) negative meniscus.

Lenses are divided into collecting lenses and scattering lenses according to their optical properties. The thickness (the distance between the confining surfaces) of a thin lens is much smaller than the radii of the surfaces that bound the lens. A straight line passing through the centres of curvature of lens surfaces is called the

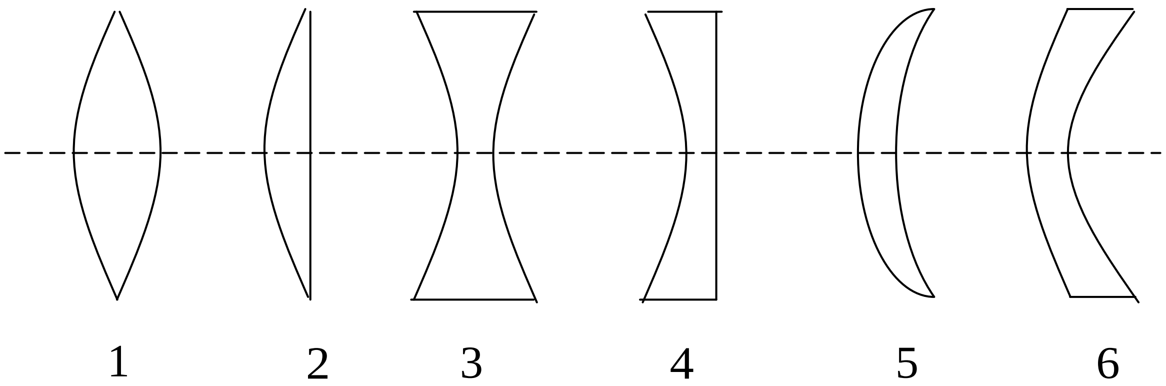


Figure 9.3. Types of lenses.

main optical axis. For any lens there is a point called the *optical centre of the lens*. The optical centre is on the main optical axis. The rays passing through the optical centre are not refracted. For simplicity, the optical centre O of the lens will be considered to coincide with the geometric centre of the middle part of the lens (this is true only for biconvex and biconcave lenses with the same radii of curvature of both surfaces). For plane-convex and flat-concave lenses, the optical centre O lies at the intersection of the main optical axis with a spherical surface.

The ratio connecting the radii of the curvature R_1 and R_2 of the lens surfaces with the distances a and b from the lens to the object and its image is called the *thin lens formula*. To derive the formula of a thin lens, we use Fermat's principle. According to the *Fermat's principle*, the path taken between two points by a ray of light is the path propagated in the minimum time.

Let us consider two trajectories of a light ray (Figure 9.4): the straight line connecting points A and B (ray AOB), and a trajectory passing through the edge of the lens (ray ACB).

We use the condition that the travel time of light along these trajectories is equal. The light passes along the path of the AOB during the time

$$t_1 = \frac{a + N(e + d) + b}{c}, \tag{9.9}$$

where $N = \dfrac{n}{n_1}$ is the *relative index of refraction* (n and n_1 are, respectively, the absolute refractive indices of the lens and the surrounding medium). The light passes along the trajectory ACB in time

$$t_2 = \frac{\sqrt{(a+e)^2 + h^2} + \sqrt{(b+d)^2 + h^2}}{c}. \tag{9.10}$$

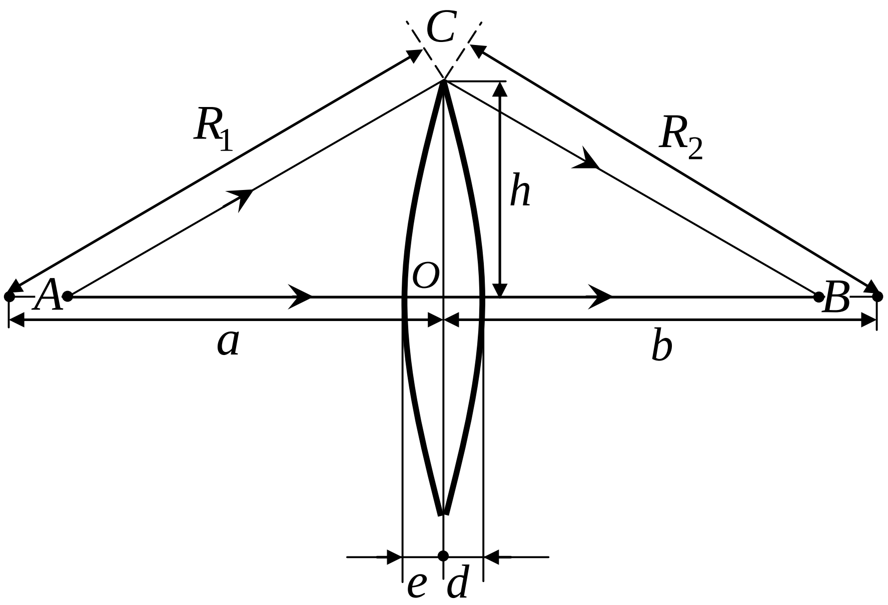


Figure 9.4. Trajectory of light rays in the lens.

Since $t_1 = t_2$, then

$$a + N(e + d) + b = \sqrt{(a+e)^2 + h^2} + \sqrt{(b+d)^2 + h^2}. \tag{9.11}$$

Let us consider paraxial rays, that is, rays forming the small angles with the optical axis. Only the paraxial rays form a stigmatic image, that is, all the rays of the paraxial beam emanating from point A intersect the optical axis at the same point B. Then

$$h << (a+e),\ h << (b+d) \tag{9.12}$$

and

$$\sqrt{(a+e)^2 + h^2} = (a+e)\sqrt{1+\frac{h^2}{(a+e)^2}} =$$

$$= (a+e)\left[1+\frac{1}{2}\left(\frac{h}{a+e}\right)^2\right] = a+e+\frac{h^2}{2(a+e)}. \tag{9.13}$$

Similarly

$$\sqrt{(b+d)^2 + h^2} = b+d+\frac{h^2}{2(b+d)}. \tag{9.14}$$

Taking these expressions into account, we obtain

$$(N-1)(e+d) = \frac{h^2}{2}\left(\frac{1}{a+e}+\frac{1}{b+d}\right). \tag{9.15}$$

For a thin lens, conditions $e << a$ and $d << b$ can be written, so the last expression can be represented in the form

$$(N-1)(e+d) = \frac{h^2}{2}\left(\frac{1}{a}+\frac{1}{b}\right). \tag{9.16}$$

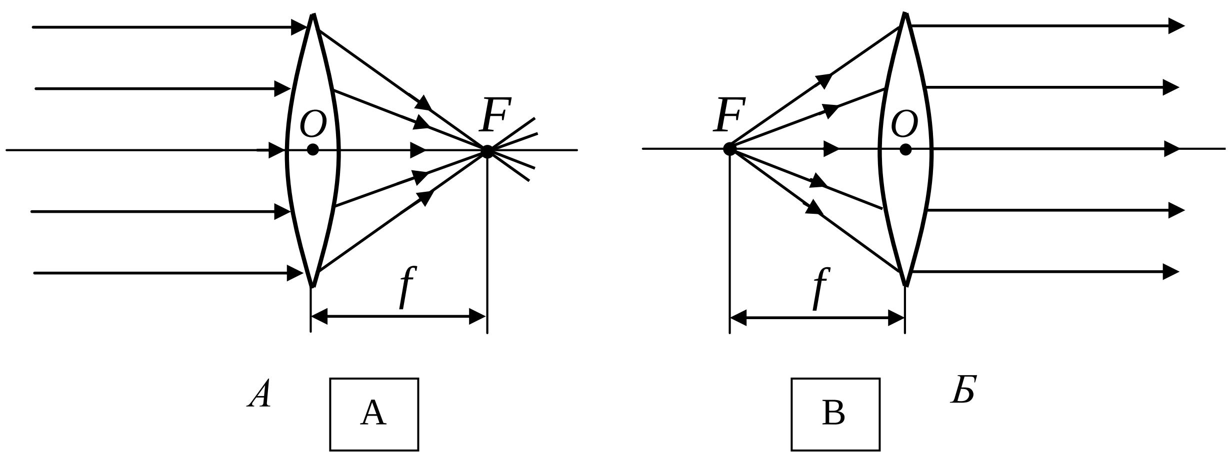


Figure 9.5. Focal points of lens.

Taking into account that

$$e = R_2 - \sqrt{{R_2}^2 - h^2} \sim \frac{h^2}{2R_2} \tag{9.17}$$

and, respectively, we obtain

$$(N-1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{1}{a}+\frac{1}{b}. \qquad (9.18)$$

This expression is the thin lens formula. The radius of curvature of the convex surface of the lens is considered as positive, and the radius of the concave lens is considered as negative.

For the case $a=\infty$ rays fall on the lens with a parallel beam (see Figure 9.5, A), then

$$\frac{1}{b}=(N-1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right). \qquad (9.19)$$

The distance

$$b=OF=f \qquad (9.20)$$

is called the <u>*focal length*</u> of the lens

$$f=\frac{1}{(N-1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}. \qquad (9.21)$$

This distance depends on the relative refractive index and the radii of curvature.

For the case $b=\infty$ the image is in infinity and, consequently, the rays emerge from the lens in a parallel beam (see Figure 9.5, B), then $a=OF=f$. Thus, the focal lengths of the lens surrounded on both sides by the same medium are equal. Points F, lying on both sides of the lens at a distance equal to the focal length, are called focal points. All the rays incident on the lens parallel to the main optical axis are collected at a single point after refraction. This point is called the <u>*focus*</u>. The value

$$(N-1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{1}{f}=\Phi \qquad (9.22)$$

is called the <u>*optical power*</u> of the lens. The optical power of the lens is measured in dioptres. Lenses with positive optical power are convergence lenses. Lenses with negative optical power are diverging lenses. The planes passing through the focal points of the lens perpendicular to its main optical axis are called <u>*focal planes*</u>. Unlike the collecting lens, the diverging lens has imaginary foci. The imaginary extensions of the rays incident on the scattering lens parallel to the main optical axis converge after refraction at the imaginary focus.

We rewrite the expression for the lens formula

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{f}. \qquad (9.23)$$

For a scattering lens, the distances f and b must be considered negative.

The image of the object in the lenses is constructed using the following rays:

1) a ray passing through the optical centre of the lens and not changing its direction;
2) a ray passing parallel to the main optical axis; after refraction in the lens, this ray (or its continuation) passes through the second focus of the lens;
3) a ray (or its continuation) passing through the first focus of the lens; after refraction in it, it leaves the lens parallel to its main optical axis.

The ratio of the linear dimensions of the image and the object is called the *linear magnification* factor of the lens. Negative values of the linear magnification correspond to the actual image (it is inverted), the positive image of the linear magnification corresponds to the imaginary image (it is direct). Combinations of collecting and scattering lenses are used in optical instruments to solve various scientific and technical problems.

9.3. Aberrations of Optical Systems

Considering the light propagation through thin lenses, we limited ourselves to the case of paraxial rays. The refractive index of the lens material was assumed to be independent of the wavelength of the incident light, and the incident light was monochromatic. Since in real optical systems these conditions are not satisfied, image distortions, called aberrations, arise in them.

1. Spherical aberration. In the case of the divergent beam of light falls on the lens, the paraxial rays after refraction intersect at point S' (at a distance of OS' from the optical centre of the lens), and the rays farther from the optical axis intersect at point S'', closer to the lens (Figure 9.6). As a result, the image of the luminous point on the screen perpendicular to the optical axis will be in the form of a fuzzy spot. This type of error in the image associated with the sphere form of the refractive surfaces is called *spherical aberration*. The quantitative measure of spherical aberration is segment $\delta = OS'' - OS'$.

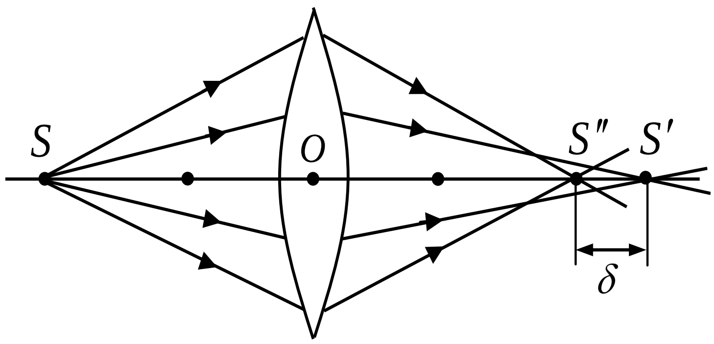


Figure 9.6. Spherical aberration.

Spherical aberration can be reduced by using diaphragms (limited by paraxial rays). However, at the same time, the light-gathering power of the lens decreases.

Spherical aberration can be practically eliminated, making up systems from collecting ($\delta < 0$) and scattering ($\delta > 0$) lenses. Spherical aberration is a particular case of astigmatism.
2. Coma. If a wide beam from a luminous point located not on the optical axis passes through the optical system, the resulting image of this point will be in the form of an illuminated speck that resembles a comet tail. Such an image error is therefore called a *coma*. Elimination of coma is performed by the same methods as spherical aberration.
3. Distortion. The error at which, at large angles of incidence of the rays on the lens, a linear increase for the points of the object, which are at different distances from the main optical axis, is different, is called *distortion*.

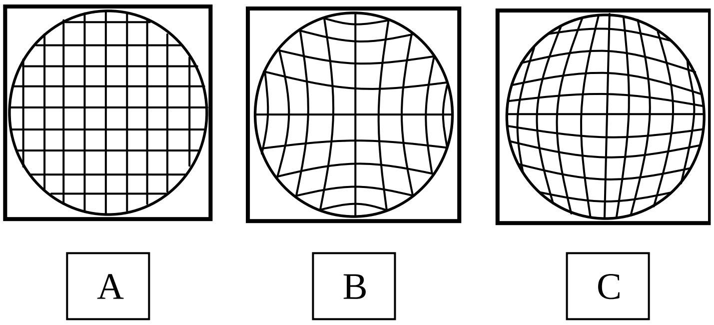


Figure 9.7. Distortion.

As a result, the geometric similarity between the object (rectangular grid, Figure 9.7, A) and its image (pillow-shaped distortion, Figure 9.7, B; barrel distortion, Figure 9.7, C) is broken. Distortion is especially dangerous in cases where optical systems are used in aerial photography and in microscopy. Distortion is corrected by the appropriate selection of the constituent parts of the optical system.
4. Chromatic aberration. Earlier we assumed that the refractive indices of the optical system are constant. However, this statement is valid only for illuminating the optical system with monochromatic light ($\lambda = const$). Non monochromatic light requires considering the dependence of the refractive index of the lens material (and the surrounding medium, if it is not air) on the wavelength (dispersion phenomenon). When a white light falls onto an optical system, the individual components of its monochromatic rays are focused at different points (red rays are the largest focal length, the smallest is the purple rays), so the image is blurred and coloured at the edges. This phenomenon is called *chromatic aberration*. Since different grades of glass have different dispersions, combining collecting and diffusing lenses from different glasses, it is possible to combine the foci of two (achromat) and three (apochromats) of different colours, thereby eliminating chromatic aberration.
5. Astigmatism. The error in the image due to the unequal curvature of the optical surface in different planes of the section of the incident light beam is called *astigmatism*. In this case, the image of a point distant from the main optical axis is observed on the screen in the form of a blurry spot of elliptical shape. This spot, depending on the distance of the screen to the optical centre of the lens, turns into either a vertical or a horizontal line. Astigmatism is corrected by selecting the radii of

curvature of the refractive surfaces and their focal distances. Systems in which spherical and chromatic aberration and astigmatism are eliminated are called anastigmatic.

Elimination of aberrations is possible only by selecting specially designed complex optical systems. Simultaneous correction of all errors is a very difficult task, and sometimes even unsolvable. Therefore, it is usually completely eliminated only those errors, which in this or that case are especially harmful.

9.4. Photometry

The optics section dealing with the measurement of the intensity of light and its sources is called *photometry*. The following quantities are used in photometry.

1. Energy values that characterize the energy parameters of optical radiation irrespective of its effect on radiation receivers.

The quantity equal to the ratio of the radiation energy W to the time t over which radiation has occurred is called the *radiant flux*

$$\Phi_e = \frac{W}{t}. \qquad (9.24)$$

The unit of radiant flux is watts (W).

The value of R_e, equal to the ratio of the radiant flux Φ_e emitted by the surface, to the cross-sectional area S through which this stream passes is called *energy luminosity*

$$R_e = \frac{\Phi_e}{S}. \qquad (9.25)$$

Energy luminosity is equal to the surface radiant flux density. The unit of energy luminosity is watt per square meter (W / m^2).

Energy intensity of light is determined by the concept of a point source of light source whose dimensions are negligible compared to the distance to the place of observation. The value equal to the ratio of the source radiant flux Φ_e to the solid angle ω, within which this radiation propagates

$$I_e = \frac{\Phi_e}{\omega} \qquad (9.26)$$

is called the *energy intensity* of light I_e.

The unit of the energy intensity of light is watt per steradian (W /sr). The value of B_e, equal to the ratio of the energy intensity of the light ΔI_e of the element of the radiating surface to the area ΔS of the projection of this element on a plane perpendicular to the direction of observation

$$B_e = \frac{\Delta I_e}{\Delta S}, \qquad (9.27)$$

is called radiance. The unit of radiance is watt per steradian-meter in square (W / (cr × m^2)). The magnitude of the radiation flux incident on a unit of the illuminated surface is called irradiance (W / m^2).

2. Light values that characterize the physiological effects of light and are estimated from the effect on the eye (come from the so-called average sensitivity of the eye) or other radiation receivers. Optical measurements use different radiation receivers (e.g., eyes, photocells, and photomultipliers) that do not have the same sensitivity to energy of different wavelengths, thus being selective. Each radiation receiver is characterized by its sensitivity curve to light of different wavelengths. Therefore, light measurements, being subjective, differ from objective, energy measurements. For light measurements, light units are introduced, used only for visible light. The main luminous unit in SI is the unit of intensity of light - candela (cd).

The luminous flux Φ is defined as the power of optical radiation by the light sensation caused by it (by its action on a selective light receiver with a given spectral sensitivity). The unit of luminous flux is lumen (lm). The luminous flux in 1 lm is emitted by a point source with a 1-cd light intensity with the radiation field uniformly inside the solid angle.

Luminosity R is determined by the relation $R = \frac{\Phi}{S}$. The unit of luminosity is lumen per square meter (lm / m^2).

The luminance B_φ of the luminous surface in a certain direction φ is a quantity equal to the ratio of the light intensity I in this direction to the area S of the projection of the luminous surface on a plane perpendicular to this direction

$$B_\varphi = \frac{I}{(S \cdot \cos\varphi)}. \qquad (9.28)$$

The unit of luminance is candela per meter per square (cd / m^2).

The illuminance E is a value equal to the ratio of the light flux Φ incident on the surface S

$$E = \frac{\Phi}{S}. \qquad (9.29)$$

A point source generates illuminance

$$E = I \cos\alpha / r^2, \qquad (9.30)$$

where I is intensity of light;

α is angle of incidence;

r is the distance between optical source and illuminated area.

The unit of illuminance is lux: 1 lux is the illuminance of the surface, on the surface in 1 m^2 of which the luminous flux in 1 lm falls (1 lx = 1 lm/ m^2).

Test questions

1. List the basic laws of geometric optics.
2. What property should the environment have in order for the light to propagate in it straightforwardly?
3. What phenomena lead to the violation of the law on non-interaction of intersecting light rays?
4. Formulate the law of reflection.
5. Write down the law of refraction for the case when the vacuum is one of the media.
6. Give a definition of the relative index of refraction.
7. Specify the units for absolute index of refraction.
8. How are the electrical and magnetic properties of the medium with an absolute refractive index interconnected?
9. Formulate the phenomenon of mutual reversibility of light rays.
10. What characteristics of the medium affect the value of the critical angle of total internal reflection?
11. What optical devices can be used to determine the refractive index of bodies?
12. Write down the thin lens formula.
13. Define the focus as well as the focal length and provide the appropriate drawing for the converging and diverging lens.
14. Formulate the rule for determining the signs of the optical power of the converging and diverging lenses.
15. What rays are used to build the image in an optical lens?
16. Determine the linear magnification factor of the lens.
17. Explain the phenomenon of spherical aberration.
18. Give an explanation of the phenomenon of spherical aberration.
19. Is the optical coma phenomenon characteristic of thin or thick optical rays?
20. Write the formula for the illumination generated by a point optical source.

Problem-solving examples

Problem 9.1

Problem description. The two media are separated by a plane-parallel plate. The refractive indices of the first medium, the second medium and the plate are responsibly equal $n_1 = 1.36$, $n_2 = 1.33$, $n = 1.51$ ($n > n_1$). A ray of light falls from the first medium onto the plate at an angle $i_1 = 35°$. Determine the angle i_2, at which a ray of light comes out of the plate.

Known quantities: $n_1 = 1.36$, $n_2 = 1.33$, $n = 1.51$ ($n > n_1$), $i_1 = 35°$.

Quantities to be calculated: i_2.

Problem solution. Passing through the plate, the light beam is refracted twice on its faces. In the case of light falling on the boundary of the first medium with the plate, we have

$$n_1 \sin i_1 = n \sin i, \qquad (9.1.1)$$

and when light falls on the border of the plate with the second medium

$$n \sin i = n_2 \sin i_2. \qquad (9.1.2)$$

Here:
n is the refraction index of the plate;
n_1 is the refraction index of the first medium;
n_2 is the refractive index of the second medium;
i is the angle of refraction of light at the boundary of the first medium with the plate;
i_1 is the angle of incidence of light on the boundary of the first medium with the plate.

Therefore, we can write

$$n_1 \sin i_1 = n_2 \sin i_2. \qquad (9.1.3)$$

Thus, the angle at which the light comes out of the plate is determined by the expression

$$i_2 = \arcsin\left(\frac{n_1 \sin i_1}{n_2}\right) = 33.6°. \qquad (9.1.4)$$

For the case when $n_1 > n_2$ it may turn out that the value calculated by the last formula exceeds unity. This means that the beam will not reach the second medium, but will completely reflect from the plate boundary with the second medium. After reflection, the beam will again be in the first medium, leaving the plate at an angle i_1.

The above equations can be used for the case when we remove the plate and place two media so that they contact. Consequently, the introduction of plastic does not change the direction of the beam in the second medium, but only in parallel displaces it. For a sufficiently small plate thickness, this displacement can be neglected.

Answer. The angle at which the light comes out of the plate is $i_2 = 33.6°$.

Problem 9.2

Problem description. Calculate the illumination of the Earth, created by normally falling sunshine. The brightness of the Sun is equal to $1.2\times10^9\ cd/m^2$. The distance from the Earth to the Sun is $1.5\times10^8\ km$. The radius of the Sun is $7\times10^5\ km$.

Known quantities: $B=1.2\times10^9\ cd/m^2$, $r=1.5\times10^8\ km$, $R=7\times10^5\ km$.

Quantities to be calculated: E.

Problem solution. Due to the large distance from the Earth to the Sun, we believe that the rays falling from the Sun to the Earth go in a parallel beam. Considering that the Sun can be viewed as a flat luminous disk, we find that its brightness is equal to

$$B=\frac{2I}{S}, \tag{9.2.1}$$

where $S=\pi R^2$ is the area of the Sun's disk;

R is the radius of the Sun;

I is the power of light.

A factor of 2 is introduced because the flat disk radiates in two directions. Then

$$B=\frac{2I}{\pi R^2}, \tag{9.2.2}$$

from where

$$I=\frac{\pi BR^2}{2}. \tag{9.2.3}$$

By the condition of the problem, $\cos\alpha=1$. Then the illumination of the Earth's surface is

$$E=\frac{I}{r^2}=\frac{\pi BR^2}{2r^2}, \tag{9.2.4}$$

where r is the distance from the Earth to the Sun.

Numerically

$$E=4.06\cdot10^4\ lx. \tag{9.2.5}$$

Answer. Earth's illumination is equal to $E=4.06\times10^4\ lx$.

Problem 9.3

Problem description. The lamp is used to print a photograph. Two cases are being studied: 1) The lamp with a light intensity of 55 cd is located at a distance of 1.6 m from the image, and the exposure time is equal to 2.5 s; 2) lamp with a light intensity of 42 cd is located at a distance of 2.3 m from the image. Determine the exposure time in the second case.

Known quantities: $I_1 = 55\ cd$, $r_1 = 1.6 m$, $t_1 = 2.5 s$, $I_2 = 42 cd$, $r_2 = 2.3 m$.

Quantities to be calculated: t_2.

Problem solution. The light energy received by the photo paper over time t is equal to the product of the light flux Φ by the exposure

$$W = \Phi t = ESt, \qquad (9.3.1)$$

where E illuminance;

S is the area of photographic paper.

Therefore, for each case, we can write

$$W_1 = E_1 S t_1,\ W_2 = E_2 S t_2, \qquad (9.3.2)$$

where indices "1" and "2" correspond to the first and second cases, which are indicated in the statement of the problem.

The quality of the photographs will be the same if in both cases the same light energy enters the photographic paper $W_1 = W_2$ or taking into account the previous formulas

$$E_1 S t_1 = E_2 S t_2. \qquad (9.3.3)$$

Then

$$t_2 = \frac{E_1 t_1}{E_2}. \qquad (9.3.4)$$

According to the law of illumination for a point source, we can write

$$E_1 = \frac{I_1}{r_1^2} \text{ and } E_2 = \frac{I_2}{r_2^2}, \qquad (9.3.5)$$

where r_1 and r_2 are the distances between the lamp and the photo paper in the first and second cases;

I_1 and I_2 luminous intensity in the first and second cases.

Therefore, for the exposure time in the second case we get

$$t_2 = \frac{I_1 r_2^2 t_1}{r_1^2 I_2}. \qquad (9.3.6)$$

Substitute numeric data: $t_2 = 6.76 s$.

Answer. The exposure time in the second case $t_2 = 6.76\ s$.

Problems

Problem A

Problem description. The refractive angle of a glass prism is equal to $\theta = 30°$. A ray of light falls on the face of the prism perpendicular to its surface and emerges into the air from the other face, deviating by an angle of $\delta = 20°$ from the original direction. Determine the refractive index n of the glass.

Answer. $n = 1.63$.

Problem B

Problem description. Determine the shortest distance L between the subject and its actual image created by the collecting lens with a main focal length of $F = 12\ cm$.

Answer. $L = 0.48\ m$.

Problem C

Problem description. The limits of accommodation of the eye of a myopic person without glasses lie between the values of $a_1 = 16\ cm$ and $a_2 = 80\ cm$. In glasses, he sees distant objects well. At what minimum distance can he keep a book when reading glasses?

Answer. $d = 0.2\ m$.

Problem D

Problem description. What current I will the galvanometer connected to the selenium photocell show if a light bulb is placed at a distance of $r = 75\ cm$ from it, the total luminous flux of which is $\Phi_0 = 1.2\, klm$? The surface area of the photocell is $S = 10\ cm^2$, the sensitivity of the photocell is $\gamma = 300\, \mu A / lm$.

Answer. $I = 5.1 \times 10^{-5}\ A$.

Problem E

Problem description. Determine the illuminance E, luminosity R and luminance B of a movie screen that uniformly scatters light in all directions, if the luminous flux falling on the screen from a camera lens (without a tape) is $\Phi = 1.75\, klm$. The screen size is $5 \times 3.6\ m$, the screen reflection coefficient is $\rho = 0.75$.

Answer. $E = 97\ lx$, $M = 73\ lx$, $B = 23\ cd / m^2$.

CHAPTER 10. INTERFERENCE OF LIGHT

10.1. Huygens' Principle

As a result of the development of ideas about light, two theories of light are used: corpuscular and wave. According to the corpuscular theory, light is a beam of particles (corpuscles) emitted by luminous bodies and propagate along rectilinear trajectories. Newton subjected the laws of mechanics to the motion of light corpuscles. Thus, the reflection of light was understood similarly to the reflection of an elastic ball when it struck a plane, where the law of equality of the angles of incidence and reflection is also observed. Newton explained the refraction of light by the attraction of corpuscles to a refractive medium, as a result of which the velocity of corpuscles changes during the transition from one medium to another. From Newton's theory followed the constancy of the ratio of the sine of the angle of incidence i_1 to the sine of the angle of refraction i_2:

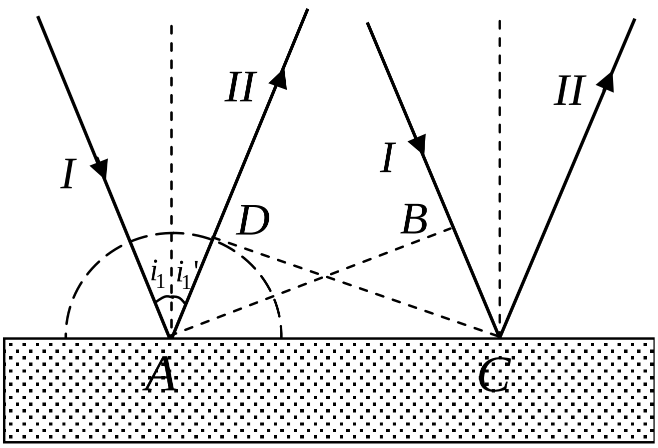
$$\frac{\sin i_1}{\sin i_2} = \frac{\upsilon}{c} = n', \tag{10.1}$$

where c is the speed of light in vacuum,

υ is the speed of light propagation in the medium.

Since the value of n' in the medium is always greater than one, then, according to Newton's theory $c < \upsilon$. The propagation velocity of light in a medium must always be greater than the speed of light propagation in a vacuum.

According to the wave theory, developed on the basis of the analogy of optical and acoustic phenomena, light is an elastic wave propagating in a special medium-ether. The ether fills all world space, permeates all bodies and possesses mechanical properties - elasticity and density.



10.1. Reflection of light waves

According to Dutch physicist Christiaan Huygens (1629–1695), the high speed of light propagation is due to the special properties of the ether. The wave theory is based on the *Huygens' principle*: every point of wave is the centre of the secondary waves, and the envelope of these secondary waves gives the position of the wave front at the next instant of time. Huygens' principle allows us to analyze the propagation of light and derive the laws of reflection and refraction.

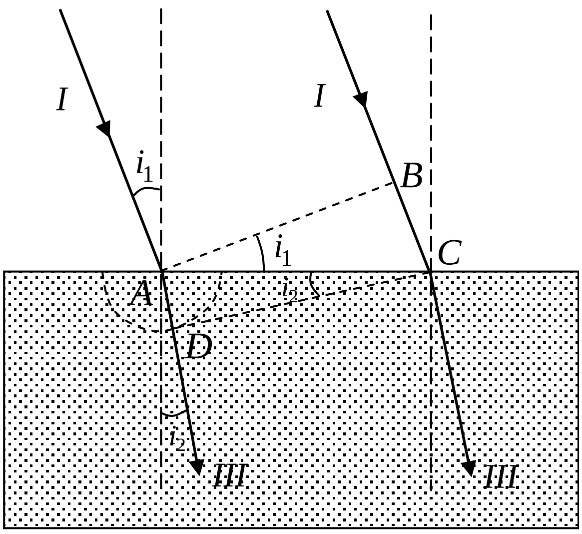
We shall deduce the laws of reflection and refraction of light, starting from the Huygens' principle. Let a plane wave be incident on the interface between two media (the wave front is the plane AB). This wave propagates along the direction I (Figure 10.1). When the wave front reaches the reflecting surface at point A, this point will begin to emit a secondary wave. The wave will pass the distance BC in time

$$\Delta t = \frac{BC}{\upsilon}. \qquad (10.2)$$

During this time, the front of the secondary wave will reach the points of the hemisphere, the radius of which is

$$\upsilon \Delta t = BC. \qquad (10.3)$$

The position of the front of the reflected wave is given by the plane DC, and the direction of propagation of this wave is given by the ray II. The equality of the triangles ABC and ADC implies the law of reflection: the reflection angle i_1' is equal to the angle of incidence i_1.



10.2. Refraction of light waves

To derive the refraction law, we assume that a plane wave (the wave front is the plane AB) propagates in a vacuum along the direction I with the speed of light c. The wave falls on the interface with the medium, in which its propagation velocity is υ (Figure 10.2). Let the wave pass the path BC in time Δt, then we can write $BC = c\Delta t$. During this time, the wave front in the medium with the speed of light υ, reaches the points of the hemisphere, the radius of which is $AD = \upsilon \Delta t$. The position of the front of the refracted wave at this time in accordance with the Huygens' principle is given by the plane DC, and the direction of its propagation is given by the ray III. Then

$$AC = \frac{BC}{\sin i_1} = \frac{AD}{\sin i_2}, \tag{10.4}$$

that is,

$$\frac{c\Delta t}{\sin i_1} = \frac{\upsilon\Delta t}{\sin i_2}, \tag{10.5}$$

whence

$$\frac{\sin i_1}{\sin i_2} = \frac{c}{\upsilon} = n. \tag{10.6}$$

Comparing similar expressions, we see that the wave theory leads to a conclusion different from the conclusion of Newton's theory. According to Huygens' theory $\upsilon < c$, that is, the speed of propagation of light in a medium must always be less than the speed of its propagation in a vacuum.

Thus, approaches to explaining the nature of light: Newton's corpuscular theory and Huygens' wave theory contradicted each other. Both these theories explained the rectilinear propagation of light, the laws of reflection and refraction. The Foucault (1819–1868) and Fizeau (1819–1896) experiments confirmed the validity of the wave theory of light.

Despite the recognition of the wave theory, this theory had a number of shortcomings. For example, the phenomena of interference, diffraction and polarization could be explained only if the light waves are assumed to be transverse. On the other hand, if light waves are transverse, then their carrier-ether must have the properties of solids. An attempt to consider ether as a substance with inherent properties of a solid body did not have success, since the ether does not have a noticeable effect on the bodies moving in it. It was found that the propagation speed of light in different media is different; therefore the ether must have different properties in different media. The Huygens' theory could not explain the physical nature of the presence of different colours.

The science of light accumulated experimental data, indicating the interconnection of light, electric and magnetic phenomena, which allowed Maxwell to create an electromagnetic theory of light. According to Maxwell's electromagnetic theory,

$$\frac{c}{\upsilon} = \sqrt{\varepsilon\mu} = n, \tag{10.7}$$

where c and υ are, respectively, the propagation speed of light in a vacuum and in a medium with a permittivity of ε and a magnetic permeability of μ. This relationship connects optical, electrical and magnetic constants. According to Maxwell, ε and μ

are values that do not depend on the wavelength of light, so the electromagnetic theory could not explain the phenomenon of dispersion (the dependence of the refractive index on the wavelength). This difficulty was overcome in Lorentz, who proposed an electronic theory, according to which the dielectric constant ε depends on the wavelength of the incident light. The theory of Lorentz introduced the idea of electrons that oscillate within an atom, and allowed to explain the phenomena of emission and absorption of light by matter.

Despite the tremendous successes of Maxwell's electromagnetic theory and the Lorentz electron theory, they were somewhat contradictory and a number of difficulties encountered in their application. Both theories were based on the ether hypothesis, only the "elastic ether" was replaced by "electromagnetic ether" (Maxwell's theory) or "fixed ether" (Lorentz theory). Maxwell's theory could not explain the processes of light emission and absorption, photoelectric effect, Compton scattering. The theory of Lorentz could not explain many phenomena associated with the interaction of light with matter, in particular the question of the distribution of energy over the wavelengths with the thermal radiation of a black body.

The above difficulties and contradictions were overcome thanks to the *Planck postulate*, according to which the emission and absorption of light is not continuous, but discrete, i.e., certain portions (quanta) whose energy is determined by the frequency ν:

$$\varepsilon_0 = h\nu\,, \tag{10.8}$$

where h is the Planck constant.

Planck postulate does not require the introduction of the concept of ether. This theory explained the thermal radiation of a black body. Einstein created a quantum theory of light, according to which not only the emission of light, but also its propagation occurs in the form of a flux of light quanta – photons whose energy is

$$\varepsilon_0 = h\nu\,, \tag{10.9}$$

and the mass is

$$m_{ph} = \frac{\varepsilon_0}{c^2} = \frac{h\nu}{c^2} = \frac{h}{\lambda c}\,. \tag{10.10}$$

Quantum concepts of light are in good agreement with the laws of radiation and absorption of light, the laws of interaction of light with matter. The phenomena of interference, diffraction, and polarization of light are easily explained on the basis of wave representations. All the variety of properties studied and the laws of light propagation, its interaction with matter, shows that light has a complex nature. It represents the unity of opposing types of motion – corpuscular (quantum) and wave (electromagnetic). A long way of development led to modern ideas about the dual corpuscular-wave nature of light.

10.2. Coherence of Light Waves

The interference of light can be explained by considering the interference of waves. A necessary condition for wave interference is their coherence, that is, the consistent flow of several oscillatory or wave processes in time and space. Monochromatic waves satisfy the condition of coherence. Unbounded waves in space of one definite and strictly constant frequency are called *monochromatic waves*. Since no real source produces strictly monochromatic light, the waves emitted by any independent light sources are always incoherent. Therefore, experiment does not observe interference of light from independent sources, for example, from two electric bulbs. To understand the physical cause of nonmonochromaticity, and consequently, incoherence, of waves emitted by two independent light sources, one can proceed from the mechanism of the emission of light by atoms. Atoms in two independent light sources emit independently of each other. The radiation process is finite and lasts a very short time ($\tau = 10^{-8}\ s$). During this time, the excited atom returns to its normal state and ceases to emit light. After a while, the atom is again excited and begins to emit light waves, but with a new initial phase. Since the phase difference between the emissions of two such independent atoms varies with each new emission event, the waves spontaneously emitted by the atoms of any light source are incoherent. Thus, the waves emitted by the atoms only have a constant amplitude and phase of oscillations during the time interval 10^{-8} s, whereas in a longer time interval both the amplitude and the phase change. The intermittent emission of light by atoms in the form of individual short pulses is called a *wave train*.

The described model of light emission is also valid for any macroscopic source, since the atoms of the luminous body emit light also independently of each other. This means that the initial phases of the wave trains corresponding to them are not connected with each other. In addition, even for the same atom, the initial phases of different trains differ for the two subsequent radiation events. Consequently, the light emitted by the macroscopic source is incoherent.

Any nonmonochromatic light can be represented as a set of successive independent harmonic trains. The average duration of one train is called the *coherence time* τ_{coh}. Coherence exists only within a single train, and the coherence time can not exceed the radiation time, that is, $\tau_{coh} < \tau$. The device will detect a clear interference pattern only when the resolution time of the device is much shorter than the coherence time of the applied light waves.

If the wave propagates in a homogeneous medium, then the oscillation phase at a certain point in space is retained only during the coherence time τ_{coh}. During this time, the wave propagates in a vacuum to a distance of

$$l_{coh} = c\tau_{coh}, \tag{10.11}$$

called the *coherence length*. Two or more waves lose coherence when they pass a distance equal to the coherence distance. Hence it follows that observation of light interference is possible only with optical path differences that are shorter than the coherence length for the light source used.

The wave becomes more monochromatic if the width of the spectrum of its frequencies $\Delta\omega$ decreases. In this case, the coherence time τ_{coh} and coherence length l_{coh} increase. The coherence of oscillations that occur at the same point in space, determined by the degree of monochromaticity of the waves, is called *temporal coherence*.

In order to describe the coherent properties of waves in a plane perpendicular to the direction of their propagation, the notion of not only temporal coherence but also spatial coherence is introduced. Two sources, whose dimensions and relative positioning allow observing interference (with the necessary degree of light monochromaticity), are called spatially coherent. The maximum distance that is directed transversely to the propagation of a wave and at which interference is possible is called the coherence radius (or the length of spatial coherence. Thus, spatial coherence is determined by the *coherence radius*.

The coherence radius is $r_{coh} = \dfrac{\lambda}{\varphi}$, where λ is the length of the light waves, φ is the angular size of the source. Thus, the minimum possible coherence radius for solar beams (with the angular size of the Sun on Earth $\varphi \approx 10^{-2}$ rad and $\lambda \approx 0.5$ μm) is ≈ 0.05 mm. With such a small coherence radius, it is impossible to directly observe the interference of the sun's rays, since the resolving power of the human eye at a distance of the best view is only 0.1 mm.

10.3. Interference of Light

Suppose that two monochromatic light waves, superimposed on each other, excite at a certain point in space the oscillations of the same direction:

$$X_1 = A_1 \cos(\omega t + \varphi_1) \tag{10.12}$$

and

$$X_2 = A_2 \cos(\omega t + \varphi_2). \tag{10.13}$$

The electric E or magnetic H fields are indicated as X in the last equation. The vectors $\vec{E}$ and $\vec{H}$ oscillate in mutually perpendicular planes. The principle of superposition is applicable to the electric and magnetic fields intensity. The amplitude of the resulting oscillation at a given point is

$$A^2 = A_1^{\,2} + A_2^{\,2} + 2A_1A_2\cos(\varphi_2 - \varphi_1). \tag{10.14}$$

Since the waves are coherent, $\cos(\varphi_2 - \varphi_1)$ does not change with time, so the intensity of the resultant wave is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\varphi_2 - \varphi_1),\ (I \sim A^2)\,. \qquad (10.15)$$

For the resulting intensity, the following inequalities can be written:

$$I > I_1 + I_2\,, \qquad (10.16)$$

for the case $\cos(\varphi_2 - \varphi_1) > 0$ and $I < I_1 + I_2$, we get $\cos(\varphi_2 - \varphi_1) < 0$.

Consequently, when two (or several) coherent light waves are superimposed, a spatial redistribution of the light flux occurs, as a result of which maxima appear in some places, while in others, intensity minima are observed. This phenomenon is called <u>*interference of light*</u>. The phase shift $\varphi_2 - \varphi_1$ continuously varies for incoherent waves, so the time average $\cos(\varphi_2 - \varphi_1)$ is zero, and the intensity of the resulting wave is everywhere the same and at $I_1 = I_2$ is $2I_1$. The intensity maxima for coherent waves are $I = 4I_1$, and for minima $I = 0$ are valid. The separation method is used to obtain coherent light waves. In this case, a wave emitted from one source is divided into two parts, which, after passing through different optical paths, are superimposed on each other and an interference pattern is observed.

Let the separation into two coherent waves occur at point O. Suppose that one wave spreads over a distance S_1 to a point M in a medium with a refractive index of n_1. The second wave passed the path S_2 in a medium with a refractive index of n_2. The phase of the oscillations is ωt at the point O. In this case, the oscillation of the first wave $A_1 \cos\omega\left(t - \frac{S_1}{\upsilon_1}\right)$ will be observed at point M. The second-wave oscillation $A_2 \cos\omega\left(t - \frac{S_2}{\upsilon_2}\right)$ will be observed at the same point. The phase velocities of the first and second waves are

$$\upsilon_1 = \frac{c}{n_1} \text{ and } \upsilon_2 = \frac{c}{n_2}\,, \qquad (10.17)$$

respectively.

The phase difference of the oscillations excited by the waves at point M is

$$\delta = \omega\left(\frac{S_2}{\upsilon_2} - \frac{S_1}{\upsilon_1}\right) = \frac{2\pi}{\lambda_0}(S_2 n_2 - S_1 n_1) = \frac{2\pi}{\lambda_0}(L_2 - L_1) = \frac{2\pi}{\lambda_0}\Delta\,. \qquad (10.18)$$

The relations

$$\frac{\omega}{c} = \frac{2\pi\nu}{c} = \frac{2\pi}{\lambda_0} \qquad (10.19)$$

(λ_0 is the wavelength in vacuum) were taken into account in the derivation of the last equation.

The product of the geometric length S of the path of a light wave in a medium by the refractive index n of this medium is called the <u>*optical path length*</u> L. The value equal to the difference of the optical path lengths

$$\Delta = L_2 - L_1 \tag{10.20}$$

is called the <u>*optical path difference*</u>.

Let us consider the case when the optical path difference is equal to an integer number of waves in a vacuum

$$\Delta = \pm m\lambda A_0 \ (m = 0,1,2,...). \tag{10.21}$$

Then the oscillations excited at point M by both waves will occur in the same phase. Consequently, equation

$$\Delta = \pm m\lambda A_0 \tag{10.22}$$

is the condition of the interference maximum. If the optical path difference is

$$\Delta = \pm(2m+1)\frac{\lambda_0}{2}, \ (m = 0, 1, 2, ...), \tag{10.23}$$

then

$$\delta = \pm(2m+1)\pi \tag{10.24}$$

and the oscillations excited at point M by both waves will occur in anti phase. Consequently, equation

$$\Delta = \pm(2m+1)\frac{\lambda_0}{2} \tag{10.25}$$

is the <u>*condition of the interference minimum*</u>.

British physician Thomas Young (1773–1829) experimentally proved that light is a wave, contrary to that most other scientists then thought [4]. The interference pattern can be calculated using two narrow parallel slits located close enough to each other (Figure 10.3).

Slits S_1 and S_2 are at a distance of d from each other and can be considered as coherent light sources. The interference is observed at point A of the screen. The screen is located parallel to both slits and removed from them by a distance of l ($l >> d$). The intensity at point A of the screen, lying at a distance X from point O, is determined by the optical path difference $\Delta = S_2 - S_1$.

Geometric construction of the figure leads to the following equations:

$$S_2{}^2 = l^2 + \left(X + \frac{d}{2}\right)^2 \tag{10.26}$$

and

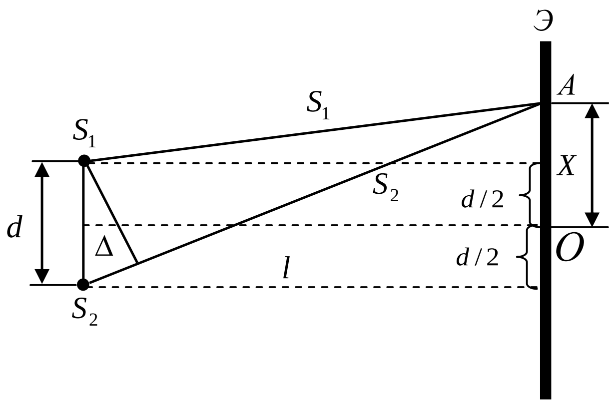
$$S_1^{\,2} = l^2 + \left(X - \frac{d}{2} \right)^2 \qquad (10.27)$$

from where

$$S_2^{\,2} - S_1^{\,2} = 2Xd \qquad (10.28)$$

or

$$\Delta = S_2 - S_1 = \frac{2Xd}{S_1 + S_2}. \qquad (10.29)$$



10.3. Young's interference experiment

The approximate equality

$$S_1 + S_2 \approx 2l \qquad (10.30)$$

follows from condition $l >> d$, so

$$\Delta = \frac{Xd}{l}. \qquad (10.31)$$

An analysis of the expression for Δ, shows that the intensity maxima will be observed when condition

$$X_{\max} = \pm m \frac{l}{d} \lambda_0 \; (m = 0, 1, 2, ...) \qquad (10.32)$$

is satisfied. The *condition for the intensity minima* takes the form

$$X_{\min} = \pm \left(m + \frac{1}{2} \right) \frac{l}{d} \lambda_0 \; (m = 0, 1, 2, ...). \qquad (10.33)$$

The distance

$$\Delta X = \frac{l}{d} \lambda_0 \qquad (10.34)$$

between two adjacent maxima (or minima) is called the *width of the interference fringe*.

The value of ΔX does not depend on the order of the interference (the value of m) and is constant for fixed values of l, d, λ_0. The value of ΔX is inversely proportional to d, therefore, for a large distance between sources, for example at $d \approx l$, individual bands become indistinguishable. For visible light, $\lambda_0 \approx 10^{-7}\ m$ is valid, so a clear interference pattern is observed under condition $l >> d$. The length of the light wave can be determined using the measured values of the quantities $l, d, \Delta X$. It follows from the earlier obtained expressions that the interference pattern produced on the screen by two coherent light sources is an alternation of light and dark bands parallel to each other. The main maximum corresponding to $m = 0$ passes through point O. Up and down from it at equal distances from each other are the maxima (minima) of the first ($m = 1$), second ($m = 2$) orders, and so on.

The described picture, however, is valid only for monochromatic light ($\lambda_0 = const$). Let's consider the use of white light for interference. White light is a continuous set of wavelengths from 0.39 μm (violet border of the spectrum) to 0.75 μm (red border of the spectrum). In this case, the interference maxima for each wavelength will be shifted relative to each other and have the appearance of iridescent bands.

Test questions

21. Formulate the Huygens' principle.
22. Explain the relationship between the speeds of light in the medium and in vacuum according to the Huygens' principle.
23. Can Maxwell's electromagnetic theory of light explain the phenomenon of dispersion?
24. List the phenomena of optics that the Lorentz electron theory could not explain.
25. Formulate Planck postulate.
26. State the postulates of Einstein's quantum theory of light.
27. What properties of light rays are necessary for observing the phenomenon of wave interference?
28. Specify the time interval during which the waves emitted by an atom can be considered coherent.
29. Give the definition of coherence time and coherence length.
30. Calculate the coherence radius of the sun's rays.
31. What practical methods are used to produce coherent light waves?
32. Write down the formula for the resulting intensity of the two light waves.
33. Describe the interference phenomenon.
34. Estimate the difference in optical path lengths for air and for vacuum.
35. Formulate the condition of interference maxima.
36. What parameters affect the width of the interference fringe?

37. Does the the width of the interference fringe depend on the order of interference?
38. What conditions correspond to the observation of a clear interference pattern for visible light?
39. Give the definition of the order of the interference pattern.
40. Consider the use of white light for interference.

Problem-solving examples

Problem 10.1

Problem description. A normally parallel beam of light with a wavelength of 510 nm is incident on a thin glass wedge. The distance between adjacent dark interference fringes in reflected light is 0.45 mm. Determine the angle between the wedge surfaces. The refractive index of the glass that makes up the wedge is 1.56.

Known quantities: $\lambda = 510\ nm$; $L = 0.45\ m$; $n = 1.56$.

Quantities to be calculated: α.

Problem solution. The optical path difference for light fringes is

$$\Delta = k\lambda, \tag{10.1.1}$$

where k is the interference order;
λ is the wavelength.
The difference in the course of 2 rays is equal to

$$\Delta = 2d_k n - \frac{\lambda}{2} = k\lambda, \tag{10.1.2}$$

where n is the wedge refractive index;
d_k is the thickness of the wedge in place of the light band with the number k.
We have

$$d_k = \frac{(2k+1)\lambda}{4n} \tag{10.1.3}$$

and, similarly,

$$d_{N+k} = \frac{(2(N+k)+1)\lambda}{4n}. \tag{10.1.4}$$

For a small wedge angle, we can write

$$tg\alpha \approx \alpha = \frac{d_{N+k} - d_k}{L} = \frac{N\lambda}{2nL}, \tag{10.1.5}$$

where L is the distance between adjacent fringes (and, therefore, $N = 1$). We have

$$\alpha = \frac{\lambda}{2nL}. \tag{10.1.6}$$

Substitute numeric data $\alpha = 3.63 \times 10^{-4}\ rad$.

Answer. The angle between the wedge surfaces is $\alpha = 3.63 \times 10^{-4}\ rad$.

Problem 10.2

Problem description. There is a liquid between the glass plate and the flat convex lens lying on it. Find the refractive index of the liquid, if the radius of the third dark Newtonian ring when observed in reflected light with a wavelength of $\lambda = 0.62\,\mu m$ is equal to $0.86\,mm$. The radius of curvature of the lens is 0.54 m.

Known quantities: $\lambda = 0.62\,\mu m$; $m = 3$; $R = 0.54\,m$; $r_3 = 0.86\,mm$.

Quantities to be calculated: n.

Problem solution. We will find first of all the optical path difference Δ. Upon reflection from the boundary, the liquid-glass phase changes to π (loss of the half-wave). The phase does not change for the case when the reflection occurs from the glass-liquid interface. Therefore, the optical path difference is

$$\Delta = 2n\delta_m + \lambda/2, \tag{10.2.1}$$

where n is the refractive index of the liquid;

δ_m is the distance between the lens and the plane for the ring with the number m.

In order for the ring to be dark, it is necessary that

$$\Delta = \frac{(2m+1)\lambda}{2}, \tag{10.2.2}$$

i.e. at thickness

$$\delta_m = \frac{m\lambda}{2n}. \tag{10.2.3}$$

Ring radius with the number m is

$$r_m^{\,2} = R^2 - (R - \delta_m)^2 \approx 2R\delta_m = 2R\frac{m\lambda}{2n}. \tag{10.2.4}$$

Here R is the lens radius. As a result, we get

$$r_m = \sqrt{\frac{Rm\lambda}{n}}, \tag{10.2.5}$$

consequently

$$n = \frac{Rm\lambda}{r_m^{\;2}} = 1.36. \tag{10.2.6}$$

Answer. The refractive index of the liquid is $n = 1.36$.

Problem 10.3

Problem description. Monochromatic light with a wavelength of 520 nm is incident on a thin film in the direction normal to its surface. Light reflected from the film is maximized due to interference. Calculate the minimum film thickness if the refractive index of the film material is 1.44.

Known quantities: $n = 1.44$; $\alpha = 90°$; $\lambda = 520\ nm$.

Quantities to be calculated: d .

Problem solution. The optical path difference of the light rays reflected from two surfaces of a thin film, on both sides of which there is air, is equal to

$$\Delta = 2d\sqrt{n^2 - \sin^2\alpha} + \frac{\lambda}{2}, \tag{10.3.1}$$

where n is the refractive index; d is the film thickness; λ is the wavelength; α is the angle of incidence. For the maximum interference condition, we can write

$$\Delta = 2d\sqrt{n^2 - \sin^2\alpha} + \frac{\lambda}{2} = \pm k\lambda, \tag{10.3.2}$$

where k is the interference order.

The minimum d value will be when $k = 1$, i.e.

$$\Delta = 2d\sqrt{n^2 - \sin^2\alpha} + \frac{\lambda}{2} = \lambda. \tag{10.3.3}$$

We obtain the formula for the thickness of the film

$$d = \frac{\lambda}{4\sqrt{n^2 - \sin^2\alpha}}. \tag{10.3.4}$$

According to the problem $\alpha = 90^0$, Consequently

$$d = \frac{\lambda}{4\sqrt{n^2 - 1}}. \tag{10.3.5}$$

Substitute numerical values in the formula $d = 1.25 \times 10^{-7}\ m$.

Answer. The minimum film thickness is $d = 1.25 \times 10^{-7}\ m$.

Problems

Problem A

Problem description. The distance between two coherent light sources with a wavelength $\lambda = 0.5\,\mu m$ is $d = 0.1\,mm$. The distance between the interference fringes on the screen in the middle part of the interference pattern is $b = 1\,cm$. Determine the distance L from the sources to the screen.

Answer. $L = 2\ m$.

Problem B

Problem description. Find all wavelengths of visible light (from 0.76 to 0.38 μm) that will be: 1) maximally amplified, 2) maximally attenuated at an optical path difference of interfering waves equal to 1.8 μm.

Answer. 1) $\lambda_1 = 0.6\ \mu m$, $\lambda_2 = 0.45\ \mu m$; 2) $\lambda_1 = 0.72\,\mu m$, $\lambda_2 = 0.51\ \mu m$, and $\lambda_3 = 0.4\ \mu m$.

Problem C

Problem description. Monochromatic light falls at an angle of $\pi/2$ on a thin glass ($n = 1.55$) wedge. The dihedral angle between the surfaces is equal to $\alpha = 2'$. Determine the length λ of the light wave if the distance between adjacent interference peaks in reflected light is $b = 0.3\,mm$.

Answer. $\lambda = 5.41 \times 10^{-7}\ m$.

Problem D

Problem description. A flat-convex lens with optical power $D = 2\ dpt$ lies on a glass plate with a convex side. The radius of the fourth dark ring of Newton in transmitted light is $r_4 = 0.7\ mm$. Determine the length of the light wave.

Answer. $\lambda = 4.9 \times 10^{-7}\ m$.

Problem E

Problem description. Determine the movement of the mirror in the Michelson interferometer, if the interference pattern has shifted to $m = 100$ bands. The experiment was conducted with light with a wavelength of $\lambda = 546\ nm$.

Answer. $\delta = 2.73 \times 10^{-5}\ m$.

CHAPTER 11. DIFFRACTION OF LIGHT

11.1 Huygens-Fresnel Principle

The diffraction of light is the phenomenon of the deviation of light from the straight direction of propagation when passing near obstacles. Thanks to diffraction, waves can fall into the region of geometric shadow, bend around obstacles, penetrate through small holes in screens, etc. For example, sound can be heard around the corner of the house, i.e. the sound wave bends around it.

The diffraction phenomenon is explained with the help of the Huygen's principle. According to Huygen's principle each point of wave front serves as the centre of the secondary waves, and the envelope of these waves gives the position of the wave front at the next instant of time.

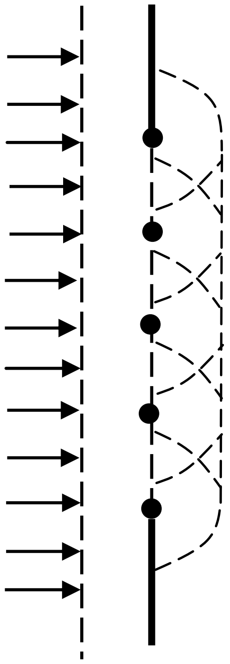


Figure 11.1. Secondary waves

Let the plane wave normally fall on the hole in the opaque screen (Figure 11.1). Each point of the wave front that passes through the hole serves as a source of secondary waves (in a homogeneous isotropic medium they are spherical). The result of constructing the envelope of the secondary waves indicates that the wave front propagates into the region of the geometric shadow, that is, the wave bends around the edges of the hole. The phenomenon of diffraction is characteristic of wave processes.

Huygen's principle solves only the problem of the direction of propagation of the new front, but does not touch upon the problem of the amplitude, and consequently, of the intensity of waves propagating in different directions. French physicist Augustin-Jean Fresnel (1788–1827) supplemented the Huygens principle with physical meaning, namely, the idea of interference of secondary waves.

According to the *Huygens–Fresnel principle*, a light wave excited by any source can be represented as a result of a superposition of coherent secondary waves "emitted" by fictitious sources. Infinitely small elements of any closed surface enclosing the source S can be considered as such sources. Usually one of the wave surfaces is chosen as this surface, therefore all fictitious sources act in phase. Thus, waves propagating from a source are the result of interference of all coherent secondary waves.

Fresnel ruled out the possibility of inverse secondary waves and suggested that if an opaque screen with an aperture is located between the source and the observation point, then the amplitude of the secondary waves is equal to zero on the screen surface.

The values of the amplitudes and phases of the secondary waves make it possible in each concrete case to find the amplitude (intensity) of the resultant wave at any point of space, that is, to determine the patterns of light propagation. In the general case, the calculation of the interference of the secondary waves is rather complicated, however, as will be shown below, for some cases the amplitude of the resulting oscillation is found by algebraic summation.

11.2. Fresnel Zones

The Huygens–Fresnel principle within the framework of the wave theory is to answer the question of the rectilinear propagation of light. Fresnel solved this problem by considering the mutual interference of the secondary waves and applying a technique called the Fresnel zone method.

Let us find at an arbitrary point M the amplitude of the light wave, which propagates in a homogeneous medium from a point source S (Figure 11.2). According to the Huygens-Fresnel principle, we replace the action of the source S by the action of imaginary sources located on the auxiliary surface Φ, which is the front of a wave coming from S (the surface of a sphere with centre S).

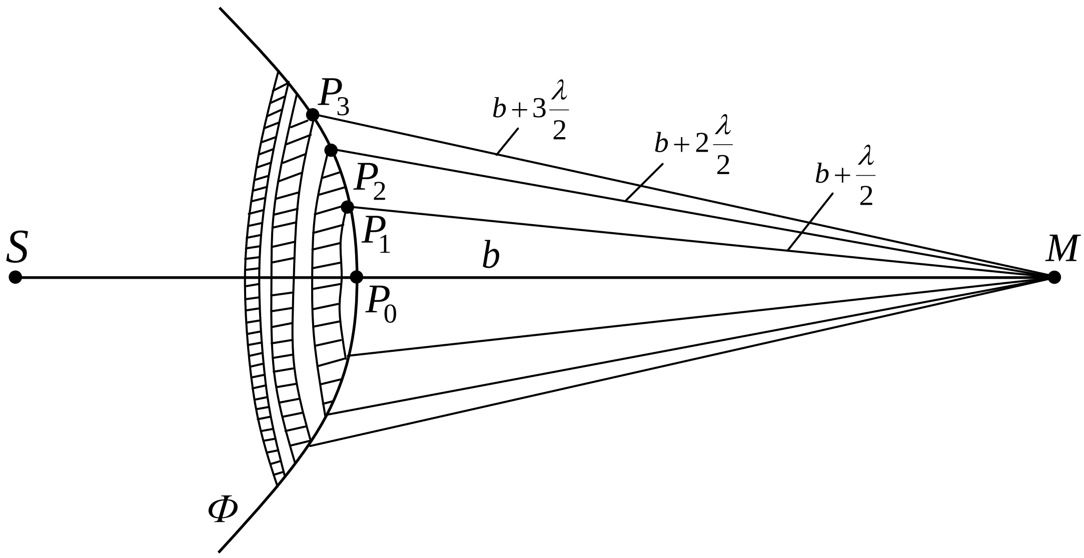


Figure 11.2. Fresnel zones

Fresnel divided the wave surface Φ into annular zones of such size that the distances from the edges of the zone to M differ by $\lambda/2$, that is, $P_1M - P_0M = P_2M - P_1M = P_3M - P_2M = ... = \lambda/2$. Such a partition of the wave front into zones can be performed by drawing spheres with radii

$$b+\lambda/2,\ b+2\lambda/2,\ b+3\lambda/2, ..., b+m\lambda/2 \qquad (11.1)$$

centred on M.

Since the oscillations from neighboring zones pass to point M of the distance, differing by $\lambda/2$, they arrive at point M in the opposite phase and, when superimposed, these oscillations mutually weaken each other. Therefore, the amplitude of the resulting light vibration at point M is

$$A = A_1 - A_2 + A_3 - A_4 + ... \pm A_m, \qquad (11.2)$$

where $A_1, A_2, ..., A_m$ is the amplitude of the oscillations excited by the 1 st, 2 nd, ..., n-th zones.

To estimate the oscillation amplitudes, we find the areas of the Fresnel zones. The outer boundary of the zone with the number m identifies a spherical height segment on the wave surface. The outer boundary of the zone with the number m singles out on the wave surface a spherical segment whose height is equal to h_m (Figure 11.3).

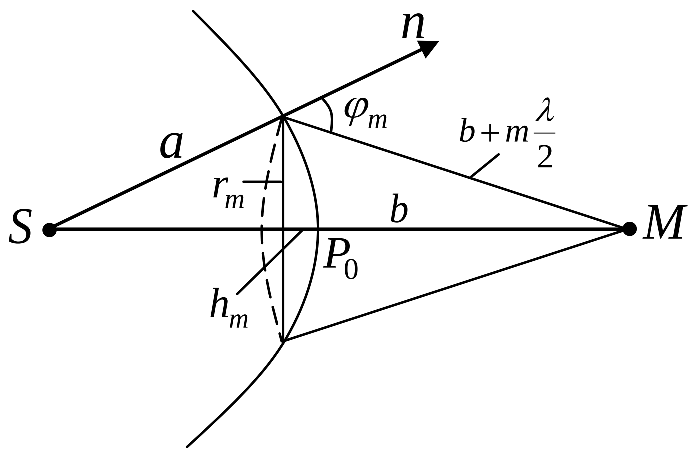


Figure 11.3. Parameters of Fresnel zones.

We denote the area of this segment by σ_m. Then the area of the Fresnel zone with number m is equal to

$$\Delta\sigma_m = \sigma_m - \sigma_{m-1}, \qquad (11.3)$$

where σ_{m-1} is the area of the spherical segment of the outer boundary of zone $(m-1)$.

It follows from the figure that

$$r_m{}^2 = a^2 - (a - h_m)^2 = \left(b + \frac{m\lambda}{2}\right)^2 - (b + h_m)^2. \qquad (11.4)$$

After elementary transformations, taking into account that $\lambda << a$ and $\lambda << b$, we obtain

$$h_m = \frac{bm\lambda}{2(a+b)}. \qquad (11.5)$$

The area of the spherical segment is

$$\sigma_m = 2n\pi h_m = \frac{\pi ab\lambda m}{a+b}, \qquad (11.6)$$

and the area of Fresnel zone with number m is

$$\Delta\sigma_m = \sigma_m - \sigma_{m-1} = \frac{\pi ab\lambda}{a+b}. \qquad (11.7)$$

The resulting expression does not depend on m. Thus, the construction of Fresnel zones splits the wave surface of a spherical wave into equal bands.

According to Fresnel's assumption, the effect of individual zones at point M decreases with an increase in the angle φ_m between the normal to the surface of the zone and the direction towards point M, i.e. the action of the zones gradually decreases from the central zone (P_0) to the peripheral ones (to zero). In addition, the radiation intensity in the direction of point M decreases with increasing m and the distance from the zone to point M. Taking both of these factors into account, we can write $A_1 > A_2 > A_3 > ...$ The total number of Fresnel zones that fit on the hemisphere is very large; for example, at $a = b = 10\,\text{cm}$ and λ = 0.5 μm we have

$$N = \frac{2\pi a^2}{\pi ab\lambda(a+b)} = 8\cdot 10^5. \qquad (11.8)$$

Therefore, as an allowable approximation, we can assume that the amplitude of oscillation A_m from a certain Fresnel zone m is equal to the arithmetic mean of the amplitudes of the adjacent bands, i.e.

$$A_m = \frac{A_m + A_{m+1}}{2}. \qquad (11.9)$$

Then the expression for the amplitude can be written in the form

$$A = \frac{A_1}{2} + \left(\frac{A_1}{2} - A_2 + \frac{A_3}{2}\right) + \left(\frac{A_3}{2} - A_4 + \frac{A_5}{2}\right) + ... = \frac{A_1}{2}. \qquad (11.10)$$

Suppose that the height of segment $h_m << a$ (for not too large m), then

$$r_m{}^2 = 2ah_m. \qquad (11.11)$$

In this case, we can write

$$r_m = \sqrt{\frac{abm\lambda}{a+b}} \qquad (11.12)$$

for the radius of the outer boundary of the Fresnel zone.

11.3. Fresnel Diffraction from Simple Barriers

Let us consider the diffraction of spherical waves, or the Fresnel diffraction, which occurs when the diffraction pattern is observed at a finite distance from the obstacle that caused the diffraction. A spherical wave propagating from a point source S meets on its way a screen with a circular aperture. The diffraction pattern is observed on the screen at a point B lying on the line connecting source S with the centre of the hole (Figure 11.4).

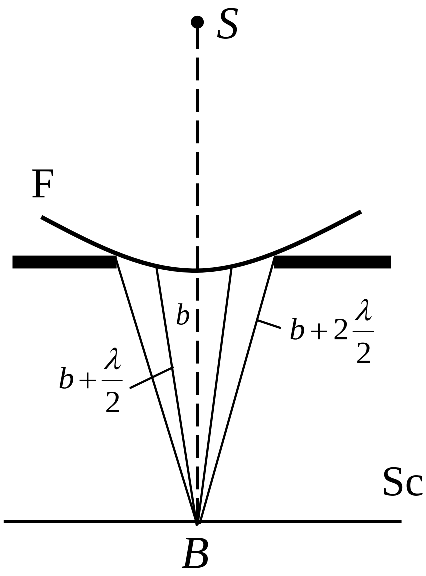


Figure 11.4. Circular-aperture diffraction.

We divide the open part of the wave surface into Fresnel zones. The form of the diffraction pattern depends on the number of Fresnel zones that fit into the hole. The amplitude of the resulting oscillation excited at point B by all bands is

$$A = \frac{A_1}{2} \pm \frac{A_m}{2}, \qquad (11.13)$$

where the plus sign corresponds to the odd m and the minus sign corresponds to even m. When the hole opens an odd number of Fresnel zones, the amplitude (intensity) at point B will be greater than when the wave propagates freely. If the hole opens an even number of Fresnel zones, then the amplitude (intensity) will be zero. If one Fresnel zone is placed in the hole, then at point B the amplitude is $A = A_1$, that is, twice as large as in the absence of an opaque screen with an aperture.

The Bavarian physicist Joseph Ritter von Fraunhofer (1787–1826) made optical glass and developed diffraction grating. *Fraunhofer diffraction*, which is of great practical importance, is observed when the light source and the observation point are infinitely removed from the obstacle that caused the diffraction. To realize this type of diffraction, it is sufficient to place a point source of light in the focus of the collecting lens, and to study the diffraction pattern in the focal plane of the second converging lens installed behind the obstacle.

Let us consider the Fraunhofer diffraction on an infinitely long slit (for this it is practically sufficient that the length of the slit is much larger than its width).

Let a plane monochromatic light wave fall normally to the plane of a narrow slit of width a (Figure 11.5). The optical path difference between the rays MC and ND from the slit in an arbitrary direction φ is $\Delta = NF = a \cdot \sin\varphi$, where F is the base of the

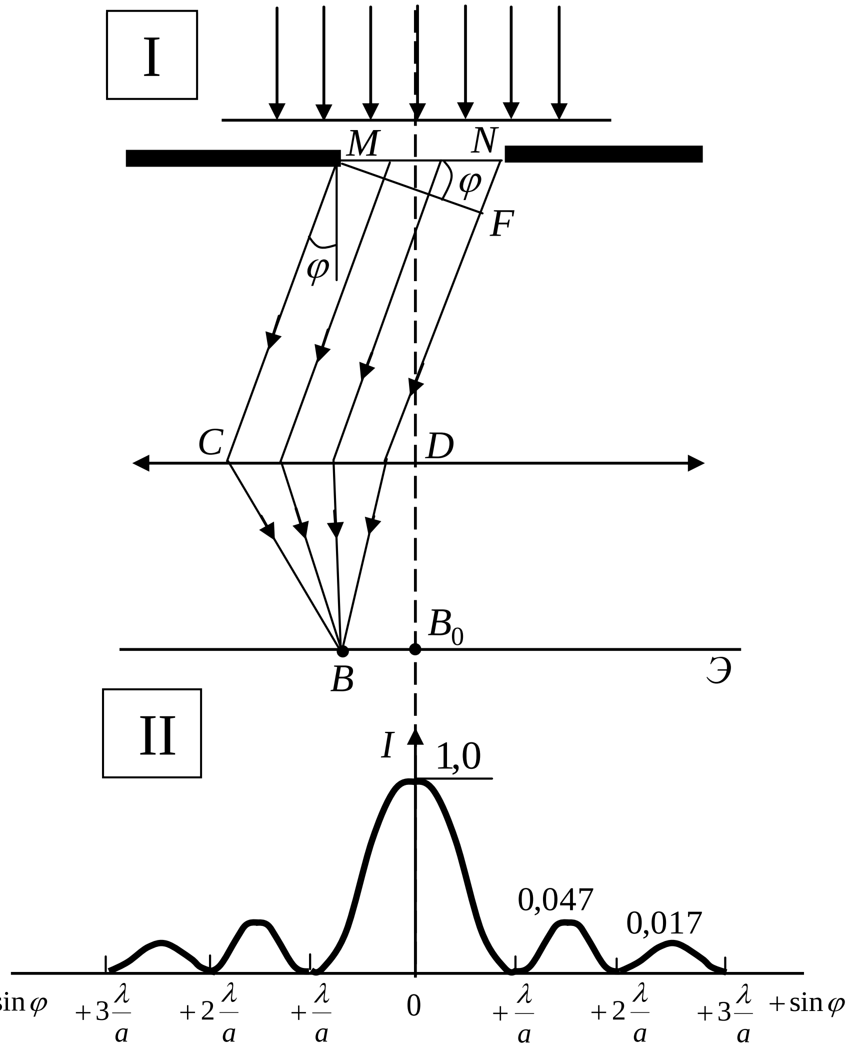


Figure 11.5. Single-slit diffraction.

perpendicular dropped from point M to beam ND.

We divide the open part of the wave surface in the plane of the slit MN into Fresnel zones, having the form of bands parallel to the edge M of the slit. The width of each zone is chosen so that the path difference from the edges of these zones is equal to $\lambda/2$, that is, a total of $\Delta : \lambda/2$ zones will fit on the width of the slit.

If the number of Fresnel zones is even then a *diffraction minimum* $a \cdot \sin\varphi = \pm 2m\lambda/2$, $(m = 1, 2, 3, \ldots)$, is observed at point B (total darkness). If the number of Fresnel zones is odd

$$a \cdot \sin\varphi = \pm(2m+1)\lambda/2, \ (m = 1, 2, 3, ...), \qquad (11.14)$$

then a *diffraction maximum* is observed at point B. In the forward direction ($\varphi = 0$), the slit acts as one Fresnel zone, and in this direction the light propagates with the greatest intensity, i.e. at the point B_0 the central diffraction maximum is observed. Calculations show that the intensities of the central and subsequent maxima are treated as 1: 0.047: 0.017: 0.0083: ..., i.e. the bulk of the light energy is concentrated at the central maximum.

The position of the diffraction maxima depends on the wavelength λ, therefore, the diffraction pattern considered here has only the case for monochromatic light. The maximums $m = 1$, $m = 2$ ... are vague, so it is impossible to obtain a distinct separation of different wavelengths by diffraction on a single slit.

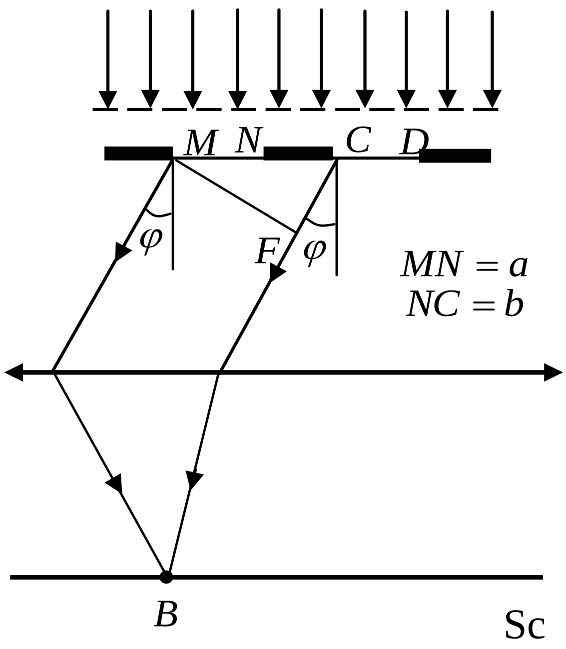


Figure 11.6. Diffraction grating.

Diffraction observed when light passes through a one-dimensional diffraction grating is of great practical importance. The *diffraction grating* is a system of parallel slits of equal width, lying in the same plane and separated by equal widths with opaque gaps. The diffraction pattern on the grating is determined as the result of mutual interference of waves coming from all the slots, i.e. in the diffraction grating, multi-beam interference of coherent diffracted light beams originating from all the slits is realized.

11.4. Diffraction Grating

Let us consider a diffraction grating. Figure 11.6 presents only two adjacent slits MN and CD.

If the width of each slit is a, and the width of the opaque sections between the slits is b, then $d = a + b$ is called the *period of the diffraction grating*.

Let a plane monochromatic wave fall normally to the plane of the lattice. Since the slits are at the same distances from each other, the path differences of the rays coming from two adjacent slits will be the same for the given direction φ within the entire diffraction grating

$$\Delta = CF = (a + b)\sin\varphi = d\sin\varphi . \qquad (11.15)$$

The main intensity minima will be observed in directions determined by condition

$$a \cdot \sin\varphi = \pm m\lambda \;\; (m = 1, 2, 3, \ldots). \qquad (11.16)$$

The effect of one slit will amplify the action of the other, if

$$d\sin\varphi = \pm 2m\lambda / 2 = \pm m\lambda , \; (m = 0, 1, 2, \ldots), \qquad (11.17)$$

that is, the resulting expression specifies the condition of the principal maxima.

Thus, the total diffraction pattern for two gaps is determined from the condition:

principal minima

$$a \cdot \sin\varphi = \lambda, 2\lambda, 3\lambda, \ldots; \qquad (11.18)$$

additional minima

$$d\sin\varphi = \lambda/2,\ 3\lambda/2,\ 5\lambda/2 \ldots; \qquad (11.19)$$

principal maxima

$$d\sin\varphi = 0, \lambda, 2\lambda, 3\lambda, \ldots \qquad (11.20)$$

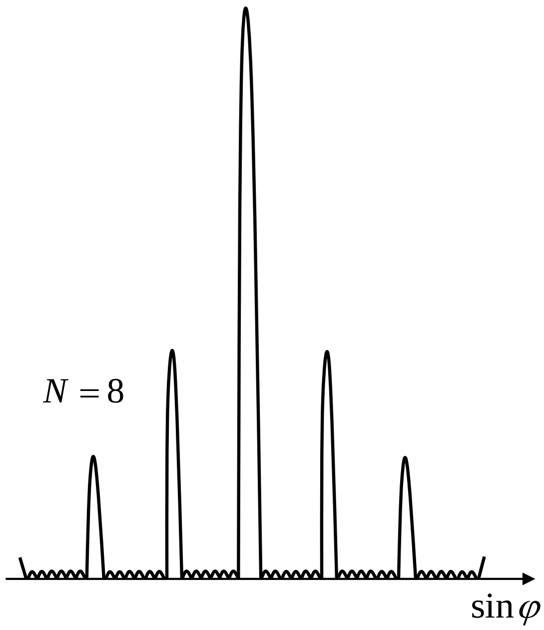


Figure 11.7. Secondary maxima background.

If the diffraction grating consists of N slits, then the condition of the principal minima is

$$a \cdot \sin\varphi = \pm m\lambda, \tag{11.21}$$

the condition of the principal maxima is

$$d \sin\varphi = \pm m\lambda, \tag{11.22}$$

and the condition of additional minima is

$$d \sin\varphi = \pm m'\lambda / N, \tag{11.23}$$

$$\left(m' = 1, 2, \ldots, N-1, N+1, \ldots, 2N-1, 2N+1, \ldots\right)$$

where m' can take all integer values except $0, N, 2N, \ldots$ under which condition $d \sin\varphi = \pm m'\lambda / N$ becomes $d \sin\varphi = \pm m\lambda$.

Consequently, in the case of N slits there are $N-1$ additional minima separated by secondary maxima. These minima create a very weak background (Figure 11.7).

The position of the main maxima depends on the wavelength. Therefore, when the white-light passing through grating, all the maxima, except the central ($m = 0$), will expand into the spectrum. The violet region of the spectrum will face the centre of the diffraction pattern, and the red one is outward. The value $\lambda/(mN)$ gives the theoretical resolution on the grating [8].

Diffraction gratings used in different spectral regions differ in size, shape, material of the surface, the profile of the slits and their frequency (from 6000 to 0.25 slits / mm, which allows to cover the spectral range from the ultraviolet to infrared part). For example, the stepped profile of the grating makes it possible to concentrate the main part of the incident energy in the direction of one certain nonzero order.

Test questions

1. Describe the phenomenon of diffraction.
2. Specify the geometric shape of the secondary wave that each wavefront point generates.
3. Formulate the law of geometric optics, which is violated in the phenomena of diffraction.
4. Give a detailed description of the Huygens–Fresnel principle.
5. Is it true to say that the distance between the edges of the Fresnel zones is λ?
6. Why do the amplitudes of the neighbouring Fresnel zones have opposite signs?
7. Write down the formula for the area of the Fresnel zone.
8. Are the effects of the Fresnel zones decreasing, increasing, or unchanged with an increase in their sequence number?
9. Calculate the number of Fresnel zones for the case when the distance between the source and the point of observation is 30 cm.
10. Identify the factors that influence the numerical value of the Fresnel radius.
11. Describe the Fresnel diffraction, which occurs when the diffraction pattern is observed at a finite distance from the obstacle that caused the diffraction.
12. Explain the nature of the influence of the number of obstructed Fresnel zones on the resulting amplitude of the light waves.
13. Give the definition of the Fraunhofer diffraction.
14. What is the shape of the Fresnel zone for the case of Fraunhofer diffraction on an infinitely long slit?
15. Write down the condition under which the number of Fresnel zones during Fraunhofer diffraction on an infinitely long slit will be odd.
16. Make a comparison of the amplitudes of the central and subsequent diffraction maxima in relative units.
17. What physical factors determine the position of the diffraction maximum?
18. State the reasons that the distance between parallel slits in the diffraction grating should not change.
19. Write the formula for the condition of the principal maxima.
20. Give examples of practical application of diffraction gratings.

Problem-solving examples

Problem 11.1

Problem description. The intensity created on the screen by a certain monochromatic light wave in the absence of obstacles is equal to I_0. Calculate the intensity in the centre of the diffraction pattern, if there is an obstacle with a circular opening in the path of the wave: a) the first Fresnel zone; b) half of the first Fresnel zone; c) Fresnel zone and a half; d) third of the first Fresnel zone.

Known quantities: I_0.

Quantities to be calculated: I.

Problem solution. The resulting amplitude of the diffraction pattern for a round hole

$$A = \frac{A_1}{2} \pm \frac{A_m}{2}, \tag{11.1.1}$$

where the sign "+" corresponds to odd m, and the sign "–" corresponds to even m. The amplitudes of the neighboring zones are about the same. The absence of a barrier corresponds to the ratio

$$A = \frac{A_1}{2} \tag{11.1.2}$$

or in intensities

$$I_a = I_0 = \frac{I_1}{4}, \tag{11.1.3}$$

i.e.

$$I_1 = 4I_0. \tag{11.1.4}$$

Half of the open first Fresnel zone corresponds to

$$A = \frac{A_1}{2} \tag{11.1.5}$$

or in intensities

$$I_b = \frac{I_1}{4} = \frac{4I_0}{4} = I_0. \tag{11.1.6}$$

One and a half open Fresnel zones corresponds to the amplitude

$$A = \frac{A_1}{2} - \frac{1}{2}\frac{A_2}{2} \approx \frac{A_1}{2} - \frac{1}{2}\frac{A_1}{2} = \frac{A_1}{4} \tag{11.1.7}$$

or in intensities

$$I_c = \frac{I_0}{4}. \qquad (11.1.8)$$

One third of the open first Fresnel zone corresponds to the amplitude

$$A = \frac{1}{3} A_1 \qquad (11.1.9)$$

or in intensities

$$I_d = \frac{I_1}{9} = \frac{4I_0}{9}. \qquad (11.1.10)$$

Answer. Intensities in the cases mentioned in the problem are equal: $I_a = I_b = I_0$, $I_c = \frac{I_0}{4}$, $I_d = \frac{4I_0}{9}$.

Problem 11.2

Problem description. Calculate the smallest number of strokes of the diffraction grating, if the spectrum of the second order can be distinguished separately two yellow sodium lines with wavelengths of 590 nm and 591 nm. Determine the length of such a grating, if the grating constant is 5.3 μm.

Known quantities: $d = 5.3\ \mu m$; $\lambda_1 = 590\ nm$; $\lambda_2 = 591\ nm$; $m = 2$.

Quantities to be calculated: N, L.

Problem solution. The position of the main maxima of the diffraction grating is determined by the ratio

$$d(\sin\theta - \sin\theta_0) = m\lambda, \qquad (11.2.1)$$

where θ_0 is the angle of incidence;
λ is the wavelength;
m is an integer (diffraction order).

The position of the diffraction minima is determined by the ratio

$$d(\sin\theta - \sin\theta_0) = \left(m + \frac{p}{N}\right)\lambda, \qquad (11.2.2)$$

where N is the number of strokes;
$p = 1, 2, \ldots N-1$.

Spectral lines are considered resolved if the main maximum for one wavelength λ_2 coincides in position with the first diffraction minimum, in the same order for another wave λ_1, therefore

$$m\lambda_2 = \left(m + \frac{1}{N}\right)\lambda_1 \ . \qquad (11.2.3)$$

Then

$$\Delta\lambda = \lambda_2 - \lambda_1 = \frac{\lambda'}{Nm} . \qquad (11.2.4)$$

Therefore, for the resolution we can write

$$R = \frac{\lambda'}{\Delta\lambda} = Nm . \qquad (11.2.5)$$

Then the number of strokes of the diffraction grating is

$$N = \frac{\lambda'}{\Delta\lambda m} , \qquad (11.2.6)$$

where $\lambda' = \dfrac{\lambda_1 + \lambda_2}{2}$ is the average wavelength.

Finally,

$$N = \frac{\lambda_1 + \lambda_2}{2\Delta\lambda m} = 295 . \qquad (11.2.7)$$

Then the length of the lattice is equal to

$$L = Nd = 1.56 \times 10^{-3} \ m . \qquad (11.2.8)$$

Answer. The number of strokes of the diffraction grating is equal to $N = 295$. The length of the diffraction grating is equal to $L = 1.56 \times 10^{-3} \ m$.

Problem 11.3

Problem description. White light propagates in the direction perpendicular to the plane of the diffraction grating. The diffraction grating contains 610 strokes per millimetre. The lens is placed near the diffraction grating and projects light coming out of the grating onto the screen. Determine the length of the spectrum of the first order on the screen, if the distance from the lens to the screen is 1.3 m. The boundaries of the visible spectrum correspond to the wavelengths $\lambda_r = 780 \ nm$, $\lambda_V = 400 \ nm$.

Known quantities: $N = 610$; $l = 1 mm$; $L = 1.3 \ m$; $\lambda_V = 400 \ nm$; $\lambda_r = 780 \ nm$.

Quantities to be calculated: Δx.

Problem solution. The condition of the main maxima in the diffraction of light on the diffraction grating

$$d\sin\theta = \pm k\lambda, \qquad (11.3.1)$$

where d is the diffraction grating constant;

λ is the wavelength;

k is the diffraction maximum order;

θ is the diffraction angle.

According to the condition of the problem $k=1$, i.e.

$$d\sin\theta = \lambda. \qquad (11.3.2)$$

The number of strokes on the length of l is equal to

$$N = \frac{1}{d}, \qquad (11.3.3)$$

then

$$d = \frac{1}{n} \text{ and } \frac{1}{N}\sin\theta = \lambda. \qquad (11.3.4)$$

The length of the spectrum is equal to

$$\Delta x = (tg\,\theta_r - tg\,\theta_V)L, \qquad (11.3.5)$$

where the indices "r" and "V" refer to the red and violet spectral regions, respectively.

For angles θ_V and θ_r we can write:

$$\sin\theta_V = \frac{N\lambda_V}{l} = 0{,}24, \text{ hence } \theta_V = 14{,}1^0. \qquad (11.3.6)$$

$$\sin\theta_r = \frac{N\lambda_r}{l} = 0{,}468, \text{ hence } \theta_r = 28{,}4^0. \qquad (11.3.7)$$

Then numerically

$$\Delta x = \left(tg27{,}9^0 - tg13{,}9^0\right)\cdot 1{,}2 = 0{,}38\ m. \qquad (11.3.8)$$

Answer. The length of the spectrum of the first order on the screen is equal to $\Delta x = 0{,}38\ m$.

Problems

Problem A

<u>Problem description</u>. Calculate the radius of the fifth Fresnel zone for a plane wave front ($\lambda = 0.5\ \mu m$), if the construction is done for the observation point located at a distance of $b = 1\ m$ from the wave front.

Answer. $\rho_5 = 1.58 \times 10^{-3}\ m$.

Problem B

Problem description. A flat light wave ($\lambda = 0.7\,\mu m$) normally falls on a diaphragm with a circular aperture of radius $r = 1.4\ mm$. Determine the distances b_1, b_2, and b_3 from the diaphragm to the three points farthest from it, where intensity minima are observed.

Answer. $b_1 = 1.4\ m$, $b_2 = 0.7\ m$, $b_3 = 0.47\ m$.

Problem C

Problem description. A monochromatic light falls at an angle of $\pi/2$ onto a diffraction grating containing $n = 100$ strokes per length $1\,mm$. The telescope of the spectrometer is aimed at a maximum of the third order. To bring the pipe to another maximum of the same order, it must be rotated at an angle of $\Delta\varphi = 20°$. Determine the wavelength λ of the light.

Answer. $\lambda = 5.8 \times 10^{-7}\ m$.

Problem D

Problem description. The diffraction pattern was obtained using a diffraction grating with a length of $L = 1.5\,cm$ and a period of $d = 5\ \mu m$. Determine the value of the smallest order $k_{\min}$ of the spectrum of the diffraction pattern at which separate images of two spectral lines with a wavelength difference of $\Delta\lambda = 0.1\,nm$ are obtained. These lines are in the extreme red part of the spectrum ($\lambda \approx 760\ nm$).

Answer. $k_{\min} = 3$.

Problem E

Problem description. A parallel X-ray beam ($\lambda = 147\ pm$) falls on the brink of a rock salt crystal. Determine the distance d between the atomic planes of the crystal, if the diffraction maximum of the second order is observed when the radiation falls at an angle of $\theta = 31°30'$ to the surface of the crystal.

Answer. $d = 2.8 \times 10^{-8}\ m$.

CHAPTER 12. POLARIZATION OF LIGHT

12.1. Natural and Polarized Light

A consequence of Maxwell's theory is the transverse nature of light waves: the vectors of the intensities of the electric $\vec{E}$ and magnetic $\vec{H}$ fields of the wave are mutually perpendicular and oscillate perpendicularly to the velocity vector $\vec{\upsilon}$ of the wave propagation (perpendicular to the ray). Therefore, in order to describe the laws of light polarization, it is sufficient to know the behaviour of only one of the vectors $\vec{E}$ and $\vec{H}$. Usually all the arguments are conducted relative to the light vector – the vector of electric field intensity $\vec{E}$.

Light is the total electromagnetic radiation of a set of atoms. Atoms emit light waves independently of each other, so the light wave emitted by the body as a whole is characterized by all possible equiprobable oscillations of the light vector (Figure 12.1, A; the ray is perpendicular to the plane of the figure).

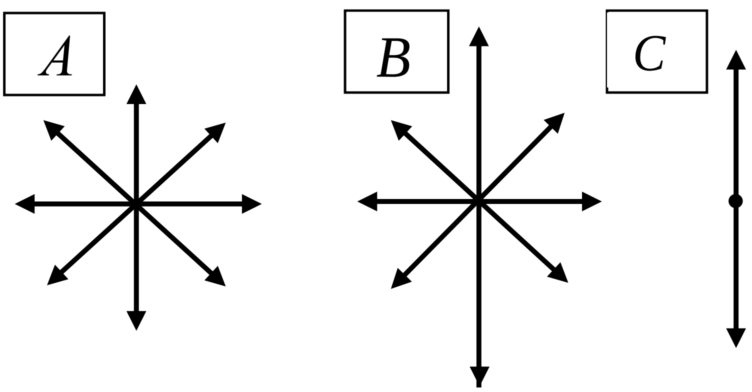


Figure 12.1. Light wave orientations.

In this case, the uniform distribution of the vectors $\vec{E}$ is explained by the large number of atomic radiators. Light with all possible equiprobable orientations of the vector $\vec{E}$ (and, therefore, $\vec{H}$) is called a *natural light*. The light, in which the directions of the oscillations of the light vector are somehow ordered, is called *polarized light*. So, if as a result of some external influences a predominant the fixed direction of vector oscillations (Figure 12.1, B), then we are dealing with partially polarized light. The light in which the vector $\vec{E}$ (and therefore $\vec{H}$) oscillates only in one direction perpendicular to the ray (Figure12.1, C), is called *plane polarized light* (linearly polarized).

The plane passing through the direction of the light vector of a plane-polarized wave and the direction of propagation of this wave is called the *plane of polarization*. Plane-polarized light is the limiting case of elliptically polarized light, i.e. light, for which the vector $\vec{E}$ (and vector $\vec{H}$) varies with time so that its end describes an ellipse lying in a plane perpendicular to the ray.

The degree of polarization is the value equals

$$P = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}, \qquad (12.1)$$

where I_{max} and I_{min} are the maximum and minimum light intensities corresponding to two mutually perpendicular components of vector $\vec{E}$.

For natural light $I_{max} = I_{min}$ and $P = 0$, for plane-polarized $I_{min} = 0$ and $P = 1$.

12.2. Polarisers. Malus's Law

Natural light can be converted to a plane-polarized light, using so-called polarisers, which transmit oscillations of only a certain direction (for example, transmitting oscillations parallel to the plane of the polarizer and completely retarding oscillations perpendicular to this plane). A natural crystal tourmaline has long been used as a polariser.

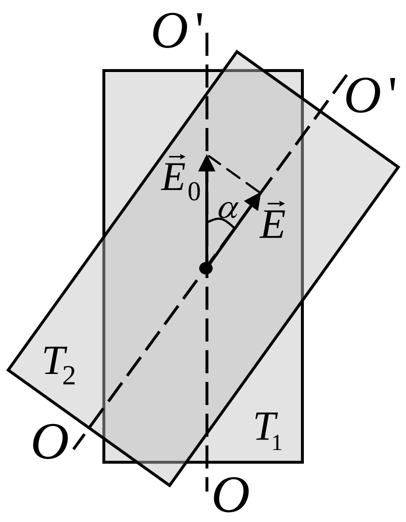


Figure 12.2. Plane-polarized light.

Let's consider classical experiments with tourmaline (Figure 12.2). We direct the natural light perpendicular to the tourmaline plate T_1 cut in parallel with the so-called optical axis OO. No changes in the intensity of the light passing through the tourmaline are observed when the tourmaline crystal T_1 rotates around the direction of the beam. For the case when a second plate of tourmaline T_2 is placed in the path of the beam and rotated around the direction of the beam, the intensity of light transmitted through the plates varies depending on the angle α between the optical axes of the crystals according to the Malus' law:

$$I = I_0 \cos^2 \alpha, \qquad (12.2)$$

where I_0 and I are, respectively, the intensity of light falling on the second crystal and emerging from it.

Consequently, the intensity that passes through the light plates varies from a minimum at $\alpha = \pi/2$ (the optical axis of the plates are perpendicular) to a maximum at $\alpha = 0$ (the optical axes of the plates are parallel). However, as it follows from the figure, the amplitude $\vec{E}$ of the light oscillations transmitted through the plate T_2 will be less than the amplitude of the light oscillations $\vec{E}_0$ that fall on the plate T_2:

$$\vec{E} = \vec{E}_0 \cos\alpha. \qquad (12.3)$$

The results of experiments with crystals of tourmaline can be explained quite simply, if we proceed from the above conditions for the transmission of light by a polarizer.

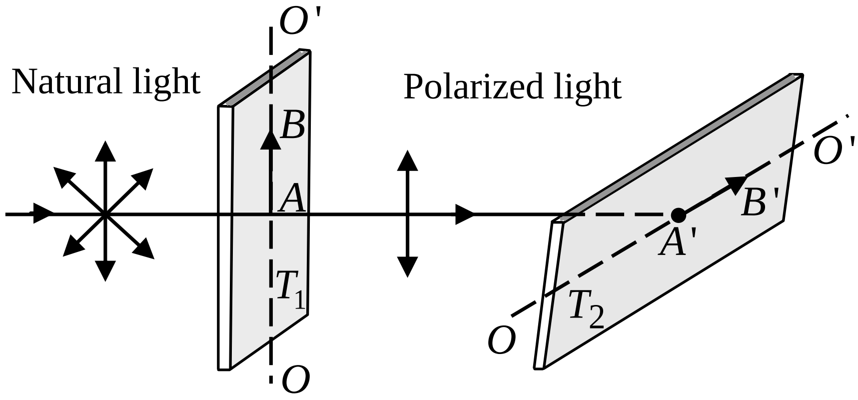


Figure 12.3. Verification of Malus's law.

The first plate of tourmaline passes vibrations only of a certain direction (in the figure this direction is indicated by an arrow AB), i.e. converts natural light into plane-polarized light. The second plate of tourmaline, depending on its orientation from the polarized light, passes a greater or lesser part of it, which corresponds to the component $\vec{E}$ parallel to the axis of the second tourmaline. In the figure, both plates are arranged so that the directions of the oscillations AB and $A'B'$ passed by them are perpendicular to each other. In this case, T_1 passes the oscillations directed at AB, and T_2 completely extinguishes them, that is, for the second plate of tourmaline, the light does not pass.

A plate that converts natural light into plane-polarized light is a *polariser*. A plate serving to analyze the degree of polarization of light is called an *analyzer*. Both plates are exactly the same (they can be interchanged).

12.3 Polarization in Reflection and Refraction

Let’s consider the case then natural light falls on the interface between two dielectrics (for example, air and glass), then part of it is reflected, and the part is refracted. Experiments have shown that oscillations perpendicular to the plane of incidence prevail in the reflected ray (they are indicated by dots in the figure), and oscillations parallel to the plane of incidence (shown by arrows) prevail in the refracted ray (Figure 12.4).

The degree of polarization (the degree of emission of light waves with a certain orientation of the electric (and magnetic) vector) depends on the angle of incidence of the rays and the refractive index.

At an angle of incidence i_B (the *Brewster’s angle*), determined by the relation

$$tg i_B = n_{21}, \tag{12.4}$$

where n_{21} is the refractive index of the second medium relative to the first, the reflected ray is plane-polarized (contains only the oscillations perpendicular to the plane of incidence) (Figure 12.5).

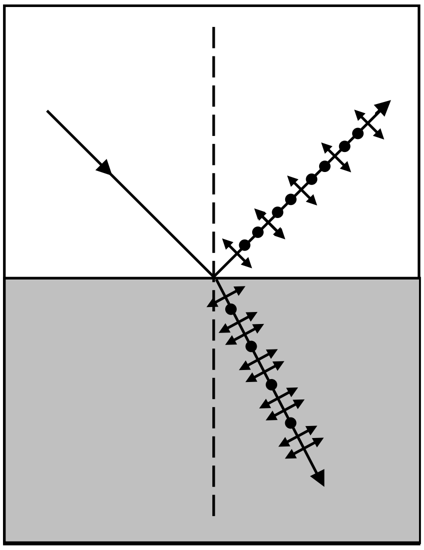


Figure 12.4. Reflection, refraction and polarization.

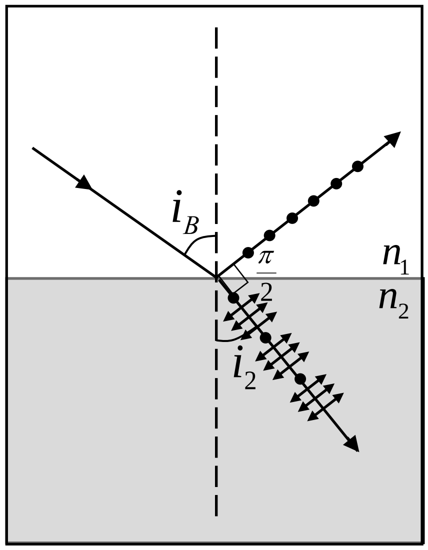


Figure 12.5. Brewster's angle.

The refracted beam at the angle of incidence i_B is polarized as much as possible, but not completely. For the case then light falls on the interface at the Brewster's angle the reflected and refracted rays are mutually perpendicular.

Indeed:

$$tgi_B = \sin i_B / \cos i_B, \tag{12.5}$$

then

$$n_{21} = \sin i_B / \sin i_2, \tag{12.6}$$

where i_2 is the angle of refraction, whence

$$\cos i_B = \sin i_2. \tag{12.7}$$

Consequently,

$$i_B + i_2 = \pi / 2, \tag{12.8}$$

but $i_B' = i_B$ (the law of reflection), so

$$i_B' + i_2 = \pi / 2. \tag{12.9}$$

The degree of polarization of reflected and refracted light at various angles of incidence can be calculated from Maxwell's equations if we take into account the boundary conditions for the electromagnetic field at the interface of two isotropic dielectrics (the so-called *Fresnel formulas*).

The degree of polarization of the refracted light can be greatly increased (by repeated refraction if the light falls every time at the Brewster's angle). The degree of polarization of the refracted beam for glass ($n = 1{,}53$) is 15%. After refraction of 8 to 10 superimposed glass plates, the light released from such a system will be almost completely polarized.

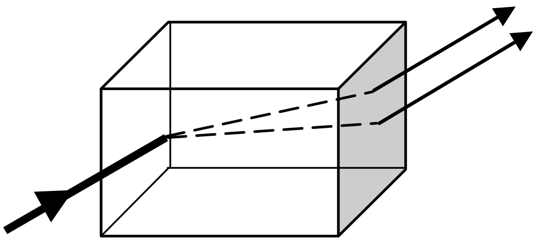


Figure 12.6. Birefringence.

All transparent crystals (except crystals of the cubic system that are optically isotropic) have the ability of *birefringence*, that is, bifurcation of each light beam incident on them. This phenomenon, observed for Iceland spar (a kind of calcite $CaCO_3$), is explained by the features of the propagation of light in anisotropic media and follows directly from Maxwell's equations. Some materials, such as glass and plastic, become birefringent when stressed. Engineers often use technique, called optical stress analysis, in designing structures ranging from bridges to small tools [5].

A narrow beam of light after passing through a thick crystal of Iceland spar is divided into two rays parallel to each other and the incident ray (Figure 12.6). Even in the case when the primary beam falls on the crystal normally, the refracted beam is divided into two, one of which is an extension of the primary beam and the other is deflected (Figure 12.7). The second of these rays is called the *extraordinary ray* (e), and the first is called the *ordinary ray* (o).

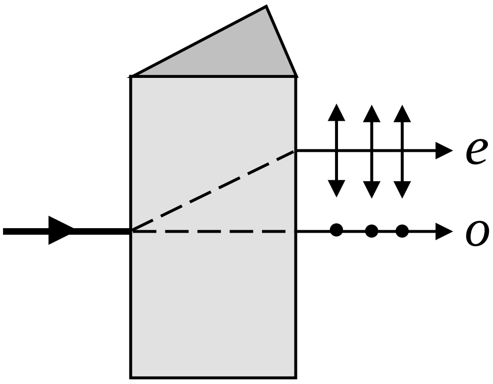


Figure 12.7. Extraordinary and ordinary rays.

In the crystal of Iceland spar, there is a single direction along which birefringence is not observed. The direction in an optically anisotropic crystal, along which a light beam propagates without experiencing birefringence, is called the *optical axis of the crystal*. In this case, we are talking about the direction, rather than the straight line, passing through some point of the crystal. Any straight line that runs parallel to this direction is the optical axis of the crystal. Crystals, depending on the type of their symmetry, are *uniaxial* and *biaxial*, that is, they have one or two optical axes (the Iceland spar belongs to the first type). The plane passing through the direction of the light ray and the optical axis of the crystal is called the principal plane (or the principal section) of the crystal.

12.4. Polarization in Double Refraction

Analysis of polarized light (for example, using tourmaline or a glass mirror) shows that the rays emerging from the crystal are plane polarized in mutually perpendicular planes. The plane of the light vector oscillations (the electric field intensity vector) in an ordinary ray is perpendicular to the principal plane. The oscillations of the light vector in an extraordinary ray are in the principal plane (Figure 12.7).

The unequal refraction of the ordinary and extraordinary rays indicates the difference in their refractive indices. Obviously, for any direction of the ordinary ray, the oscillations of the light vector are perpendicular to the optical axis of the crystal, so the ordinary ray propagates in all directions at the same speed and, consequently, the refractive index for it is constant. For an extraordinary beam, the angle between the direction of the oscillations of the light vector and the optical axis is different from the direct one and depends on the direction of the beam, so the extraordinary rays propagate in different directions at different speeds. Consequently, the refractive index n_e of an extraordinary ray is a variable quantity, depending on the direction of the ray.

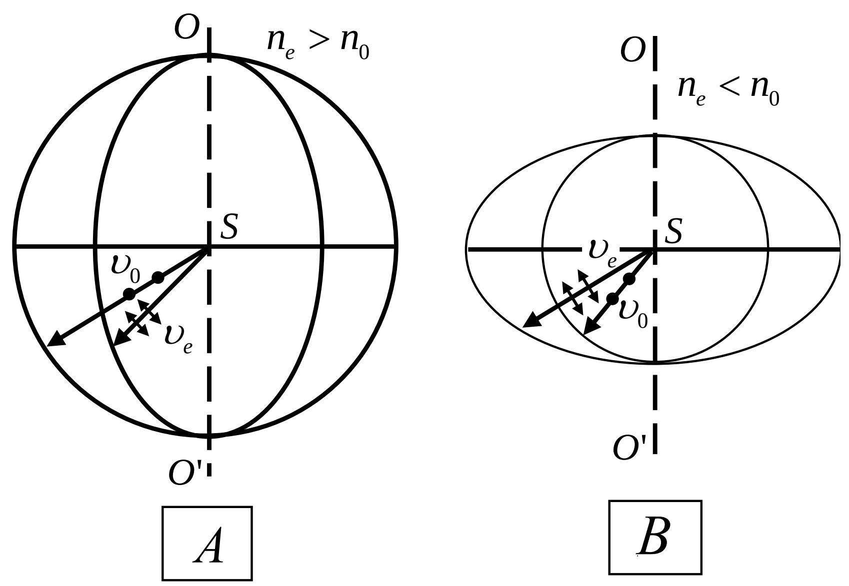


Figure 12.8. Extraordinary wave refractive index.

Ordinary rays propagate in the crystal in all directions at the same velocity

$$\upsilon_0 = c / n_0 , \tag{12.10}$$

while the extraordinary rays propagate at different velocities

$$\upsilon_e = c / n_e \tag{12.11}$$

(depending on the angle between the vector $\vec{E}$ and the optical axis).

Let us assume that a point light source is located inside the uniaxial crystal at point S. Figure 12.8 presents the propagation of ordinary and extraordinary rays in a crystal. The main plane coincides with the plane of the drawing, the direction of the optical axis is denoted by OO'. The wave surface of an ordinary ray (it propagates at a velocity of $\upsilon_0 = const$) is a sphere. The wave surface of an extraordinary ray ($\upsilon \neq const$) is an ellipsoid. The ellipsoid and the sphere touch each other at the points of their intersection with the optical axis OO'. For the case when $\upsilon_e < \upsilon_0$, $n_e > n_0$, the ellipsoid of an extraordinary ray is inscribed in the sphere of an ordinary ray (the ellipsoid of velocities is stretched relative to the optical axis) and the uniaxial crystal is called positive (Figure 12.8, A). Consider the inverse relation $\upsilon_e > \upsilon_0$, $n_e << n_0$. In this case, the ellipsoid is described around the sphere (the ellipsoid of velocities is stretched in a direction perpendicular to the optical axis) and the uniaxial crystal is called negative (Figure 12.8, B).

As an example of the construction of ordinary and extraordinary rays, we consider the refraction of a plane wave at the boundary anisotropic medium.

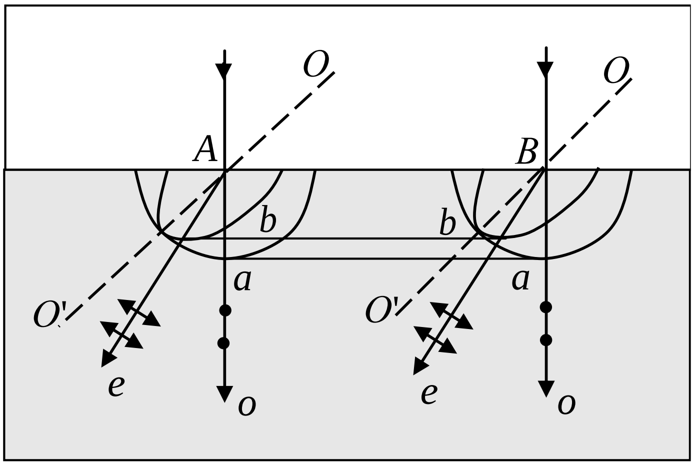


Figure 12.9. Light propagation in uniaxial crystals.

Let the light fall normally to the refracting face of the crystal, and the optical axis OO' makes with this face some angle different from zero (Figure 12.9). At points A and B, we construct spherical wave surfaces corresponding to an ordinary ray, and ellipsoidal surfaces corresponding to an extraordinary ray. At a point lying on line OO', these surfaces are in contact. According to the Huygens' principle, the surface tangent to the spheres is the front ($a - a$) of the ordinary wave, and the surface tangent to the ellipsoids will be the ($b - b$) front of the extraordinary wave.

Drawing straight lines to the points of tangency, we obtain the directions of propagation of ordinary (o) and extraordinary (e) rays. Thus, in this case the ordinary ray will go along the original direction, the extraordinary one will deviate from the original direction.

The phenomenon of birefringence is based on the work of polarization devices used to produce polarized light. Prisms and polaroids are used for this purpose. Prisms are divided into two classes: 1) prisms, at the output of which only plane-polarized beam is observed (polarization prisms); 2) prisms that produce two beams polarized in mutually perpendicular planes (birefringent prisms).

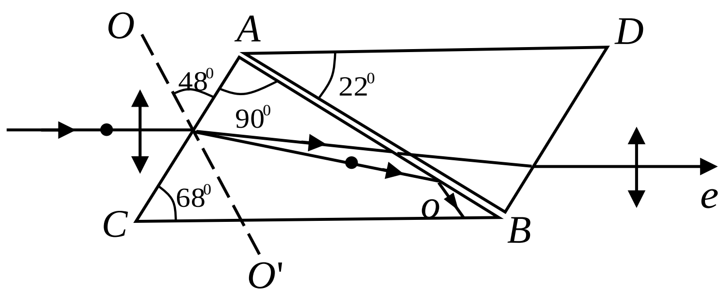


Figure 12.10. Nichol prism.

Polarization prisms are constructed by the principle of full reflection of one of the rays (for example, ordinary) from the interface, while another beam with a different refractive index passes through this boundary. A typical representative of polarization prisms is the prism, often called <u>*Nichol prism*</u>. This prism is consists of a rhombohedral crystal of Iceland spar that has been cut as shown in Figure 12.10 and then glued along the AB line by a Canadian balm with a refractive index of $n = 1.55$. The optical axis OO' of the prism makes an angle of 48° with the input face. On the front face of the prism a natural ray parallel to the edge CB divides into two rays: ordinary ($n_0 = 1.66$) and extraordinary ($n_e = 1.51$).With an appropriate selection of the angle of incidence equal to or greater than the limiting angle, the ordinary ray undergoes complete reflection (the Canadian balm for it is an optically less dense medium), and then absorbed by the blackened lateral surface CB. The extraordinary beam emerges from the crystal parallel to the incident ray, slightly shifted relative to it (due to refraction on the inclined faces AC and BD).

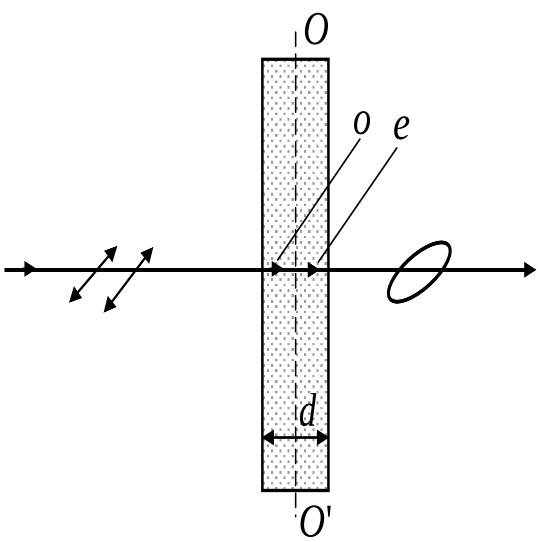


Figure 12.11. Waveplate.

Let a plane-polarized light normally fall on a crystal plate cut out parallel to the optical axis (Figure 12.11). Inside the plate, it is divided into ordinary (o) and extraordinary (e) rays, which in the crystal are not spatially separated (but move with different velocities). The electrical intensity vector $\vec{E}$ of these waves (and, consequently, the vector $\vec{H}$) varies with time so that its end describes an ellipse oriented arbitrarily with respect to the coordinate axes.

The equation of this ellipse is:

$$\frac{X^2}{E_O{}^2} - \frac{2XY}{E_O E_e}\cos\varphi + \frac{Y^2}{E_e{}^2} = \sin^2\varphi, \quad (12.12)$$

where E_O and E_e are, respectively, the components of the electric field intensity of the wave in the ordinary and extraordinary rays,
φ is the phase difference of the oscillations.

Thus, as a result the plane-polarized light becomes elliptically polarized.

Between the ordinary and extraordinary rays in the plate there is an optical path difference or phase difference

$$\varphi = (2\pi / \lambda_0)(n_O - n_e)d, \quad (12.13)$$

where d is the thickness of the plate,
λ_0 is the wavelength in vacuum.

Let’s consider the plate cut out parallel to the optical axis, for which the optical path difference is

$$\Delta = (n_O - n_e)d = \pm\left(m + \frac{1}{4}\right)\lambda_0, \quad (m = 0,1,2,...). \quad (12.14)$$

This plate is called a *quarter-wave plate*. The plane-polarized light, passing the quarter-wave plate, transforms at the output into elliptically polarized light.

A plate for which

$$(n_O - n_e)d = \pm\left(m + \frac{1}{2}\right)\lambda_0, \quad (m = 0,1,2,...) \quad (12.15)$$

is called a *half-wave plate*.

Test questions

1. Specify the component of the electromagnetic field, which is usually used to describe the phenomenon of polarization.
2. Are the processes of radiation of electromagnetic waves atoms dependent?
3. Give the definition of natural light.
4. Is it possible to say that the direction of the oscillations of the electric field intensity vector is equally probable for polarized light?
5. Specify the angle between the electric field strength vector of the electromagnetic wave and the light beam for linearly polarized light.
6. Calculate the degree of polarization for natural light.
7. Can polarisers be used to analyze the polarized light?

8. Formulate the Malus's law.
9. Calculate the intensity of light at the exit of two tourmaline crystals, the optical axes of which are perpendicular.
10. Indicate the direction of polarization that is predominant in the refracted rays.
11. Under what conditions the reflected light is completely polarized?
12. Specify the angle between the reflected and refracted rays for the case when light falls on the interface between two dielectrics at the Brewster's angle.
13. Describe the method by which you can significantly increase the degree of polarization of the same light beam.
14. Define the birefringence property.
15. Is there a separation for the light beam into two beams in the case of movement along the optical axis?
16. Specify the difference between uniaxial and biaxial crystals.
17. Is it true to say that a beam incident on a uniaxial crystal is perpendicular to the principal plane of the crystal?
18. Explain the inequality of the refractive indices of ordinary and extraordinary rays.
19. Describe the Nichol prism.
20. Write the expression for the optical path length in the quarter-wave plate.

Problem-solving examples

Problem 12.1

Problem description. A parallel beam of light passes from glycerol to glass so that the beam reflected from the interface between these media is as polarized as possible. Determine the angle between the incident and refracted beams.

Known quantities: $n_1 = 1.47$; $n_2 = 1.5$; $\alpha = i_B$.

Quantities to be calculated: φ.

Problem solution. From the law of refraction of light we get

$$\frac{\sin\alpha}{\sin\beta} = \frac{n_2}{n_1}, \tag{12.1.1}$$

where α is the angle of incidence;

β is the angle of refraction;

n_1, n_2 are absolute refractive indices in which the incident (glycerin) and refracted (glass) rays passed, respectively.

Therefore, we get

$$\sin\beta = \frac{n_1 \sin\alpha}{n_2}. \tag{12.1.2}$$

Reflected light is completely polarized in the event of a Brewster falling:

$$\alpha = i_B, \tag{12.1.3}$$

where $i_Б$ is the Brewster's angle.

According to Brewster's law

$$tgi_Б = n_{21} = \frac{n_2}{n_1}, \tag{12.1.4}$$

where n_{21} is a relative refractive index. Numerically:

$$\alpha = arctg\frac{n_2}{n_1} = 45.57^0. \tag{12.1.5}$$

Then

$$\beta = \arcsin\left(\frac{n_1 \sin\alpha}{n_2}\right) = 44.42^0. \tag{12.1.6}$$

We can write for the angles

$$\varphi = 180^0 - \alpha + \beta = 178.85^0. \tag{12.1.7}$$

Answer. The angle between the incident and refracted rays is $\varphi = 178.85^0$.

Problem 12.2

Problem description. A beam of light passes successively through two nicols, whose transmission planes form an angle of 40^0 between them. Assuming that the absorption coefficient of each nicole is 0.15, calculate how many times the beam of light coming out of the second nicole is weakened compared to the beam incident on the first nicole.

Known quantities: $\alpha = 40^0$; $k = 0.15$.

Quantities to be calculated: I_0 / I_2.

Problem solution. For the case of the passage of light through the first nicole (in the case of natural light), we obtain

$$I_1 = \frac{1}{2} I_0 (1-k), \tag{12.2.1}$$

where I_0 is the intensity of natural light; k is the absorption coefficient. Then

$$\frac{I_0}{I_1} = \frac{2}{1-k} = 2.35. \tag{12.2.2}$$

After the light has passed through the second nicole, the light intensity is equal to

$$I_2 = I_1 \cos^2 \alpha (1-k) = \frac{I_0 \cos^2 \alpha (1-k)^2}{2}. \tag{12.2.3}$$

Then

$$\frac{I_0}{I_2} = \frac{2}{\cos^2 \alpha (1-k)^2} = 4.72. \tag{12.2.4}$$

Answer. The relative attenuation of the beam is $\frac{I_0}{I_2} = 4.72$.

Problem 12.3

Problem description. The angle of refraction of the beam in the liquid is 35^0. Determine the refractive index of the fluid, if it is known that the reflected light beam is as polarized as possible.

Known quantities: $i_2 = 35^0$.

Quantities to be calculated: n.

Problem solution. A beam of light reflected from a dielectric is maximally polarized if the tangent of the angle of incidence is numerically equal to the relative refractive index

$$tgi_1 = n. \tag{12.3.1}$$

The law of refraction can be written in the form

$$\frac{\sin i_1}{\sin i_2} = n, \tag{12.3.2}$$

where i_1, i_2 are angles of incidence and refraction;
n is a relative refractive index.

Equate the left parts of the equations

$$\frac{\sin i_1}{\cos i_1} = \frac{\sin i_1}{\sin i_2}. \quad (12.3.3)$$

Substitute numeric data

$$\sin i_2 = \cos i_1 = 0.57 . \quad (12.3.4)$$

Since

$$tg i_1 = n, \quad (12.3.5)$$

then

$$\frac{\sqrt{1-\cos^2 i_1}}{\cos i_1} = n, \quad (12.3.6)$$

i.e. $n = 1.44$.

Answer. The refractive index of the liquid is $n = 1.44$

Problems

Problem A

Problem description. The Brewster angle when light falls from the air onto a rock salt crystal is $i_B = 57°$. Determine the speed of light in this crystal.

Answer. $\upsilon = 1.94 \times 10^8\ m/s$.

Problem B

Problem description. The analyzer reduces the intensity of the light coming to it from the polarizer by 2 times. Determine the angle between the transmittance planes of the polarizer and the analyzer. The loss of light intensity in the analyzer is neglected.

Answer. $\varphi = 45°$.

Problem C

Problem description. Determine the relative attenuation of the intensity of light passing through two nicols whose transmission planes form an angle of $\alpha = 30°$. The incident light loses 10% of the intensity as each of the nicols passes.

Answer. $\mathrm{N} = 3.3$.

Problem D

Problem description. The analyzer is placed on the path of partially polarized light. The degree of polarization of light is $P = 0.6$. The intensity of the light passing through the analyzer is maximum. How many times will the light intensity decrease if the analyzer transmittance plane is rotated through an angle of $\alpha = 30°$?

Answer. $\mathrm{N} = 1.23$.

Problem E

Problem description. Plate of quartz with a thickness of $d = 2\ mm$, cut perpendicular to the optical axis. The plate is placed between parallel nikole crystals. As a result, the plane of polarization of light turned at an angle of $\varphi = 53°$. Determine the thickness of the plate at which this monochromatic light does not pass through the analyzer.

Answer. $h = 3.4 \times 10^{-3}\ m$.

CHAPTER 13. DISPERSION OF LIGHT

13.1. Dispersive Prism

A *dispersion of light* is the dependence of the refractive index n of a substance on the frequency ν (wavelength λ) of light or the dependence of the phase velocity of light waves on its frequency ν. Dispersion of light is represented as a relationship $n = f(\lambda)$. A consequence of the dispersion is the expansion into the spectrum of a beam of white light when passing through a prism.

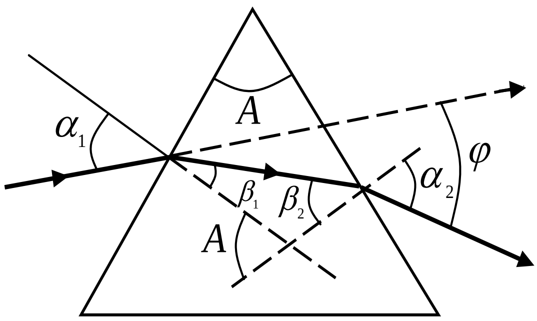


Figure 13.1. Dispersive prism.

Consider the light dispersion in the prism. Let a monochromatic beam of light fall on a prism with a refractive index n (Figure 13.1) at an angle of α_1. After double refraction (on the left and right sides of the prism), the beam is deflected from the original direction by an angle of φ.

It follows from the figure that

$$\varphi = (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) = \alpha_1 + \alpha_2 - A. \quad (13.1)$$

Suppose that the angles A and α_1 are small, then the angles $\alpha_2, \beta_1 \beta_2$ will also be small, and instead of the sinus of these angles, one can use their values. Therefore,

$$\alpha_1 / \beta_1 = n,\ \alpha_2 / \beta_2 = 1/n \quad (13.2)$$

and since

$$\beta_1 + \beta_2 = A, \quad (13.3)$$

then

$$\alpha_1 + \alpha_2 = nA. \quad (13.4)$$

Consequently,

$$\varphi = A(n-1), \quad (13.5)$$

that is, the angle of deflection of the rays by the prism is greater the larger the refracting angle of the prism.

The angle of deflection of the rays by the prism depends on the value of $n-1$, and n is a function of the wavelength, so the rays of different wavelengths after passing through the prism will be rejected at different angles, i.e. the beam of white light behind the prism splits into a spectrum. Thus, with the help of a prism, as well as with the aid of a diffraction grating, it is possible to determine its spectral composition of light by expanding it into a spectrum.

Consider the differences in the diffraction and prismatic spectra.

1. The diffraction grating decomposes the incident light directly over the wavelengths, therefore, using the values of the measured angles (along the directions of the corresponding maxima), one can calculate the wavelength. Decomposition of light into the spectrum in the prism occurs according to the values of the refractive index.
2. Compound colours in the diffraction and prismatic spectra are arranged in different ways. The sine of the deflection angle in the diffraction grating is proportional to the wavelength. Consequently, the red rays, having a longer wavelength than the violet ones, are deflected more strongly by the diffraction grating. The prism also decomposes the rays into the spectrum according to the values of the refractive index, which decreases monotonically for all transparent substances with wavelength increasing. Consequently, the red rays, which have a smaller refractive index than the violet ones, are deflected less by the prism.

13.2. Normal and Anomalous Dispersion

The quantity $D = \frac{dn}{d\lambda}$ is called the <u>*dispersion*</u> of matter. Dispersion of the substance shows how quickly the refractive index changes with the wavelength. It follows from the figure that the refractive index for transparent substances decreases monotonically with decreasing wavelength; consequently, the value $dn/d\lambda$ modulo also increases with decreasing λ.

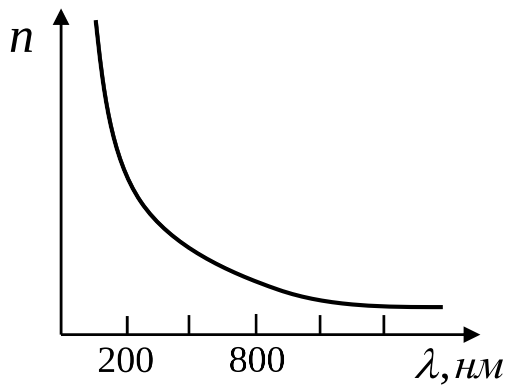


Figure 13.2. Normal dispersion.

Such dispersion is called *normal*. The shape of the curve $n(\lambda)$, called the *dispersion curve*, near the lines and absorption bands will be different: the value of n decreases with decreasing λ. This type of dependence n on λ is called *anomalous dispersion*.

The effect of prism spectrographs is based on the phenomenon of normal dispersion. Despite their certain shortcomings (for example, the need for calibration, different dispersion in different parts of the spectrum) in determining the spectral composition of light, prism spectrographs find wide application in spectral analysis. This is because the manufacture of good prisms is much simpler than the production of good diffraction gratings. Prism spectrographs, moreover, have a greater light-gathering power.

It follows from Maxwell's macroscopic electromagnetic theory that the absolute refractive index of the medium is $n = \sqrt{\varepsilon\mu}$, where ε is the dielectric constant of the medium, μ is the magnetic permeability.

In the optical region of the spectrum, for all substances $\mu \approx 1$, therefore, $n = \sqrt{\varepsilon}$. These formulas contain some contradictions with experience: the quantity n, being a variable, remains at the same time equal to a certain constant $\sqrt{\varepsilon}$. In addition, the values of n obtained from this expression do not agree with the experimental values.

13.3. Electron Theory of Dispersion

The difficulties of explaining the dispersion of light from the point of view of Maxwell's electromagnetic theory are eliminated by the electron Lorentz theory. The dispersion of light is regarded in Lorentz theory as the result of the interaction of electromagnetic waves with charged particles that make up the substance and perform forced oscillations in the alternating electromagnetic field of the wave.

We apply the electronic theory of the dispersion of light for a homogeneous dielectric, assuming formally that the dispersion of light is a consequence of the dependence of ε on the frequency ω of light waves. The permittivity of a substance is, by definition,

$$\varepsilon = 1 + \chi = 1 + P/(\varepsilon_0 E), \tag{13.6}$$

where χ is the dielectric susceptibility of the medium,

ε_0 is the electric constant,

P is the instantaneous value of polarization.

Consequently,

$$n^2 = 1 + P/(\varepsilon_0 E), \tag{13.7}$$

that is, the refractive index depends on P. In this case, the main significance is due to electron polarization that is forced oscillations of electrons under the influence of the electric component of the wave field. For orientation polarization of molecules, the frequency of oscillations in the light wave is very high $\nu \approx 10^{15}$ Hz.

In the first approximation, we can assume that the forced oscillations are performed only by the outer electrons, which are the weakest in the nucleus, namely the optical electrons. For simplicity, let us consider the oscillations of only one optical electron. The induced dipole moment of an electron making forced oscillations is

$$p = eX, \tag{13.8}$$

where e is the charge of an electron,

X this is the displacement of an electron under the action of the electric field of a light wave.

For the case when the concentration of atoms in the dielectric is n_0, the instantaneous value of the polarization is

$$P = n_0 p = n_0 eX. \tag{13.9}$$

Then we can obtain

$$n^2 = 1 + n_0 eX/(\varepsilon_0 E). \tag{13.10}$$

Consequently, the problem reduces to determining the displacement X of an electron under the action of an external field E. The field of a light wave will be considered a function of frequency ω, that is, varying in harmonic order:

$$E = E_0 \cos \omega t . \tag{13.11}$$

The equation of forced oscillations of an electron for the simplest case (without taking into account the resistance force, which determines the absorption of the energy of the incident wave) will be written in the form

$$\ddot{X} + \omega_0{}^2 X = \frac{F_0}{m}\cos\omega t = \frac{e}{m} E_0 \cos\omega t , \tag{13.12}$$

where $F_0 = \varepsilon E_0$ is the amplitude value of the force acting on the electron from the side of the wave field,

$\omega_0 = \sqrt{\frac{k}{m}}$ is the eigenfrequency of the electron's oscillations,

m is the mass of the electron.

Solving the equation, we find the solution $\varepsilon = n^2$ as a function of the atom constants (e, m, ω_0) and the frequency ω of the external field, that is, we solve the dispersion problem.

The solution of the equation can be written in the form

$$X = A\cos\omega t , \tag{13.13}$$

where $A = \dfrac{eE_0}{m\left(\omega_0{}^2 - \omega^2\right)}$.

Substituting the solution into the expression for n^2, we obtain

$$n^2 = 1 + \frac{n_0 e^2}{\varepsilon_0 m}\frac{1}{\left(\omega_0{}^2 - \omega^2\right)}. \tag{13.14}$$

If there are different charges e_i in the substance that perform forced oscillations with different natural frequencies ω_{0i}, then

$$n^2 = 1 + \frac{n_0}{\varepsilon_0}\sum_i \frac{e_i{}^2/m}{\omega_{0i}{}^2 - \omega^2}, \tag{13.15}$$

where m_i is the mass of the charge e_i.

From these expressions it follows that the refractive index n depends on the frequency ω of the external field, that is, the functional dependences do confirm the phenomenon of light dispersion, although under the assumptions indicated above, which in the future must be eliminated.

In the range from $\omega = 0$ to $\omega = \omega_0$, the value of n^2 is greater than unity and increases with increasing ω (normal dispersion); at $\omega = \omega_0$ we have $n^2 = \pm\infty$; in the range from $\omega = \omega_0$ to $\omega = \infty$, the value of n^2 is less than unity and increases from $-\infty$ to 1 (normal dispersion).

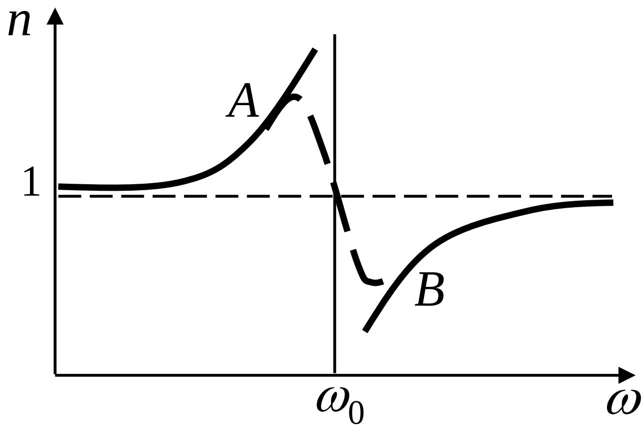


Figure 13.3. Anomalous dispersion.

Figure 13.3 presents the dependence of refractive index n on the value of ω. Similar behaviour of n near the eigenfrequency ω_0 was obtained as a result of the assumption that there are no resistance forces for electron oscillations. If this circumstance is taken into account, then the graph of the function $n(\omega)$ near ω_0 is given by the dashed line AB. Region AB describes to the anomalous dispersion (n decreases with increasing ω), the remaining sections of the dependence n of ω describe the normal dispersion (n increases with increasing ω).

13.4. Absorption of Light

Absorption of light is the phenomenon of energy loss by a light wave passing through a substance, due to the conversion of wave energy into other forms (internal energy of matter and secondary energy of other directions and spectral composition). As a result of absorption, the intensity of light when passing through matter decreases.

The absorption of light in matter is described by *Bouguer's law* (the law was discovered by French geophysicist Pierre Bouguer (1698–1758)):

$$I = I_0 e^{-\alpha X}, \tag{13.16}$$

where I_0 and I are the intensities of a plane monochromatic light wave at the entrance and exit of a layer of an absorbing substance of thickness X,

α is the absorption coefficient, depending on the wavelength of light, chemical nature and state of matter and not which depends on the intensity of light. At $X = 1/\alpha$, the intensity I of light decreases by a factor of e compared with I_0.

The absorption coefficient depends on the wavelength λ (or frequency ω) and depends on the nature of the substance. For example, monatomic gases and metal vapours (that is, substances in which the atoms are located at considerable distances from each other and can be considered isolated) have a near-zero absorption coefficient and only for very narrow spectral regions (approximately 10^{-12} – 10^{-11} m) there are sharp maxima (so-called <u>*line absorption spectrum*</u>). These lines correspond to the frequencies of the natural oscillations of the electrons in atoms. The absorption spectrum of molecules, determined by the oscillations of atoms in molecules, is characterized by absorption bands (about 10^{-10} – 10^{-7} m).

The absorption coefficient for dielectrics is small (about 10^{-3} – 10^{-5} cm^{-1}), however, selective absorption of light is observed in certain intervals of wavelengths, when α sharply increases, and relatively wide absorption bands are observed, i.e. dielectrics have a continuous absorption spectrum. This is due to the fact that there are no free electrons in dielectrics and the absorption of light is due to the phenomenon of resonance under forced oscillations of electrons in atoms and atoms in dielectric molecules.

The absorption coefficient for metals has large values (about 10^3 – 10^5 cm^{-1}) and therefore metals are opaque to light. There are free electrons in metals. These electrons move under the action of the electric field of the light wave, which leads to the appearance of rapidly varying currents. The presence of currents in metals is accompanied by the release of Joule heat. Therefore, the energy of the light wave decreases rapidly, turning into the internal energy of the metal.

The typical dependence of the absorption coefficient α on the wavelength λ of light and the dependence of the refractive index n on λ in the region of the absorption band is shown in Figure 13.4. Anomalous dispersion is observed inside the absorption band (n decreases with decreasing λ). However, the absorption of matter must be significant in order to influence the refractive index.

The coloring of absorbing bodies can be explained by the dependence of the absorption coefficient on the wavelength. For example, glass, slightly absorbing red and orange rays and strongly absorbing green and blue, when illuminated with white light, will appear red. If you send green and blue light to such a glass, then because of the strong absorption of light of these wavelengths, the glass will appear black.

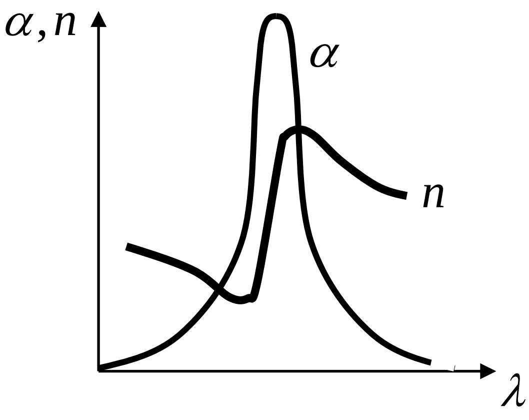


Figure 13.4. Absorption band.

The absorption phenomenon is widely used in the absorption spectral analysis of a gas mixture based on measurements of the frequency spectra and intensities of absorption lines (bands).

The structure of the absorption spectra is determined by the composition and structure of the molecules, and therefore the study of absorption spectra is one of the main methods of quantitative and qualitative study of substances.

Test questions

1. Does the phase velocity of light waves depend on their frequency?
2. Can we talk about the implementation of the law of refraction for white light as it passes through the prism?
3. What physical factors affect the angle of deflection of rays when passing through a prism?
4. Consider the case of rays passing through a prism for the case when the refractive index of the prism is less than the refractive index of the environment.
5. Describe the differences in the diffraction and prismatic spectra.
6. Is it true that the red rays deviate less when passing through the prism and diffraction grating?
7. Write a formula to determine the dispersion of matter.
8. Draw a graph of the refractive index versus wavelength when passing through transparent media.
9. Define the normal dispersion.
10. Consider the behaviour of the dispersion curve near the absorption bands.
11. What type of dispersion is called anomalous dispersion?
12. What physical phenomenon is used when working with spectrographic prisms?
13. Specify the shortcomings in determining the spectral composition of light by spectrographic prisms.
14. Indicate the reason that in the optical region of the spectrum it is sufficient to use the dielectric constant to determine the refractive index of the medium.
15. Point out the theory that eliminates the difficulties of explaining the dispersion of light from the point of view of Maxwell's electromagnetic theory.
16. Write a formula for the relationship of the refractive index and the polarization value.
17. Indicate the effect of electron displacement due to external force on refractive index.
18. Write a formula that determines the eigenfrequency of the electron's oscillations according to the Lorentz theory.
19. Describe the functional dependence of the refractive index on the frequency of the external field.
20. Write down the Bouguer's law.

Problem-solving examples

Problem 13.1

Problem description. Determine the refractive index of carbon dioxide under normal conditions. The polarizability of the CO_2 molecule is $\beta = 3.3 \times 10^{-29}\ m^3$.

Known quantities: $\beta = 3.3 \times 10^{-29}\ m^3$.

Quantities to be calculated: n.

Problem solution. The refractive index n and the dielectric constant ε are related by

$$n = \sqrt{\varepsilon}\,. \tag{13.1.1}$$

For the dielectric constant, we can write the formula

$$\varepsilon = 1 + \chi\,, \tag{13.1.2}$$

where χ is dielectric susceptibility.

The dielectric susceptibility is proportional to the concentration of gas molecules

$$\chi = N\beta\,, \tag{13.1.3}$$

where N is the number of gas molecules,

β is the polarizability of a single molecule.

In this way

$$n = \sqrt{1 + N\beta}\,. \tag{13.1.4}$$

Under normal conditions, the concentration of CO_2 molecules is $N = 2.687 \cdot 10^{25}\ m^{-3}$.

Substitute the numeric values in the last formula

$$n = 1.00044. \tag{13.1.5}$$

Answer. The refractive index of carbon dioxide is $n = 1.00044$.

Problem 13.2

Problem description. The study of the passage of an electromagnetic wave with a frequency of $\nu = 8\ MHz$ through a flat homogeneous plasma layer with a free-electron concentration of $N = 10^{12}\ m^{-3}$ showed that with an increase in the layer thickness $\eta_d = 2$ times, the energy transmission coefficient τ changes $\eta_\tau = 10$ times. Neglecting the reflection of the wave at the boundaries, find the thickness d of the plasma layer.

Known quantities: $\nu = 8\ MHz$, $N = 10^{12}\ m^{-3}$, $\eta_d = 2$, $\eta_\tau = 10$.

Quantities to be calculated: d.

Problem solution. According to Bouguer's law, the intensity of a wave in a medium is

$$I(d) = I_0 \exp(-\alpha d),$$
$$I(\eta_d d) = I_0 \exp(-\alpha \eta_d d) \qquad (13.2.1)$$

where I_0 is the intensity of the wave outside the medium,
d is the thickness of the layer,
$\alpha = 2\frac{\omega}{c}n''$ is the absorption coefficient,
ω is the frequency of the electromagnetic wave,
c это скорость света в вакууме,
n'' is an indicator of absorption,
η_d is the multiplicity of change in the energy transmittance.
According to the problem

$$\frac{I(d)}{I(\eta_d d)} = \eta_\tau. \qquad (13.2.2)$$

The concentration of free electrons is $N = 10^{12}\ m^{-3}$. Then the plasma frequency is

$$\omega_p = \frac{e}{\sqrt{\varepsilon_0 m}}\sqrt{N} = 56.7 \cdot 10^6\ s^{-1}, \qquad (13.2.3)$$

where ε_0 is an electrical constant;
e is an electron charge,
m is the electron mass.
The cyclic frequency of the wave is

$$\omega = 2\pi\nu = 50.3 \cdot 10^6\ s^{-1}, \qquad (13.2.4)$$

where ν is the linear frequency of the wave.

Consequently, the dielectric constant of the plasma corresponding to this frequency is equal to

$$\varepsilon = 1 - \frac{\omega_p^{\ 2}}{\omega^2} = -0.272 < 0 . \tag{13.2.5}$$

Thus, for the indices of refraction and absorption, we obtain, respectively $n' = 0$, $n'' = \sqrt{|\varepsilon|}$.

Finally

$$d = \frac{\ln \eta_\tau}{\eta_d \alpha} = \frac{\ln 10}{2\frac{\omega}{c} n''} = 13.5 . \tag{13.2.6}$$

Answer. The thickness of the plasma layer is equal to $d = 13.5$.

Problem 13.3

Problem description. The refractive index of a transparent medium near the frequency $\omega*$ varies according to the law $n(\omega) = n_0 - \dfrac{A}{\omega - \omega_0}$, where $n_0 = 1.5$, $\omega_0 = 4 \times 10^{14}\ s^{-1}$, $A = const$, $\omega < \omega_0$. A short light pulse passes through a layer of matter whose thickness is $l = 3\ cm$. The spectral composition $\left(\omega * - \dfrac{\Delta\omega}{2}, \omega * + \dfrac{\Delta\omega}{2}\right)$ of this light pulse is rather narrow $\Delta\omega << |\omega * - \omega_0|$. Estimate the time τ it takes for a pulse to pass through a layer if $|\omega * - \omega_0| \approx 10^{12}\ s^{-1}$ and $|n(\omega *) - n_0| = 0.01$.

Known quantities: $n_0 = 1.5$, $\omega_0 = 4 \times 10^{14}\ s^{-1}$, $A = const$, $l = 3\ cm$, $|\omega * - \omega_0| \approx 10^{12}\ s^{-1}$, $|n(\omega *) - n_0| = 0.01$.

Quantities to be calculated: τ.

Problem solution. The time it takes for a pulse to pass through a layer

$$\tau = \frac{l}{u}, \tag{13.3.1}$$

where l is the thickness of the layer,

u is the group velocity of light in the medium.

The group velocity of light in the environment is represented by the formula

$$u = \frac{c}{n + \omega^* \dfrac{dn}{d\omega}}. \qquad (13.3.2)$$

where ω^* is a fixed frequency,
c is the speed of light in a vacuum,
n is the refractive index,
ω is the frequency of the electromagnetic wave, which is not much different from the frequency ω^*.

Using numerical data from the problem statement, we get

$$\frac{dn}{d\omega} = \frac{n_0 - n^*}{\omega^* - \omega_0} = 10^{-14}\ s,$$
$$\omega^* \approx \omega_0 = 4 \cdot 10^{14}\ s^{-1}. \qquad (13.3.3)$$

In this approximation, the group velocity is $u = 0.545\ m/s$. The time it takes for a pulse to pass through a layer is equal to $\tau = 5.5 \times 10^{-10}\ s$.

Answer. The time it takes for a pulse to pass through a layer is $\tau = 5.5 \times 10^{-10}\ s$.

Problems

Problem A

Problem description. Determine the refractive index of carbon dioxide under normal conditions. The polarizability of the molecule CO_2 is $\beta = 3.3 \times 10^{-29}\ m^3$.

Answer. $n = 1.00044$.

Problem B

Problem description. The concentration of electrons in the Sun at a distance of $r = 0.06R$ from the border of the photosphere ($R = 6.95 \times 10^8\ m$ is the radius of the Sun) is approximately equal to $N = 2 \times 10^{14}\ m^{-3}$. Find the maximum wavelength that can reach the Earth from this region of the Sun.

Answer. $\lambda_{\max} = 2.3\ m$.

Problem C

<u>*Problem description*</u>. A plane electromagnetic wave with a frequency of $\nu = 8\ MHz$ passes through a flat homogeneous plasma layer with a free electron concentration of $N = 10^{12}\ m^{-3}$. An increase in the plasma layer thickness by 2 times leads to a change in the transmittance by 10 times. Find the thickness d of the plasma layer. The reflection of the wave at the boundaries of the plasma layer can be neglected.

<u>*Answer*</u>. $d = 13.5\ m$.

Problem D

<u>*Problem description*</u>. Pulsed radiation from one of the pulsars at frequency $\nu_1 = 80\ MHz$ reaches Earth at time $\Delta t = 7\ s$ later than the pulse at frequency $\nu_2 = 2000\ MHz$. Determine the distance to the pulsar if the average concentration of electrons in interstellar space is $N = 0.05\ cm^{-2}$.

<u>*Answer*</u>. $L = 7 \times 10^{18}\ m$.

Problem E

<u>*Problem description*</u>. The light beam propagates parallel to the surface of the earth. Considering the air still, calculate the deviation Δh of the beam on the path $L = 1\ km$, if the air pressure is $P_0 = 1\ at$, the temperature is $T = 300\ K$, and the refractive index of air is $n = 1 + 3 \times 10^{-4}$.

<u>*Answer*</u>. $\Delta h = 1.7 \times 10^{-2}\ m$.

CHAPTER 14. QUANTUM NATURE OF RADIATION

14.1. Thermal Radiation. Kirchhoff's Law

Heating the bodies to sufficiently high temperatures leads to their luminescence. The glow of bodies due to heating is called *thermal radiation* is the radiation of electromagnetic waves generated by the thermal motion of particles in matter. Thermal radiation, being the most widespread in nature, is accomplished at the expense of the energy of thermal motion of atoms and molecules of matter (i.e., due to its internal energy) and is characteristic of all bodies at temperatures above 0 K. Thermal radiation is characterized by a continuous spectrum, the position of the maximum of which depends on the temperature. At high temperatures, short (visible and ultraviolet) electromagnetic waves are emitted, at low temperatures, long (infrared) radiation is emitted.

Thermal radiation is practically the only type of radiation that can be in equilibrium. Suppose that a heated (radiating) body is placed in a cavity bounded by an ideally reflecting shell. As a result of the continuous exchange of energy between the body and the radiation, there will be equilibrium, that is, the body per unit time will absorb as much energy as it radiate. Let us assume that the equilibrium between the body and the radiation is broken for some reason and the body emits more energy than absorbs. If the body radiates more per unit of time than absorbs (or vice versa), then the body temperature will begin to drop (or increase). As a result, the amount of energy emitted by the body will be weakened (or increase) until finally equilibrium is established. All other types of radiation are non equilibrium.

The quantitative characteristic of thermal radiation is the spectral density of the energy luminosity (emissivity) of the body, i.e. the radiation power per unit surface area of the body in the frequency interval of unit width

$$R_{\nu,T} = \frac{dW_{\nu,\nu+d\nu}}{d\nu}, \tag{14.1}$$

where $dW_{\nu,\nu+d\nu}$ is the energy of electromagnetic radiation emitted per unit of time (radiation power) per unit surface area of the body in the frequency range from ν to $\nu + d\nu$.

The recorded formula can be represented as a function of the wavelength

$$dW_{\nu,\nu+d\nu}{}^{em} = R_{\nu,T} = R_{\lambda,T} d\lambda. \tag{14.2}$$

Since $c = \lambda / \nu$, then

$$d\lambda / d\nu = -c/\nu^2 = -\lambda^2 / c, \tag{14.3}$$

where the minus sign indicates that with increasing one of the values (ν or λ) another value decreases. Therefore, in the sequel the minus sign will be omitted. In this way,

$$R_{\nu,T} = R_{\lambda,T}\left(\lambda^2 / c\right). \tag{14.4}$$

Using this formula, you can go from $R_{\nu,T}$ to $R_{\lambda,T}$, and vice versa.

Using the value of the spectral density of the energy luminosity, it is possible to calculate the integral energy luminosity (integral radiation) (it is simply called the energy luminosity of the body), summing over all frequencies

$$R_T = \int_0^\infty R_{\nu,T} d\nu . \tag{14.5}$$

The ability of bodies to absorb the radiation incident on them is characterized by the *spectral absorption capacity*

$$A\nu,T = \frac{dW_{\nu,\nu+d\nu}{}^{ab}}{dW_{\nu,\nu+d\nu}}, \tag{14.6}$$

which shows how much of the energy delivered per unit time per unit area of the body by electromagnetic waves incident on it with frequencies from ν to $\nu + d\nu$ is absorbed by the body. The spectral absorption capacity is a dimensionless quantity. The values $R_{\nu,T}$ and $A_{\nu,T}$ depend on the nature of the body, its thermodynamic temperature, and at the same time depend on the frequency of radiation. Therefore, these values are attributed to certain T and ν (or rather, to a fairly narrow frequency range from ν to $\nu + d\nu$).

The body, capable of absorbing completely all the radiation incident on it at any temperature of any frequency, is called *black body*.

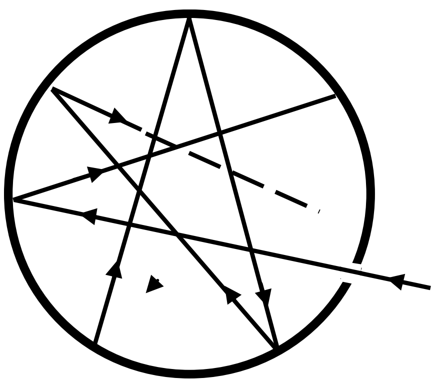


Figure 14.1. Black body model.

Consequently, the spectral absorption capacity of a black body for all frequencies and temperatures is identically equal to unity ($A_{\nu,T}{}^{b} = 1$). Absolutely black bodies are not found in nature, however, such bodies as soot, platinum black, black velvet and some others, in a certain frequency range, are similar in properties to them.

The ideal model of the black body is a closed cavity with a small aperture O, the inner surface of which is blackened (Figure 14.1). A beam of light trapped inside such a cavity undergoes multiple reflections from the walls, as a result of which the intensity of the outgoing radiation is practically zero. Experience shows that when the size of the hole is smaller 0.1 of the cavity diameter, the incident radiation of all frequencies is "completely absorbed". Because of this, the open windows of the houses from the street side seem to be black, although inside the rooms it is light enough because of the reflection of light from the walls.

In addition to the concept of a black body, the concept of a gray body is also used. Body, the absorption capacity of which is less than one, but is the same for all frequencies and depends only on the temperature, material and state of the surface of the body is called *gray body*. Thus, for a gray body

$$A_{\nu,T}{}^{c} = A_T = const < 1. \tag{14.7}$$

The study of thermal radiation played an important role in the creation of the quantum theory of light. German physicist Gustav Robert Kirchhoff (1824–1887), using the second law of thermodynamics and analyzing the conditions of equilibrium radiation in an isolated system of bodies, established a quantitative relationship between the spectral density of energy luminosity and the spectral absorptive capacity of bodies. The ratio of the spectral density of the energy luminosity to the spectral absorption capacity does not depend on the nature of the body; it is for all bodies a universal function of frequency (wavelength) and temperature (*Kirchhoff's law*)

$$R_{\nu,T} / A_{\nu,T} = r_{\nu,T}. \tag{14.8}$$

For the black body

$$A_{\nu,T}{}^{b} = 1, \tag{14.9}$$

therefore, it follows from Kirchhoff's law that the value of $R_{\nu,T}$ for a black body is $r_{\nu,T}$. Thus, the universal *Kirchhoff function* is nothing else than the spectral density of the energy luminosity of a black body. Consequently, according to Kirchhoff's law, the ratio of the spectral density of the energy luminosity to the spectral absorption capacity for all bodies is equal to the spectral density of the energy luminosity of the black body at the same temperature and frequency.

It follows from Kirchhoff's law that the spectral density of the energy luminosity of any body in any region of the spectrum is always less than the spectral density of the energy luminosity of the black body (for the same values of T and ν), since $A_{\nu,T} < 1$ and therefore

$$R_{\nu,T} < r_{\nu,T}. \tag{14.10}$$

In addition, if the body does not absorb electromagnetic waves of any frequency, then it does not emit them, since at $A_{\nu,T} = 0$ we get $R_{\nu,T} = 0$.

Using the Kirchhoff law, the expression for the energy luminosity of a body can be given in the form

$$R_T = \int_0^\infty A_{\nu,T} \cdot r_{\nu,T} d\nu \,. \qquad (14.11)$$

For a gray body

$$R_T{}^c = A_T \int_0^\infty r_{\nu,T} d\nu = A_T R_e \,, \qquad (14.12)$$

where

$$R_e = \int_0^\infty r_{\nu,T} d\nu \qquad (14.13)$$

is the energy luminosity of a black body (depends only on temperature).

Kirchhoff's law describes only thermal radiation. Radiation that is not described by Kirchhoff's law is not thermal radiation.

14.2. Stefan-Boltzmann Law and Wien's Displacement Law

From Kirchhoff's law it follows that the spectral density of the energy luminosity of a black body is a universal function, so the finding of its explicit dependence on frequency and temperature is an important task of the theory of thermal radiation.

According to the *Stefan-Boltzmann law*, the dependence of the energy luminosity R_e on temperature T is as follows

$$R_e = \sigma T^4 \,, \qquad (14.14)$$

i.e. the energy luminosity of a black body is proportional to the fourth power of its thermodynamic temperature; and σ is the Stefan-Boltzmann constant.

The Stefan-Boltzmann law, determining the temperature dependence of R_e, does not give a reply regarding the spectral composition of the blackbody radiation.

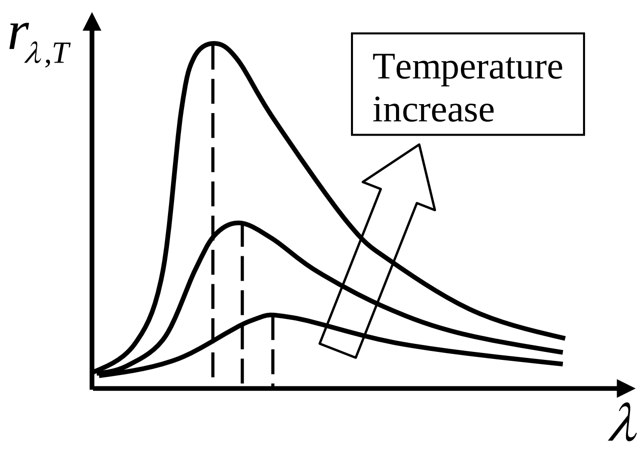


Figure 14.2. Dependence of $r_{\lambda,T}$ on the wavelength λ.

From the experimental curves of the dependence of the function $r_{\lambda,T}$ ($r_{\lambda,T} = \left(c/\lambda^2\right) r_{\nu,T}$) on the wavelength λ at different temperatures (Figure 14.2) it follows that the energy distribution in the spectrum of the black body is non uniform. All curves have an explicit maximum, which, with increasing temperature, shifts toward shorter wavelengths. The area bounded by the curve of the dependence of $r_{\lambda,T}$ on λ and the abscissa axis is proportional to the energy luminosity R_e of the black body and, consequently, according to the Stefan-Boltzmann law, the fourth power of temperature.

According to the Wien's displacement law, the dependence of the wavelength $\lambda_{\max}$ corresponding to the maximum of the function $r_{\lambda,T}$ on the temperature T has the form

$$\lambda_{\max} = b/T, \tag{14.15}$$

that is, the wavelength $\lambda_{\max}$ corresponding to the maximum value of the spectral density $r_{\lambda,T}$ of the energy luminosity of a black body is inversely proportional to its thermodynamic temperature, where b is the Wien's displacement constant. The expression is therefore called the *Wien's displacement law*, that it shows the displacement of the position of the maximum of the function $r_{\lambda,T}$ as the temperature increases to a region of short wavelengths. Wien's displacement law explains why, when the temperature of heated bodies decreases, long wave radiation predominates (for example, the transition of white heat into red during the cooling of the metal).

14.3. Planck's Formula

Despite the fact that the Stefan-Boltzmann law and Wien's displacement law play an important role in the theory of thermal radiation, they are particular laws, since they do not give a general picture of the energy distribution with respect to frequencies at different temperatures.

German physicist Max Karl Ernst Ludwig Planck (1858–1947) gave the correct expression consistent with the experimental data for the spectral density of the energy luminosity of a black body. Planck abandoned the established position of classical physics, according to which the energy of any system can change continuously, i.e. it can take any arbitrarily close values. According to the quantum hypothesis put forward by Planck, atomic oscillators emit energy not continuously, but in certain portions, quanta, and the quantum energy is proportional to the oscillation frequency

$$\varepsilon_0 = h\nu = hc/\lambda, \tag{14.16}$$

where h is Planck's constant.

Since the radiation is emitted by quanta, the oscillator energy can only take certain discrete values, which are a multiple of the whole number of elementary portions of energy

$$\varepsilon = nh\nu . \quad (14.17)$$

In this case, the average energy $<e>$ of the oscillator can not be taken equal to kT. The probability that the oscillator is in a state with an energy of ε_0 is proportional to $e^{-\varepsilon n/(kT)}$, but in calculating the average values (for discrete energy values) the integrals are replaced by sums. Under this condition, the average energy of the oscillator is

$$<\varepsilon>= \frac{h\nu}{exp(h\nu / kT)-1}, \quad (14.18)$$

and the spectral density of the energy luminosity of the blackbody is

$$r_{\nu,T} = \frac{2\pi h\nu^3}{c^2[exp(h\nu / kT)-1]}. \quad (14.19)$$

Thus, Planck derived a formula for the universal Kirchhoff's function, which brilliantly agrees with the experimental data on the energy distribution in the blackbody radiation spectra in the entire frequency and temperature range.

In the low-frequency region, i.e. when $h\nu << kT$ (the quantum energy is very small in comparison with the thermal motion energy kT), Planck's formula coincides with the Rayleigh-Jeans formula. From Planck's formula one can obtain the Stefan-Boltzmann law and the Wien's displacement law.

Thus, the Planck formula not only agrees well with the experimental data, but also contains the particular laws of thermal radiation, and also makes it possible to calculate constants in the laws of thermal radiation. Consequently, the Planck's formula is a complete solution of the main problem of thermal radiation, set by Kirchhoff.

The laws of thermal radiation are used to measure the temperature of incandescent and self-luminous bodies (for example, stars). Methods for measuring high temperatures using the dependence of the spectral density of the energy luminosity or the integral energy luminosity of bodies on temperature are called *optical pyrometry*.

14.4. Optical Pyrometry

Instruments for measuring the temperature of heated bodies by the intensity of their thermal radiation in the optical range of the spectrum are called *pyrometers*.

Depending on the law of thermal radiation used to measure the temperature of bodies, distinguish radiation, colour and brightness temperatures.

Radiation temperature is the temperature of a black body, at which its energy luminosity R_e is equal to the energy luminosity of the R_T. In this case, the energy luminosity of body is recorded and, according to the Stefan-Boltzmann law, its radiation temperature is calculated

$$T_r = \sqrt[4]{R_T / \sigma}\,. \tag{14.20}$$

The radiation temperature T_r of a body is always less than its true temperature T. Let's consider the gray body. Then we can write

$$R_T{}^c = A_T R_e = A_T \sigma T^4\,. \tag{14.21}$$

On the other hand,

$$R_T{}^c = \sigma T_r{}^4\,. \tag{14.22}$$

From a comparison of these expressions it follows that

$$T_r = \sqrt[4]{A_T}\,T\,. \tag{14.23}$$

Since $A_T < 1$, then

$$T_r < T\,, \tag{14.24}$$

i.e. the true temperature of the body is always higher than the radiation temperature.

The spectral density of energy luminosity for gray bodies (or bodies close to them in terms of properties) is

$$R_{\lambda,T} = A_T R_{\lambda,T}\,, \tag{14.25}$$

where $A_T = const < 1$. Consequently, the energy distribution in the emission spectrum of a gray body is the same as in the spectrum of a black body having the same temperature. Therefore, the Wien's displacement law applies to gray bodies, that is, knowing the wavelength λ_{max} corresponding to the maximum spectral density of the energy luminosity $R_{\lambda,T}$ of the body, one can determine its temperature

$$T_C = b / \lambda_{max}\,. \tag{14.26}$$

This temperature is called the *colour temperature*. For gray bodies, the colour temperature coincides with the true one. For bodies that are very different from gray (for example, having selective absorption), the concept of colour temperature loses its meaning. In this way, the temperature at the surface of the sun ($T_C \approx 6500\ K$) and the stars is determined.

The *brightness temperature* T_b is the temperature of a blackbody, at which, for a certain wavelength, its spectral density of energy luminosity is equal to the spectral density of the energy luminosity of the body: $r_{\lambda,T_b} = R_{\lambda,T}$, where T is the true temperature of the body. According to Kirchhoff's law we get

$$R_{\lambda,T} / A_{\lambda,T} = r_{\lambda,T} \text{ or } A_{\lambda,T} = r_{\lambda,T_b} / r_{\lambda,T}. \qquad (14.27)$$

Since for non-black bodies $A < 1$, then $r_{\lambda,T_b} < r_{\lambda,T}$ and, therefore,

$$T_b < T, \qquad (14.28)$$

i.e. the true temperature of the body is always higher than the brightness temperature.

14.5. Photoelectric Effect

Planck's hypothesis, which solved the problem of the thermal radiation of a black body, was confirmed and further developed in explaining the photoelectric effect. The discovery and study of the photoelectric effect played an important role in the development of quantum theory.

An *external photoelectric effect* (photoelectric effect) is the emission of electrons by matter under the action of electromagnetic radiation. The external photoelectric effect is observed in solids (metals, semiconductors, dielectrics), as well as in gases on individual atoms and molecules (photoionization).

The first fundamental studies of the photoelectric effect were performed by Russian physicist Alexander Stoletov (1839–1896). The basic scheme for investigating the photoelectric effect is shown in Figure 14.3.

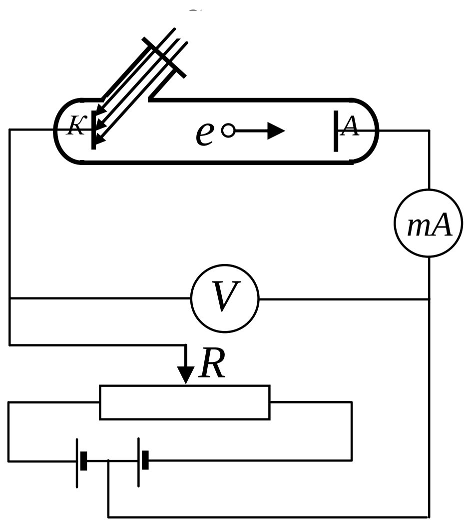


Figure 14.3. Photoelectric effect experiment.

Two electrodes (metal cathode K and a metal mesh anode A) are connected to the battery in a vacuum tube so that not only the value, but also the sign of the voltage applied to them can be changed with the help of the potentiometer R. The current that occurs when the cathode is illuminated by monochromatic light (through a quartz window) is measured by an ammeter.

By irradiating the cathode with light of different wavelengths, Stoletov established the following regularities: 1) ultraviolet radiation has the most effective effect; 2) under the influence of light the substance loses only negative charges; 3) the intensity of the current produced by the action of light is directly proportional to light intensity.

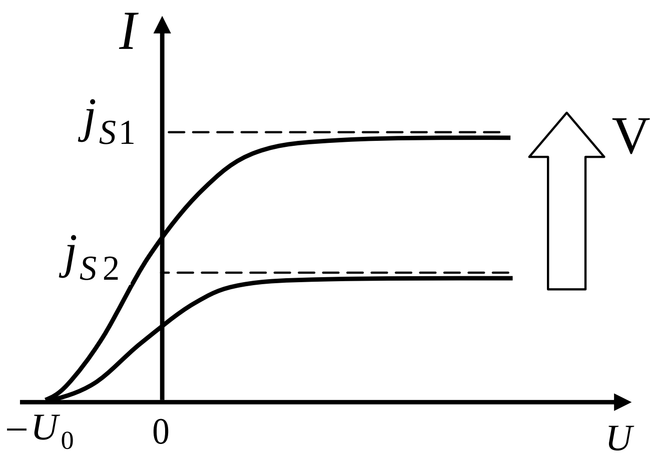


Figure 14.4. The photocurrent current-voltage characteristic.

The *internal photoelectric effect* is the electron-induced transitions of electrons inside a semiconductor or dielectric from bound states to free ones without outward emission. As a result, the concentration of carriers in the body increases, which leads to the appearance of photoconductivity (an increase in the electrical conductivity of the semiconductor or dielectric in its illumination) or to the formation of EMF.

A *barrier-layer photoelectric effect* is the emergence of EMF (photo-EMF) under illumination of a contact between two different semiconductors or a semiconductor and a metal (in the absence of an external electric field). The barrier-layer photoelectric effect thus opens the way for direct conversion of solar energy into electrical energy.

The current-voltage characteristic is the dependence of the photocurrent I (formed by the flux of electrons emitted by the cathode under the action of light) on the voltage between the electrodes. The maximum current value, i.e. the saturation photocurrent is determined by a value of I_S at which all the electrons emitted by the cathode reach the anode

$$I_S = en, \qquad (14.29)$$

where n is the number of electrons emitted by the cathode in 1 s. In order for the photocurrent to become equal to zero, it is necessary to apply a holding voltage of U_0 (Figure 14.4).

Einstein showed that the phenomenon of the photoelectric effect and its regularities can be explained on the basis of the quantum theory of the photoelectric effect proposed by him. According to Einstein, light with frequency ν is not only emitted, as Planck suggested, but also spreads in space and is absorbed by matter in separate portions (quanta), whose energy is

$$\varepsilon_0 = h\nu . \tag{14.30}$$

These quanta of electromagnetic radiation are called *photons*.

The energy of the incident photon is expended on the work A of the electron to work out of the metal and to transfer kinetic energy $m\upsilon^2_{\max}/2$ to the photoelectron. According to the law of conservation of energy

$$h\nu = A + m\upsilon^2_{\max}/2. \tag{14.31}$$

This equation is called the *Einstein equation* for the external photoelectric effect. The Einstein equation allows us to explain the experimental laws of the photoelectric effect.

Test questions

21. Point out the differences between thermal radiation and luminescence.
22. Describe the change in the spectrum of radiation during the transition to higher temperatures.
23. Can we talk about the equilibrium states with luminescent radiation?
24. Give the definition of the spectral density of the body's energy luminosity (emissivity).
25. Write the expression for the spectral density of the body's energy luminosity as a function of wavelength and as a function of frequency.
26. Calculate the integral energy luminosity (integral radiation).
27. What physical quantity expresses the body's ability to absorb radiation incident on it?
28. Specify the units of the spectral absorption capacity.
29. What parameters affect the integral energy luminosity and spectral absorption capacity?
30. Give the definition of the black body.
31. Consider the ideal model of a black body.
32. Does the spectral absorption capacity of a gray body depend on the frequency of the incident radiation?
33. Formulate the Kirchhoff's law.

34. Does the body emit the electromagnetic waves in the case then spectral absorption capacity is zero?
35. Formulate the Stefan-Boltzmann law.
36. Indicate the characteristic features of the distribution of energy in the black body emission spectrum.
37. Formulate the Wien's displacement law.
38. Describe Planck's quantum hypothesis.
39. Write down the Planck's formula for the spectral density of the energy luminosity of the blackbody.
40. Analyze the Einstein equation for the external photoelectric effect in terms of energy balance.

Problem-solving examples

Problem 14.1

Problem description. Determine the maximum spectral density of the energy luminosity, calculated at 1 nm in the emission spectrum of an absolutely black body. Body temperature is 1 K.

Known quantities: $T = 1K$; $\lambda_m = 1\ nm$.

Quantities to be calculated: $r_{\lambda,Tm}$.

Problem solution. According to the Wien's displacement law

$$\lambda_m T = b, \tag{14.1.1}$$

where λ_m is the wavelength corresponding to the maximum spectral density of the absolutely black body radiant exitance;
T is an absolutely black body temperature;
b is the Wien's displacement constant.

Taking into account the numerical data of the problem, for the temperature we obtain

$$T = \frac{b}{\lambda_m} = 2.9 \cdot 10^6\ K. \tag{14.1.2}$$

According to Kirchhoff's law

$$\frac{r_{\lambda,T}}{a_{\lambda,T}} = f(\lambda, T), \tag{14.1.3}$$

where $r_{\lambda,T}$ is the spectral density of the energy luminosity;

$a_{\lambda,T}$ is the absorption capacity of an absolutely black body ($a_{\lambda,T} \equiv 1$);

$f(\lambda, T)$ is the Kirchhoff's function.

Consequently

$$(r_{\lambda,T})_{\max} = f(\lambda_m, T). \tag{14.1.4}$$

The Planck formula we can write in the form

$$f(\lambda_m, T) = \frac{4\pi^2 \hbar c^2}{\lambda^5} \frac{1}{\exp\left(\frac{2\pi \hbar c}{kT\lambda_m}\right) - 1}, \tag{14.1.5}$$

where $\hbar$ is Planck's constant with the dash;

c is the speed of light in a vacuum;

k is the Boltzmann constant.

Substituting the numerical data of the problem, we get

$$(r_{\lambda,T})_{\max} = 3.26 \times 10^{27} \ W/m^3. \tag{14.1.6}$$

Answer. The maximum spectral density of the energy luminosity calculated for 1 nm in the emission spectrum of an absolutely black body is $(r_{\lambda,T})_{\max} = 3.26 \times 10^{27} \ W/m^3$.

Problem 14.2

Problem description. Determine the temperature and energy luminosity (radiance) of an absolutely black body, if the maximum radiation energy falls at a wavelength of 600 nm.

Known quantities: $\lambda = 600 \ nm$.

Quantities to be calculated: T, R_T.

Problem solution. The radiation intensity is

$$R_T = \frac{P}{S}, \tag{14.2.1}$$

where P is power, i.e. the radiation energy for 1 s;

S is the surface through which energy passes.

The energy emitted in 1 second per unit of the surface of an absolutely black body is determined by the Stefan-Boltzmann formula

$$R_T = \sigma T^4, \tag{14.2.2}$$

where T is thermodynamic temperature;

σ is the Stephen – Boltzmann constant.

Thermodynamic temperature can be found from the Wien's displacement law

$$\lambda = \frac{C_1}{T}, \tag{14.2.3}$$

where C_1 is the first Wien's displacement constant.

Then

$$T = \frac{C_1}{\lambda}. \tag{14.2.4}$$

In this case, we can write

$$R_T = \sigma T^4 = \sigma\left(\frac{C_1}{T}\right)^4. \tag{14.2.5}$$

Substitute the numerical data

$$T = 4833\ K; \quad R_T = 3.1\times10^7\ W/m^2. \tag{14.2.6}$$

Answer. The temperature and energy luminosity of an absolutely black body are equal, respectively $T = 4833\ K$ and $R_T = 3.1\times10^7\ W/m^2$.

Problem 14.3

Problem description. Monochromatic light with a wavelength of 0.12 μm is incident on the metal surface. The red border of the photoelectric effect is 0.34 μm. What fraction of photon energy is expended to impart kinetic energy to an electron?

Known quantities: $\lambda_0 = 0.34\ \mu m$; $\lambda = 0.12\ \mu m$.

Quantities to be calculated: T_m / ε.

Problem solution. Einstein's equation for the external photoelectric effect has the form

$$h\nu = A + T_m = A + \frac{m\upsilon_m^2}{2}, \tag{14.3.1}$$

where $\varepsilon = h\nu = \frac{hc}{\lambda}$ is photon energy;

ν is the photon frequency;
λ is the photon wavelength;
c is the speed of light in a vacuum;
A is the photoelectron work function;
m is the electron mass;
υ_m is the electron maximum velocity.

The maximum kinetic energy of an electron is

$$T_m = \varepsilon - A. \tag{14.3.2}$$

For the case of $\upsilon = 0$, we get

$$h\nu_0 = A, \tag{14.3.3}$$

where ν_0 is the red border of the photoelectric effect.

The red border of the photoelectric effect is

$$\nu_0 = \frac{c}{\lambda_0}, \tag{14.3.4}$$

then

$$A = \frac{hc}{\lambda_0}. \tag{14.3.5}$$

Finally

$$\frac{T_m}{\varepsilon} = \frac{\varepsilon - A}{\varepsilon} = 1 - \frac{A}{\varepsilon} = 1 - \frac{\left(\frac{hc}{\lambda_0}\right)}{\left(\frac{hc}{\lambda}\right)} = 1 - \frac{\lambda}{\lambda_0}. \tag{14.3.6}$$

Substitute numeric data

$$\frac{T_m}{\varepsilon} = 0.647. \tag{14.3.7}$$

Answer. The fraction of photon energy is expended to impart kinetic energy to an electron is $\frac{T_m}{\varepsilon} = 0.647$.

Problems

Problem A

<u>*Problem description*</u>. A muffle furnace consumes a power equal to $P = 1\ kW$. The temperature of its inner surface with an open hole is $T = 2.2\ kK$. The hole area is

$S = 15\ cm^2$. Assuming that the furnace hole radiates as a completely black body, determine what part ε of the power is dissipated by the walls.

Answer. $\varepsilon = 0.71$.

Problem B

Problem description. The thermodynamic temperature of the black body has doubled. In this case, the wavelength decreased by $\Delta\lambda = 400\ nm$. This length corresponds to the maximum of the spectral density of the radiance. Calculate the initial temperature T_1 and the final temperature T_2.

Answer. $T_1 = 3.62 \times 10^3\ K$, $T_2 = 7.24 \times 10^3\ K$.

Problem C

Problem description. The photo effect caused by irradiation with the ultraviolet spectrum of the platinum plate stops at a retarding potential difference of $U_1 = 3.7\ V$. For the case when the platinum plate is replaced with another plate, the retarding potential difference will have to be increased to $U_2 = 6\ V$. Calculate the work output from the surface of this plate.

Answer. $A = 6.4 \times 10^{-19}\ J$.

Problem D

Problem description. The energy flow emitted by an electric lamp is $\Phi_E = 600\ W$. At a distance of $r = 1\ m$ from the lamp, a round flat mirror with a diameter of $d = 2\ cm$ is located perpendicular to the incident beams. Assuming that the lamp's radiation is the same in all directions and that the mirror fully reflects the light incident on it, determine the force F of light pressure on the mirror.

Answer. $F = 1 \times 10^{-8}\ N$.

Problem E

Problem description. A photon with a wavelength of $\lambda = 1\ pm$ scattered on a free electron at an angle of $90°$. Calculate what percentage η of its energy the photon transferred to the electron?

Answer. $\eta = 70\%$.

CHAPTER 15. QUANTUM MECHANICS

15.1. de Broglie's Hypothesis

French physicist Louis Victor Pierre Raymond de Broglie (1892–1987), realizing the symmetry existing in nature and developing ideas about the dual corpuscular-wave nature of light, put forward the hypothesis of the universality of *corpuscular-wave dualism*. de Broglie argued that not only photons, but also electrons and any other particles of matter have wave properties.

Each micro object can be correlated with corpuscular characteristics: energy E and momentum P, and wave characteristics: frequency ν and wavelength λ. The quantitative relationships connecting the particle and wave properties of particles are the same as for photons (*de Broglie equations*):

$$E = h\nu,$$
$$P = h/\lambda. \qquad (15.1)$$

Thus, a particle with momentum is associated with a wavelength determined by the de Broglie formula:

$$\lambda = h/P. \qquad (15.2)$$

This relation is valid for any particle with momentum P.

Let us consider a particle of mass m moving freely with velocity υ. We calculate the phase and group velocities for this particle. The phase velocity is

$$\upsilon_{ph} = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{E}{P} = \frac{mc^2}{m\upsilon} = \frac{c^2}{\upsilon} \ (E = h\omega \text{ and } P = hk), \qquad (15.3)$$

where $k = 2\pi/\lambda$ is the wave number.

Since $c > \upsilon$, the phase velocity of de Broglie waves is greater than the speed of light in vacuum (the phase velocity of the waves can be either smaller or larger than c, in contrast to the group velocity of the waves). The group velocity is

$$u = \frac{d\omega}{dk} = \frac{d(\hbar\omega)}{d(\hbar k)} = \frac{dE}{dP}. \qquad (15.4)$$

For free particle

$$E = \sqrt{m^2c^4 + P^2c^2} \qquad (15.5)$$

and

$$\frac{dE}{dP} = \frac{Pc^2}{\sqrt{m_0{}^2c^4 + P^2c^2}} = \frac{Pc^2}{E} = \frac{m\upsilon c^2}{mc^2} = \upsilon. \qquad (15.6)$$

Consequently, the group velocity of the de Broglie waves is equal to the velocity of the particle.

The group velocity of the photon is $u = c$, that is, it is equal to the speed of the photon itself.

According to the dual corpuscular-wave nature of the particles of matter, for the description of micro particles, wave and corpuscular representations are used. Therefore, it is impossible to ascribe to them all the properties of particles and all the properties of waves. Naturally, it is necessary to introduce some restrictions in the application to the objects of the micro world of the concepts of classical mechanics.

15.2. Uncertainty Principle

According to the postulates of classical mechanics every particle moves along a definite trajectory, so that its coordinate and impulse are accurately fixed at any time. However, all micro particles, because of their wave properties, differ significantly from classical particles. One of the main differences is that one can not talk about the movement of a micro particle along a certain trajectory and it is wrong to speak of simultaneous exact values of its coordinate and momentum. This follows from the corpuscular-wave dualism. Thus, the concept of "wavelength at a given point" is devoid of physical meaning, and since the pulse is expressed in terms of the wavelength, it follows that the micro particle with a certain momentum has a completely undefined coordinate. Conversely, if the micro particle is in a state with the exact value of the coordinate, then its momentum is completely undefined.

German physicist Werner Karl Heisenberg (1901–1976), taking into account the wave properties of micro particles and the constraints related to wave properties in their behaviour, came to the conclusion that the object of the micro world can not be characterized simultaneously with any pre-determined accuracy by both the coordinate and the momentum. According to the Heisenberg *uncertainty relations*, a micro particle (micro object) can not have both a definite coordinate (X, Y, Z), and a certain corresponding projection of the momentum (P_X, P_Y, P_Z), and the uncertainties of these quantities satisfy the conditions

$$\Delta X \Delta P_X \geq h,$$
$$\Delta Y \Delta P_Y \geq h,$$
$$\Delta Z \Delta P_Z \geq h; \tag{15.7}$$

that is, the product of the uncertainties of the coordinate and the corresponding momentum projection can not be less than a quantity of the order of h.

It follows from the uncertainty relations that, for example, if the micro particle is in a state with the exact value of the coordinate ($\Delta X = 0$), then in this state the corresponding projection of its momentum is completely indeterminate ($\Delta P_X \to \infty$), and vice versa. Thus, for a micro particle, there are no states in which its coordinates and momentum have simultaneously exact values. This implies the fact that it is impossible at the same time to measure the coordinate and momentum of the micro object simultaneously with any given accuracy.

In quantum theory, the uncertainty relation for energy E and time t is also considered, that is, the uncertainties of these quantities satisfy condition $\Delta E \Delta t \geq h$. The value ΔE is the uncertainty of the energy of the system at the time of its measurement, and the value Δt is the uncertainty of the duration of the measurement process. Consequently, a system having an average lifetime Δt can not be characterized by a certain value of energy; the energy spread $\Delta E = h / \Delta t$ increases with decreasing average life time. From the last expression it follows that the frequency of the emitted photon must also have an uncertainty $\Delta \nu = \Delta E / h$, i.e. the spectral lines should be characterized by a frequency equal to $\nu \pm \Delta E / h$. The experience does show that all the spectral lines are blurred. Measuring the width of the spectral line, we can estimate the order of the time of existence of the atom in the excited state.

Theories based on de Broglie's idea led to a new stage in the development of quantum theory - the creation of quantum mechanics, describing the laws of motion and interaction of micro particles with regard to their wave properties.

At this stage of development, new fundamental problems arose, in particular the problem of the physical nature of the de Broglie waves. To clarify this problem, let us compare the diffraction of light waves and micro particles. The diffraction pattern observed for light waves is characterized by the fact that as a result of superposition of diffracting waves on each other at different points of space, amplification or weakening of the amplitude of oscillations occurs. According to the wave concept of the nature of light, the intensity of the diffraction pattern is proportional to the square of the light wave amplitud. According to the concepts of the photon theory, the intensity is determined by the number of photons falling at a given point of the diffraction pattern. Consequently, the number of photons at a given point in the diffraction pattern is given by the square of the amplitude of the light wave, while for one photon the square of the amplitude determines the probability of the photon falling into one or another point.

The diffraction pattern observed for micro particles is also characterized by a non-uniform distribution of the fluxes of micro particles scattered or reflected in different directions, in some directions a larger number of particles is observed than in others. The presence of maxima in the diffraction pattern from the point of view of the wave theory means that these directions correspond to the highest intensity of de Broglie waves. On the other hand, the de Broglie wave intensity is greater where there is a larger number of particles, that is, the de Broglie wave intensity at a given point in space determines the number of particles trapped at that point. Thus, the diffraction pattern for micro particles is a manifestation of a statistical (probabilistic) regularity, according to which particles fall into those places where the intensity of de Broglie waves is greatest.

15.3. Wave Function

The necessity of a probabilistic approach to the description of micro particles is the most important distinguishing feature of quantum theory. Can de Broglie waves be interpreted as the probability waves, that is, to consider that the probability of detecting a micro particle at different points in space varies according to the wave law? This interpretation of de Broglie waves is already incorrect. To eliminate these difficulties, German physicist Max Born (1882–1970) suggested that the wave law does not change the probability itself, but a quantity called the amplitude of the probability and denoted by $\psi(X,Y,Z,t)$. This quantity is also called the *wave function* (or ψ - function). The probability amplitude can be complex, and the probability W is proportional to the square of its modulus:

$$W \sim |\psi(X,Y,Z,t)|^2 \; (|\psi|^2 = \psi\psi^*)\,, \tag{15.8}$$

where ψ^* is a function complex conjugate to ψ.

Thus, the description of the state of a micro object with the help of a wave function has a statistical, probabilistic character: the square of the modulus of the wave function (the square of the modulus of the de Broglie wave amplitude) determines the probability of finding the particle at a time t in the region with coordinates X and $X+dX$, Y and $Y+dY$, Z and $Z+dZ$. So the state of micro particles is described in a fundamentally new way, namely, using the wave function, which is the main carrier of information about particle and wave properties. The probability of finding a particle in an element of volume dV is

$$dW = |\psi|^2 dV\,. \tag{15.9}$$

The value of

$$|\psi|^2 = dW / dV \tag{15.10}$$

(the square of the ψ-function modulus) has the meaning of the probability density, that is, it determines the probability of finding a particle in a unit volume in a neighbourhood of a point with coordinates X, Y, Z. Thus, the physical meaning is not the ψ-function itself, but the square of its modulus $|\psi|^2$, which determines the intensity of the de Broglie waves.

The probability of finding a particle at a time t in a finite volume V is

$$W = \int_V dW = \int_V |\psi|^2 dV\,. \tag{15.11}$$

Since $|\psi|^2 dV$ is defined as probability, it is necessary to normalize the wave function ψ so that the probability of a reliable event is converted to 1 if the volume of V assumes an infinite volume of the whole space. This means that under this condition the particle must be somewhere in space. Consequently, the condition for the normalization of probabilities is

$$\int_{-\infty}^{\infty}|\psi|^2 dV = 1, \qquad (15.12)$$

where the integral is computed over the entire infinite space, that is, in coordinates X, Y, Z from $-\infty$ to ∞.

15.4. Schrodinger Equation

The statistical interpretation of de Broglie waves and the Heisenberg uncertainty relations led to the conclusion that the equation of motion in quantum mechanics describing the motion of micro particles in different force fields should be an equation from which the observed wave properties of the particles would follow. The basic equation should be an equation for the wave function $\psi(X,Y,Z,t)$ or more precisely, for the value of $|\psi|^2$, that determines the probability of finding a particle at time t in a volume of dV, that is, in the region with coordinates X and $X + dX$, Y and $Y + dY$, Z and $Z + dZ$. Since the desired equation must take into account the wave properties of the particles, it must be a wave equation, similar to the equation describing electromagnetic waves. The basic equation of non relativistic quantum mechanics is formulated by Austrian physicist Erwin Rudolf Josef Schrödinger (1887–1961).

The Schrodinger equation, like all the basic equations of physics (for example, Newton's equations in classical mechanics and Maxwell's equations for the electromagnetic field), is not deduced, but postulated. The correctness of this equation is confirmed by agreement with the experience of the results obtained with its help, which, in turn, gives it the character of the law of nature.

The *Schrodinger equation* has the form

$$-\frac{\hbar^2}{2m}\Delta\psi + U(X,Y,Z,t)\psi = i\hbar\frac{\partial\psi}{\partial t}, \qquad (15.13)$$

where $\hbar = h/(2\pi)$ is reduced Planck's constant,

m is a particle mass,

Δ is Laplace operator ($\Delta\psi = \partial^2\psi/\partial X^2 + \partial^2\psi/\partial Y^2 + \partial^2\psi/\partial Z^2$),

i is imaginary unit,

$U(X,Y,Z,t)$ is the particle potential function,

$\psi(X,Y,Z,t)$ is the particle wave function.

The above equation is valid for any particle (with a spin equal to 0) moving with a small velocity (in comparison with the speed of light), i.e. with a velocity

$\upsilon << c$. It is supplemented by the conditions imposed on the wave function: 1) the wave function must be finite, single-valued and continuous; 2) the derivatives $\partial\psi/\partial X$, $\partial\psi/\partial Y$, $\partial\psi/\partial Z$, $\partial\psi/\partial t$ must be continuous; 3) the function $|\psi|^2$ must be integrable; in the simplest cases this condition reduces to the condition of normalization of probabilities.

Equation

$$-\frac{\hbar^2}{2m}\Delta\psi + U(X,Y,Z,t)\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{15.14}$$

is a general Schrödinger equation. It is also called *time-dependent Schrödinger equation*. For many physical phenomena occurring in the micro world, the Schrödinger equation can be simplified by eliminating the dependence of y on time, in other words, to find the Schrödinger equation for stationary states, i.e. states with fixed values of energy. This is possible if the force field in which the particle moves is stationary, that is, the function $U = U(X,Y,Z)$ does not depend explicitly on time and has the meaning of potential energy. In this case, the solution of the Schrödinger equation can be represented as a product of two functions, one of which is a function of coordinates only and the other only of time, and the time dependence is expressed by a factor

$$e^{-i\omega t} = e^{-i(Et/h)}, \tag{15.15}$$

so that

$$\psi(X,Y,Z,t) = \psi(X,Y,Z)e^{-i(Et/h)}, \tag{15.16}$$

where E is the total particle energy, constant in the case of stationary field.

Substituting the expression for $\psi(X,Y,Z,t)$ into the general Schrodinger expression, we can obtain equation

$$\Delta\psi + \frac{2m}{\hbar^2}(E-U)\psi = 0. \tag{15.17}$$

The last equation is called the *time-independent Schrödinger equation*.

In this equation, the total energy E of the particle enters as a parameter. In the theory of differential equations, it is proved that such equations have an infinite number of solutions. These solutions, having a physical meaning, are selected by imposing boundary conditions. The Schrödinger equation use the conditions for the regularity of the wave functions: the wave functions must be finite, single-valued and continuous along with their first derivatives. Thus, only those solutions that are expressed by regular functions ψ have real physical meaning. But regular solutions do not occur for any values of the parameter, but only for a certain set of them, characteristic of the given problem. These energy values are called *energy* *eigenvalues*. Solutions, which correspond to the energy eigenvalues, are called *eigenfunctions*. The energy eigenvalues E can form both a continuous and a discrete series. In the first case we speak of a continuous spectrum, in the second as a discrete spectrum.

Test questions

1. What corpuscular and wave characteristics can be correlated with each micro-object?
2. Write down the de Broglie equations.
3. Specify the nature of the change in wavelength, which can be compared with the micro particle, for the case when the impulse of the micro particle increases.
4. Specify the relationship between the phase velocity and the wave number.
5. Is it possible that the value of the phase velocity will be greater than the velocity of light in a vacuum?
6. Write down the differential ratio for group velocity.
7. What is the value of the group velocity for a photon?
8. Formulate the dual corpuscular-wave nature of the particles of matter.
9. Is it true that both particles and micro particles move along a certain trajectory?
10. Formulate the uncertainty relations.
11. Is it possible to simultaneously measure the coordinate and momentum of a particle with not very high accuracy?
12. Write down the uncertainty relations for energy and time.
13. Calculate the frequency uncertainty magnitude of a particle for the case of known value of the energy spread.
14. Compare the results of diffraction of waves and micro particles.
15. Formulate the Born hypothesis.
16. Write down the relationship that determines the wave function.
17. Formulate the physical meaning of the wave function.
18. Is the statement true that the normalization condition holds for any part of the system volume?
19. Write down the general Schrödinger equation.
20. Give a definition of energy eigenvalues.

Problem-solving examples

Problem 15.1

Problem description. Calculate the potential difference that an electron must pass in order for its de Broglie wavelength to be equal to its Compton wavelength.

Known quantities: $\lambda_C = \lambda_B$.

Quantities to be calculated: U.

Problem solution. The work performed by the electric field is numerically equal to the kinetic energy that the electron received after it passed the accelerating potential difference

$$eU = E_K\,, \tag{15.1.1}$$

where e is an electron charge,

U is the potential difference,

E_K is the kinetic energy of an electron.

The kinetic energy of a relativistic electron can be found from the equation

$$mc^2 + E_K = \sqrt{p^2c^2 + m^2c^4}\,, \tag{15.1.2}$$

where p is relativistic momentum of the electron,

c is speed of light in a vacuum,

m is an electron mass.

Let us transform the previous equation and find the solution of the resulting quadratic equation

$$E_K{}^2 + 2mc^2E_K - p^2c^2 = 0,$$
$$D = 4m^2c^4 + 4p^2c^2,$$
$$E_K = \frac{-2mc^2 + 2c\sqrt{m^2c^2 + p^2}}{2} = c\left(\sqrt{m^2c^2 + p^2} - mc\right). \tag{15.1.3}$$

The Compton wavelength of an electron is

$$\lambda_C = \frac{2\pi\hbar}{mc}\,, \tag{15.1.4}$$

where $\hbar$ is reduced Planck's constant.

The de Broglie electron wavelength is

$$\lambda_B = \frac{h}{p}. \tag{15.1.5}$$

Considering that

$$\frac{2\pi\hbar}{mc} = \frac{h}{p}\,, \tag{15.1.6}$$

we calculate the momentum of such an electron

$$p = mc. \tag{15.1.7}$$

Then

$$E_K = c\left(\sqrt{m^2c^2 + p^2} - mc\right) = c\left(mc\sqrt{2} - mc\right) = mc^2\left(\sqrt{2} - 1\right). \quad (15.1.8)$$

Consequently,

$$U = \frac{E_K}{e} = \frac{mc^2\left(\sqrt{2} - 1\right)}{e}. \quad (15.1.9)$$

Substitute numeric data

$$U = 2.1 \times 10^5 \ V. \quad (15.1.10)$$

Answer. The potential difference is equal to $U = 2.1 \times 10^5 \ V$.

Problem 15.2

Problem description. The electron has a kinetic energy $E_K = 4 \ eV$ and is localized in a region of size $l = 1 \mu m$. Calculate the relative uncertainty of electron velocity using the uncertainty relation.

Known quantities: $E_K = 4 \ eV$, $l = 1 \mu m$.

Quantities to be calculated: $\frac{\Delta \upsilon_x}{\upsilon_x}$.

Problem solution. Let us write the uncertainty relation

$$\Delta x \cdot \Delta p_x \geq \hbar \quad (15.2.1)$$

or

$$\Delta x \cdot m \Delta \upsilon_x \geq \hbar, \quad (15.2.2)$$

where Δx is the x-coordinate uncertainty,

Δp_x is the uncertainty of the projection of the momentum on the x-axis,

$\Delta \upsilon_x$ is the uncertainty of the velocity projection on the x-axis,

m is an electron mass,

$\hbar$ is reduced Planck’s constant.

We take into account that

$$\Delta x \leq l, \quad (15.2.3)$$

where l is the size of the area.

Then

$$lm\Delta\upsilon_x \sim \hbar, \tag{15.2.4}$$

and

$$\Delta\upsilon_x \sim \frac{\hbar}{lm}. \tag{15.2.5}$$

The velocity of an electron with a kinetic energy of E_K is equal to

$$\upsilon_x = \sqrt{\frac{2E_K}{m}}. \tag{15.2.6}$$

Hence

$$\frac{\Delta\upsilon_x}{\upsilon_x} \sim \frac{\hbar}{lm}\sqrt{\frac{m}{2E_K}} = \frac{\hbar}{l\sqrt{2mE_K}}. \tag{15.2.7}$$

Substitute numeric data

$$\frac{\Delta\upsilon_x}{\upsilon_x} \sim 0.0001. \tag{15.2.8}$$

Answer. The relative uncertainty of the electron velocity is $\frac{\Delta\upsilon_x}{\upsilon_x} \sim 0.0001$.

Problem 15.3

Problem description. Calculate the possible energy values of particle of mass m, which is in a spherically symmetric potential well $U(r)=0$ at $r<r_0$ and $U(r)=\infty$, for the case when the particle motion is described by a wave function $\psi(r)$ depending only on r.

Known quantities: $U(r)=0$ at $r<r_0$.

Quantities to be calculated: E.

Problem solution. Let us write time-independent Schrödinger equation for the region $r<r_0$ ($U(r)=0$):

$$\Delta\psi + \frac{2m}{\hbar^2}E\psi = 0, \tag{15.3.1}$$

where E is the total energy of the system,
ψ is the wave function,
$\hbar$ is reduced Planck's constant,
m is a particle mass.

Since the particle is in a spherically symmetric potential well, we will use the Laplace operator in spherical coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \cdot \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \cdot \frac{\partial^2}{\partial \varphi^2}, \qquad (15.3.2)$$

where r is the radial distance,
θ is the polar angle,
φ is the azimuthal angle.

According to the condition of the problem, the wave function depends only on the radius r and does not depend on the angular coordinates θ and φ. Therefore, we will use only the radial component of the Laplace operator

$$\Delta = \frac{\partial}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial}{\partial r}. \qquad (15.3.3)$$

Schrödinger equation takes the form

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial \psi}{\partial r} + \frac{2m}{\hbar^2} E\psi = 0. \qquad (15.3.4)$$

We introduce the notation

$$k^2 = \frac{2mE}{\hbar^2}, \qquad (15.3.5)$$

then

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial \psi}{\partial r} + k^2 \psi = 0. \qquad (15.3.6)$$

We use the substitution

$$\psi(r) = \frac{\chi(r)}{r}, \qquad (15.3.7)$$

then

$$\frac{\partial \psi}{\partial r} = \frac{1}{r} \cdot \frac{\partial \chi}{\partial r} - \frac{1}{r^2}\chi \qquad (15.3.8)$$

and

$$\frac{\partial^2\psi}{\partial r^2}=-\frac{1}{r^2}\cdot\frac{\partial\chi}{\partial r}+\frac{1}{r}\cdot\frac{\partial^2\chi}{\partial r^2}+\frac{2}{r^3}\cdot\chi-\frac{1}{r^2}\cdot\frac{\partial\chi}{\partial r}=\frac{1}{r}\cdot\frac{\partial^2\chi}{\partial r^2}-\frac{2}{r^2}\cdot\frac{\partial\chi}{\partial r}+\frac{2}{r^3}\cdot\chi . \quad (15.3.9)$$

Therefore, the Schrödinger equation takes the form

$$\frac{1}{r}\cdot\frac{\partial^2\chi}{\partial r^2}-\frac{2}{r^2}\cdot\frac{\partial\chi}{\partial r}+\frac{2}{r^3}\cdot\chi+\frac{2}{r}\left(\frac{1}{r}\cdot\frac{\partial\chi}{\partial r}-\frac{1}{r^2}\chi\right)+k^2\frac{\chi}{r}=0 . \quad (15.3.10)$$

Simplify the expression

$$\frac{\partial^2\chi}{\partial r^2}+k^2\chi=0 . \quad (15.3.11)$$

We write the characteristic equation $\lambda^2+k^2=0$ and calculate its roots $\lambda^2=-k^2$, then $\lambda_1=ik$; $\lambda_2=-ik$.

The solution of the equation is

$$\chi=Ae^{\lambda_1 r}+Be^{\lambda_2 r}=(A+B)\cos(kx)+(A-B)i\sin(kx) . \quad (15.3.12)$$

We use boundary conditions: $\psi(r_0)=\psi(0)=0$, then $\chi(0)=r\psi(0)=0$ and $\chi(r_0)=r\psi(r_0)=0$.

In this case, we can write that $A=-B$ and $(A-B)i\neq 0$, hence $\sin kr_0=0$,

$$kr_0=\pi n \text{ and } k^2=\frac{\pi^2n^2}{r_0{}^2}=\frac{2mE}{\hbar^2} . \quad (15.3.13)$$

This implies

$$E=\frac{\hbar^2\pi^2n^2}{2mr_0{}^2} . \quad (15.3.14)$$

Answer. The values of the particle energy are equal $E=\dfrac{\hbar^2\pi^2n^2}{2mr_0{}^2}$.

Problems

Problem A

Problem description. Estimate how many times the de Broglie wavelength λ of a particle is less than the uncertainty Δx of its coordinate, which corresponds to the relative uncertainty of a pulse of 1%.

Answer. $N = 160$.

Problem B

Problem description. Estimate the relative width $\Delta\omega/\omega$ of the spectral line if the lifetime of an atom in the excited state ($\tau = 10^{-8}\ s$) and the wavelength of the emitted photon ($\lambda = 0.6\ \mu m$) are known.

Answer. $\Delta\omega/\omega = 3\times 10^{-8}$.

Problem C

Problem description. An electron with a kinetic energy of $E_k = 15\ eV$ is in a metal particle with a diameter of $d = 1\ \mu m$. Assess the relative inaccuracy $\Delta\upsilon$ with which the electron velocity can be determined.

Answer. $\Delta\upsilon/\upsilon = 10^{-4}$.

Problem D

Problem description. The electron is in an infinitely deep one-dimensional rectangular potential box of width L. Calculate the probability that an electron in an excited state ($n = 2$) will be detected in the middle third of the box.

Answer. $P = 0.195$.

Problem E

Problem description. The monoenergetic flow of electrons ($E = 100\ eV$) falls on a low rectangular potential barrier of infinite width. Determine the height U of the potential barrier if it is known that 4% of the electrons falling on the barrier are reflected.

Answer. $U = 8.896\times 10^{-18}\ J$.

CHAPTER 16. ATOMIC AND NUCLEAR PHYSICS

16.1. Hydrogen Spectral Series

Investigations of the emission spectra of rarefied gases (that is, the emission spectra of individual atoms) have shown that each gas has a well-defined line spectrum consisting of separate spectral lines or groups of closely distributed lines. The most studied is the spectrum of the hydrogen atom.

Swiss physicist Johann Jacob Balmer (1825–1898) derived an empirical formula describing all the known spectral lines of the hydrogen atom in the visible region of the spectrum:

$$\frac{1}{\lambda} = R'\left(\frac{1}{2^2} - \frac{1}{n^2}\right), \quad n = 3, 4, \ldots, \tag{16.1}$$

where $R' = 1{,}1 \cdot 10^7 \ m^{-1}$ is the *Rydberg constant*.

Since $\nu = c/\lambda$, the *Balmer formula* can be written for the frequencies:

$$\nu = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right), \ n = 3, 4, \ldots \ , \tag{16.2}$$

where $R = R'c = 3{,}29 \cdot 10^{15} \ s^{-1}$ is also the Rydberg constant.

It follows from the above expressions that the spectral lines differing by different values of n form a group or series of lines, called the *Balmer series*. As the n increases, the lines of the series converge; the value $n = \infty$ determines the boundary of the series, to which a continuous spectrum adjoins the high-frequency side. Later on, several more series were found in the spectrum of the hydrogen atom. In the ultraviolet region of the spectrum is the *Lyman series*:

$$\nu = R\left(\frac{1}{1^2} - \frac{1}{n^2}\right), \ (n = 2, 3, 4, \ldots). \tag{16.3}$$

The series named after its discoverer U.S. physicist Theodore Lyman IV (1874–1954). German physicist Louis Carl Heinrich Friederich Paschen (1865–1947) derived an empirical formula describing spectrum in the infrared region of (*Paschen series*):

$$\nu = R\left(\frac{1}{3^2} - \frac{1}{n^2}\right), \quad (n = 4, 5, 6, \ldots); \tag{16.4}$$

American physicist Frederick Summer Brackett (1896–1988) derived an empirical formula describing spectrum in *Brackett series*

$$\nu = R\left(\frac{1}{4^2} - \frac{1}{n^2}\right), \quad (n = 5,6,7,...); \tag{16.5}$$

American physicist August Herman Pfund (1879–1949) derived an empirical formula describing spectrum in *Pfund series*

$$\nu = R\left(\frac{1}{5^2} - \frac{1}{n^2}\right), \quad (n = 6,7,8,...); \tag{16.6}$$

American physicist Curtis Judson Humphreys (1898–1986) derived an empirical formula describing spectrum in *Humphreys series*

$$\nu = R\left(\frac{1}{6^2} - \frac{1}{n^2}\right), \quad (n = 7,8,9,...). \tag{16.7}$$

All the above series in the spectrum of the hydrogen atom can be described by a single formula, called the *generalized Balmer formula*:

$$\nu = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right), \tag{16.8}$$

where m has a constant value ($m = 1,2,3,...$) in each given series, (defines a series), n takes integer values starting at $m+1$ (determines the individual lines of this series).

The study of more complex spectra, namely the spectra of alkali metal vapors (for example, Li, Na, K) showed that they are represented by a set of irregularly arranged lines. Swedish physicist (Janne) Robert Rydberg (1854–1919) managed to divide them into three series, each of which is similar to the lines of the Balmer series.

The above serial formulas were chosen empirically and for a long time had no theoretical basis, although they were confirmed experimentally with very high accuracy.

16.2. Bohr Atomic Model

The first attempt to build a qualitatively new, quantum theory of the atom was undertaken by Bohr. He set himself the goal of linking the empirical regularities of line spectra, the nuclear model of the Rutherford (derived by Ernest Rutherford, 1st Baron Rurherford of Nelson (1871–1937)) atom and the quantum character of radiation. Based on his theory, Bohr put two postulates (*Bohr's postulates*).

The first postulate (the postulate of stationary states): there are stationary (not changing with time) atom states without radiate. Stationary states of atom correspond to stationary orbits along which electrons move. The motion of electrons along stationary orbits is not accompanied by the emission of electromagnetic waves. In the stationary state of an atom, an electron moving in a circular orbit should have discrete quantized values of the angular momentum satisfying condition

$$m_e \upsilon r_n = n\hbar, \ (n = 1, 2, 3, ...), \tag{16.9}$$

where m_e is the electron mass,

υ is its velocity along the orbit with number n and radius r_m, $\hbar = h/(2\pi)$.

The second postulate (frequency rule): in the transition of an electron from one stationary orbit to the other, one photon with an energy

$$h\nu = E_n - E_m \tag{16.10}$$

equal to the energy difference of the corresponding stationary states (E_n and E_m, respectively, the energy of the stationary states of the atom before and after the radiation (absorption)) is emitted (absorbed)). A photon is emitted at $E_m < E_n$ (the transition of an atom from a state with a higher energy to a state with a lower energy, i.e. the transition of an electron from a more distant orbit to a nearer one). The photon is absorbed at $E_m > E_n$ (the transition of the atom to a state with higher energy, i.e. the transition of an electron to a more distant orbit from the nucleus). The set of possible discrete frequencies

$$\nu = (E_n - E_m)/h \tag{16.11}$$

of quantum transitions determines the line spectrum of the atom.

The postulates advanced by Bohr made it possible to calculate the spectrum of the hydrogen atom and hydrogen-like systems, i.e. systems consisting of a nucleus with a charge of Ze and one electron (for example, He + ions, Li_2 +), and also theoretically calculate the Rydberg constant.

Considering the motion of an electron in a hydrogen-like system with circular stationary orbits, it is possible, according to Bohr's theory, to obtain an expression for the radius of the n-th stationary orbit:

$$r_n = n^2 \frac{\hbar^2 4\pi\varepsilon_0}{m_e Z e^2}, \tag{16.12}$$

where $n = 1, 2, 3, ...$

For the hydrogen atom ($Z = 1$), the radius of the first orbit of an electron at $n = 1$, called the <u>*Bohr radius*</u> (a), is

$$r_1 = a = \frac{\hbar^2 4\pi\varepsilon_0}{m_e e^2} = 0.528 \times 10^{-10}\ m. \tag{16.13}$$

This corresponds to calculations based on the kinetic theory of gases.

Since the radii of stationary orbits can not be measured, to test the theory it is necessary to turn to such quantities that can be measured experimentally. Such a quantity is the energy produced and absorbed by the hydrogen atoms.

The total energy of an electron in a hydrogen-like system consists of its kinetic energy ($m_e\upsilon^2/2$) and potential energy in the electrostatic field ($-Ze^2/(4\pi\varepsilon_0 r)$) of the nucleus. The total energy is

$$E=\frac{m_e\upsilon^2}{2}-\frac{Ze^2}{4\pi\varepsilon_0 r}=-\frac{1}{2}\frac{Ze^2}{4\pi\varepsilon_0 r}. \qquad (16.14)$$

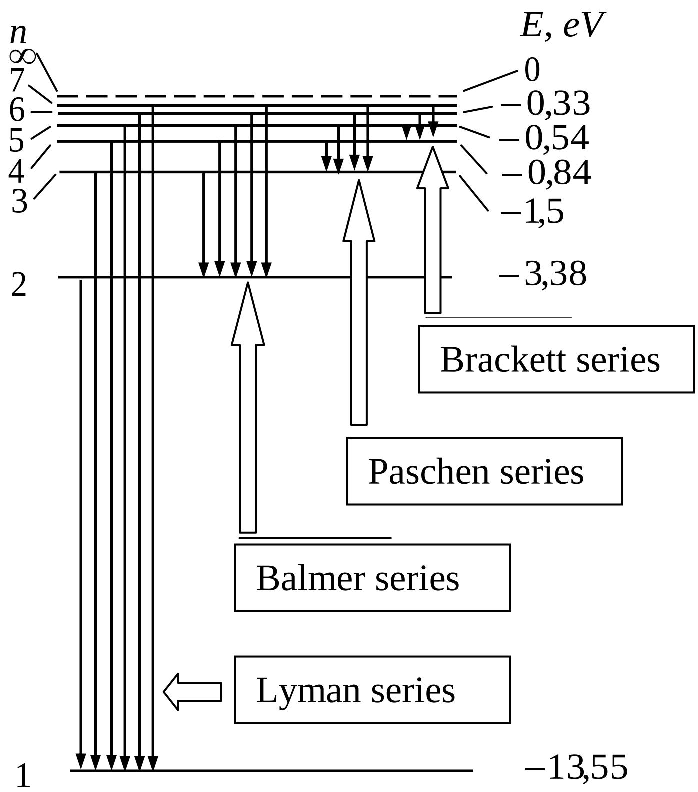


Figure 16.1. Hydrogen energy levels diagram.

Taking into account the values quantized for the radius of the n-th stationary orbit, we obtain that the electron energy can take only the following allowed discrete values:

$$E_n = -\frac{1}{n^2}\frac{Z^2 m_e e^4}{8h^2 \varepsilon_0{}^2}, \ (n = 1, 2, 3, ...), \tag{16.15}$$

where the minus sign means that the electron is in a bound state.

It follows from the last formula that the energy states of an atom form a sequence of energy levels. The position of the levels depends on the value of n. Integer value n, which determines the energy levels of the atom, is called the _principal quantum number_. The energy state with $n = 1$ is the main (normal) state. The states with $n > 1$ are excited.

The change in the integer values leads to the appearance of different energy levels for the hydrogen atom with Z = 1 (Figure 16.1).

The energy of the hydrogen atom increases with increasing n (its negative value decreases) and the energy levels approach the boundary corresponding to the value of $n = \infty$. The hydrogen atom thus possesses a minimum energy ($E_1 = -13.55\ eV$) for n = 1 and a maximum ($E_1 = 0$) for $n = \infty$ (when the electron is removed from the atom). Consequently, the value $E_\infty = 0$ corresponds to ionization of the atom (emission of an electron from an atom).

According to the second postulate, when a hydrogen atom (Z = 1) passes from a stationary state n with a higher energy to a stationary state with a lower energy, a quantum is emitted

$$h\nu = E_n - E_m = -\frac{m_e e^4}{8h^2 \varepsilon_0{}^2}\left(\frac{1}{n^2} - \frac{1}{m^2}\right). \tag{16.16}$$

In this case the frequency of the radiation is

$$\nu = \frac{m_e e^4}{8h^3 \varepsilon_0{}^2}\left(\frac{1}{m^2} - \frac{1}{n^2}\right) = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right), \tag{16.17}$$

where $R = \dfrac{m_e e^4}{8h^3 \varepsilon_0{}^2}$.

According to Bohr's theory, quantitatively explaining the spectrum of the hydrogen atom, the spectral series correspond to the radiation produced as a result of the transition of an atom to a given state from excited states located above this.

Bohr's theory considered the spectra of the hydrogen atom and hydrogen-like systems and calculated the frequencies of the spectral lines, but could not explain their intensity and answer the question: why are these or those transitions performed? A serious shortcoming of Bohr's theory was the impossibility of describing with its help the spectrum of the helium atom - one of the simplest atoms immediately following the hydrogen atom.

Rutherford, investigating 4-particles with energy of several megaelectron volts, which passed through thin gold films, came to the conclusion that the atom consists of a positively charged nucleus and surrounding electrons. The atomic nucleus consists of elementary particles: protons and neutrons. Protons and neutrons are called *nucleons*. The total number of nucleons in an atomic nucleus is called the *mass number* A.

16.3. Characteristics of Atomic Nucleus

The atomic nucleus is characterized by the charge Ze, where e is the charge of the proton, Z is the charge number of the nucleus equal to the number of protons in the nucleus and coinciding with the atomic number of the chemical element in the of Mendeleev's periodic system of elements. The nucleus is denoted by the same symbol as the neutral atom: ${}_Z^AX$, where X is the symbol of the chemical element, Z is the atomic number (the number of protons in the nucleus), The value of A is the mass number (the number of nucleons in the nucleus). Nuclei with the same Z, but different A (i.e. with different neutron numbers $N = A - Z$) are called *isotopes*, and nuclei with the same A, but different Z are called *isobars*.

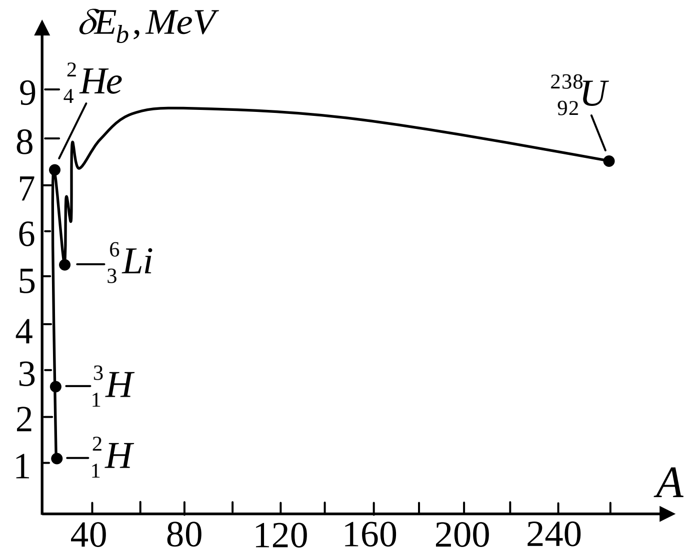


Figure 16.2. Nuclear binding energy curve.

The law of conservation of energy indicates that to divide the nucleus into constituent parts, it is necessary to expend the same amount of energy that is released during its formation. The energy that must be expended to split the nucleus into individual nucleons is called the *nuclear binding energy, namely*

$$E_b = \left[Zm_P + (A - Z)m_n - m_N\right]c^2, \qquad (16.18)$$

where m_P, m_n, m_N are, respectively, the masses of the proton, neutron, and nucleus.

The quantity

$$\Delta m = Zm_P + (A - Z)m_n - m_N \qquad (16.19)$$

is called the *nuclear mass defect*. Instead of the binding energy, the specific binding energy δE_b is often considered, i.e. the binding energy per one nucleon. It characterizes the stability (strength) of atomic nuclei, that is, the larger the δE_b, the more stable the nucleus. The specific binding energy depends on the mass number A of the element (Figure 16.2). The most stable are the so-called *magic nuclei*, in which the number of protons or the number of neutrons is equal to one of the magic numbers: 2, 8, 20, 28, 50, 82, 126. The magic nuclei are particularly stable, in which the number of protons is magical, and number of neutrons (there are only five of these nuclei: ${}^{4}_{2}He$, ${}^{16}_{8}O$, ${}^{40}_{20}Ca$, ${}^{48}_{20}Ca$, ${}^{208}_{82}Pb$).

Researchers use the high-resolution devices and special sources of spectral excitation to detect the hyperfine structure of the spectral lines.

The intrinsic angular momentum of the nucleus (the spin of the nucleus) is composed of the nucleon spins and the orbital angular momentum of the nucleons (momentum moments due to the motion of the nucleons inside the nucleus). Both these quantities are vectors, so the nuclear spin represents their vector sum. The spin of the nucleus is quantized according to the law

$$L_n = h\sqrt{I(I+1)}, \qquad (16.20)$$

where I is the spin nuclear quantum number (it is often called simply the nuclear spin), which takes integer or half-integral values 0, $^1/_2$, 1, $^3/_2$, ... Nuclei with even A have integers I, and with odd A have half-integral I.

The atomic nucleus except the spin has a magnetic moment P_{mn}. The magnetic moment of the nucleus is related to the nuclear spin by the relation

$$P_{mn} = g_n L_n, \qquad (16.21)$$

where g_n is the proportionality coefficient, called the *nuclear gyromagnetic ratio*.

16.4. Radioactive Decay

Radioactive decay, or simply decay, is the natural radioactive transformation of nuclei, which occurs spontaneously. Nearly 90% of the 2500 known nuclides are radioactive [6]; they are not stable but decay into other nuclides.The theory of radioactive decay is based on the assumption that radioactive decay is a spontaneous process that obeys the laws of statistics. Since the individual radioactive nuclei decay independently of each other, we can assume that the number of nuclei dN decaying on the average over a time interval from t to $t + dt$ is proportional to the time interval dt and the number N of nondecaying nuclei at the time t:

$$dN = -\lambda N dt, \qquad (16.22)$$

where λ is the *radioactive decay constant*. The minus sign indicates that the total number of radioactive nuclei decreases during the decay process.

Dividing the variables and integrating, that is,

$$\frac{dN}{N} = -\lambda dt,$$

$$\int_{N_0}^{N} \frac{dN}{N} = -\lambda \int_{0}^{t} dt,$$

$$\ln \frac{N}{N_0} = -\lambda t, \qquad (16.23)$$

we get

$$N = N_0 \exp(-\lambda t), \qquad (16.24)$$

where N_0 is the initial number of nondecayed nuclei at time $t = 0$,

N is the number of nondecayed nuclei at time t.

Last formula expresses the *radioactive decay law*, according to which the number of undecomposed nuclei decreases exponentially with time.

The intensity of the radioactive decay process is characterized by two values: the half-life $T_{1/2}$ and the mean lifetime t of the radioactive nucleus. The half-life $T_{1/2}$ is the time for which the initial number of radioactive nuclei decreases on average by half. Then, according to the law of radioactive decay

$$N_0 / 2 = N_0 \exp(-\lambda T_{1/2}) \qquad (16.25)$$

and

$$T_{1/2} = \frac{\ln 2}{\lambda} = 0{,}693/\lambda. \qquad (16.26)$$

The total lifetime of the dN nuclei is

$$t|dN| = \lambda N t dt. \qquad (16.27)$$

Integrating this expression over all possible t (i.e. from 0 to ∞) and dividing by the initial number of nuclei N_0, we obtain the mean lifetime of the radioactive nucleus

$$\tau = \frac{1}{N_0} \int_0^\infty \lambda N t dt = \frac{1}{N_0} \int_0^\infty \lambda N_0 t e^{-\lambda t} dt = \lambda \int_0^\infty t e^{-\lambda t} dt = \frac{1}{\lambda}. \qquad (16.28)$$

Thus, the average lifetime τ of a radioactive nucleus is the reciprocal of the constant of radioactive decay λ.

The *nuclide activity* A (nuclide is the general name of atomic nuclei, differing in the number of protons Z and neutrons N) in the radioactive source is the number of decays occurring with the nuclei of the sample in 1 s:

$$A = \left|\frac{dN}{dt}\right| = \lambda N . \tag{16.29}$$

Radioactive decay occurs in accordance with the so-called *radioactive displacement law of Fajans and Soddy*, which make it possible to establish which nucleus arises from the decay of a given nucleus. *Radioactive displacement law for α-decay*:

$$^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}He . \tag{16.30}$$

Radioactive displacement law for β-decay:

$$^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + {}^{0}_{-1}e , \tag{16.31}$$

where $^{A}_{Z}X$ is the nucleus before decay,
Y is the symbol of the nucleus after decay,
$^{4}_{2}He$ this is the helium nucleus (α-particle),
$^{0}_{-1}e$ is the symbolic designation of the electron (its charge is –1 and the mass number is zero).

Test questions

1. Is it possible to relate a continuous spectrum to the emission of gases?
2. Write down the Balmer empirical formula.
3. Formulate the concept of the Balmer series.
4. Specify the nature of lines location of the Balmer series with increasing main quantum number.
5. What formula determines the Lyman series?
6. Specify the values of the main quantum number for the Humphrey series.
7. Write down the generalized Balmer formula.
8. Explain the physical meaning of the Rydberg constant.
9. Formulate the first Bohr's postulate.
10. What physical quantity determines the value of the main quantum number?
11. Formulate the second Bohr's postulate.
12. Specify the ratio between the energies of the atom stationary states at which radiation occurs.
13. Write down the formula according to which the radius of the electron orbit is determined.

14. Indicate factors affecting the value of the Bohr radius.
15. Calculate the total energy of an electron in a hydrogen-like atom.
16. What parameter determines the position of the atom energy levels?
17. Specify the minimum value of the hydrogen atom energy.
18. Write down the formula for the frequency of the emission of a hydrogen atom.
19. Formulate the concept of binding energy.
20. Analyze the radioactive decay law.

Problem-solving examples

Problem 16.1

Problem description. The alpha particle counter, installed near the radioactive isotope, at the first measurement recorded 1400 particles per minute, and after a time t = 4 h, only 400 particles per minute. Determine the half-life of the isotope.

Known quantities: $T = 4\ h$; $N_1 = 1400\ \min^{-1}$; $N_2 = 400\ \min^{-1}$.

Quantities to be calculated: $T_{1/2}$.

Problem solution. The activity of the radioactive substance is equal to

$$A = A_0 e^{-\lambda t}, \tag{16.1.1}$$

where A_0 is the activity at the initial time;
λ is the radioactive decay constant;
t is a time.
For the radioactive decay constant we get

$$\lambda = \frac{\ln 2}{T_{1/2}}, \tag{16.1.2}$$

where $T_{1/2}$ is the half life.
Then

$$A = A_0 \exp\left(-\frac{\ln 2}{T_{1/2}} t\right), \tag{16.1.3}$$

from where

$$A_1 = A_0 \exp\left(-\frac{\ln 2}{T_{1/2}} t\right), \tag{16.1.4}$$

where t is the time elapsed from the beginning of the decay to the moment of the first observation; and

$$A_2 = A_0 \exp\left(-\frac{\ln 2}{T_{1/2}}(t+T)\right). \tag{16.1.5}$$

The result of dividing the last two equations by each other

$$\frac{A_1}{A_2} = \frac{A_0 \exp\left(-\frac{\ln 2}{T_{1/2}}t\right)}{A_0 \exp\left(-\frac{\ln 2}{T_{1/2}}(t+T)\right)} = \exp\left(\frac{\ln 2}{T_{1/2}}T\right). \tag{16.1.6}$$

Then the half-life is

$$T_{1/2} = \frac{\ln 2}{\ln\left(\frac{A_1}{A_2}\right)}T = 2.2\ h. \tag{16.1.7}$$

Answer. The half-life is $T_{1/2} = 2.2\ h$.

Problem 16.2

Problem description. How many times will change the period of rotation of an electron in a hydrogen atom, if during the transition to the unexcited state the atom emitted a photon with a wavelength of 97.5 nm?

Known quantities: H, $k = 1$; $\lambda = 97.5\ nm$.

Quantities to be calculated: T_2 / T_1.

Problem solution. The serial formula that determines the wavelength λ of light emitted or absorbed by a hydrogen atom during the transition of an electron from one orbit to another, has the form

$$\frac{1}{\lambda} = R\left(\frac{1}{k^2} - \frac{1}{n^2}\right), \tag{16.2.1}$$

where R is Rydberg's constant;

k, n are numbers of electronic levels.

Then

$$\frac{1}{n^2}=\frac{1}{k^2}-\frac{1}{\lambda R}=\frac{1}{1^2}-\frac{1}{97.5\cdot 10^{-9}\cdot 1.097\cdot 10^7}=0.065. \quad (16.2.2)$$

In this case

$$n=4. \quad (16.2.3)$$

Circulation period is

$$T=\frac{2\pi r}{\upsilon}, \quad (16.2.4)$$

where r is the orbit radius;

υ – is the speed of an electron in orbit with number n.

Radius of the orbit and the electron velocity are

$$r=\frac{h^2\varepsilon_0}{\pi me^2}n^2 \quad (16.2.5)$$

and

$$\upsilon=\frac{e^2}{2\varepsilon_0 h}\frac{1}{n}, \quad (16.2.6)$$

where h is the Planck's constant;

ε_0 is the electrical constant;

m is an electron mass.

Rewrite the formula for the period of circulation

$$T=\frac{2\pi\left(\frac{h^2\varepsilon_0}{\pi me^2}\right)}{\left(\frac{e^2}{2\varepsilon_0 h}\right)}n^3=4\frac{h^3\varepsilon_0{}^2}{me^4}n^3. \quad (16.2.7)$$

Therefore, the desired ratio of electron rotation periods is

$$\frac{T_2}{T_1}=\frac{n^3}{k^3}=n^3=4^3=64, \quad (16.2.8)$$

therefore, the rotation period will increase 64 times.

Answer. The rotation period will increase $\frac{T_2}{T_1}=64$ times

Problem 16.3

Problem description. Within what limits should the wavelength of monochromatic light lie, so that when hydrogen atoms are excited by quanta of this light, the radius of the electron orbit increases 16 times?

Known quantities: H; $r/r_0 = 16$.

Quantities to be calculated: $\Delta\lambda$.

Problem solution. The serial formula that determines the wavelength of light emitted or absorbed by a hydrogen atom when an electron moves from one orbit to another has the form

$$\frac{1}{\lambda} = R\left(\frac{1}{k^2} - \frac{1}{n^2}\right), \qquad (16.3.1)$$

where R is Rydberg's constant;
k and n are numbers of electronic orbits;
λ is the wavelength of radiation or absorption.

The radius of the Bohr orbit of the hydrogen atom

$$r = \frac{h^2\varepsilon_0}{\pi m e^2} n^2 = r_0 n^2, \qquad (16.3.2)$$

where r_0 is the Bohr radius;
ε_0 is the electrical constant;
m is the electron mass;
h is the Planck's constant;
e is an electron charge.

For n we get

$$n^2 = \frac{r}{r_0} = 16, \qquad (16.3.3)$$

from where

$$n = 4. \qquad (16.3.4)$$

Then

$$\frac{1}{\lambda_1} = R\left(\frac{1}{1^2} - \frac{1}{4^2}\right) = 1.028 \times 10^7 \ m^{-1} \qquad (16.3.5)$$

hence

$$\lambda_1 = 9.75 \times 10^{-8} \ m. \qquad (16.3.6)$$

This is the wavelength for the first boundary. The second boundary is determined from the condition that energy should not be enough to transfer to the fifth level. Consequently

$$\frac{1}{\lambda_2} = R\left(\frac{1}{1^2} - \frac{1}{5^2}\right) = 1.053 \times 10^7 \ m^{-1}, \qquad (16.3.7)$$

from where

$$\lambda_2 = 9.5 \times 10^{-8} \ m. \qquad (16.3.8)$$

Calculations show that the wavelength should be within

$$95 \ nm \le \lambda \le 97.5 \ nm. \qquad (16.3.9)$$

Answer. The wavelength should be within $95 \ nm \le \lambda \le 97.5 \ nm$.

Problems

Problem A

Problem description. Boron is a mixture of two isotopes with relative atomic masses of $A_{r,1} = 10.013$ and $A_{r,2} = 11.009$. Determine the mass fractions of the first (ω_1) and second (ω_2) isotopes in natural boron. The relative atomic mass of boron is $A_r = 10.811$.

Answer. $\omega_1 = 0.2$, $\omega_2 = 0.8$.

Problem B

Problem description. The nucleus of the cobalt isotope ${}^{60}_{27}Co$ has ejected a negatively charged β-particle. Identify the nucleus that was formed as a result of this nuclear decay.

Answer. ${}^{60}_{28}Ni$.

Problem C

Problem description. During the decay of radioactive polonium ${}^{210}Po$, helium ${}^{4}He$ was formed over time $t = 1 \ h$, which under normal conditions took up a volume of $V = 89.5 \ cm^3$. Determine the half-life $T_{1/2}$ of polonium.

Answer. $T_{1/2} = 1.19 \times 10^{7}\ s$.

Problem D

Problem description. The radioactive isotope ${}^{22}_{11}Na$ emits gamma rays with an energy of $\varepsilon = 1.28\ MeV$. Determine the energy W emitted during time $t = 5\ \min$ by the isotope of sodium, which has a mass of $m = 5\ g$. Note that the each decay is accompanied by the emission of one gamma photon with the specified energy.

Answer. $W = 7.06 \times 10^{4}\ J$.

Problem E

Problem description. The binding energy of a nucleus consisting of two protons and one neutron is $W_b = 7.72\ MeV$. Determine the mass m_a of a neutral atom having this nucleus.

Answer. $m_a = 5.008436 \times 10^{-27}\ kg$.

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APPENDICES

Table A1. SI prefixes

Prefix		Representation	Prefix		Representation
Name	Symbol	Base 10	Name	Symbol	Base 10
yotta	Y	10^{24}	deci	d	10^{-1}
zeta	Z	10^{21}	centi	c	10^{-2}
exa	E	10^{18}	milli	m	10^{-3}
peta	P	10^{15}	micro	μ	10^{-6}
tera	T	10^{12}	nano	n	10^{-9}
giga	G	10^{9}	pico	p	10^{-12}
mega	M	10^{6}	femto	f	10^{-15}
kilo	k	10^{3}	atto	a	10^{-18}
hecto	h	10^{2}	zepto	z	10^{-21}
deca	da	10^{1}	yocto	y	10^{-24}

Table A2. SI base units

Unit name	Unit symbol	Quantity name	Definition
1	2	3	4
metre	m	length	The distance traveled by light in vacuum in 1/299792458 second.
kilogram	kg	mass	The kilogram is defined by taking the fixed numerical value of the Plank constant *h* to be 6.62607015×10^{-34} when expressed in the unit J×s, which is equal to $kg\times m^2\times s^{-1}$, where the metre and the second are defined in terms of *c* and $\Delta\nu_{Cs}$.
second	s	time	The second is define by taking the fixed numerical value of the caesium frequency $\Delta\nu_{Cs}$, the unperturbed ground-state hyperfine transition frequency of the ^{133}C atom, to be 9192631770 when expressed in the unit *Hz*, which is equal to s^{-1}.

1	2	3	4
ampere	A	electric current	The ampere is defined by taking the fixed numerical value of the elementary charge *e* to be $1.602176634 \times 10^{-19}$ when expressed in unit *C*, which is equal to A×s, where the second is defined in terms of $\Delta \nu_{Cs}$.
kelvin	K	thermodynamic temperature	The kelvin is defined by taking the fixed numerical value of the Boltzmann constant *k* to be 1.380649×10^{-23} $J \times K^{-1}$ ($J = kg \times m^2 \times s^{-2}$), given the definition of the kilogram, the metre, and the second.
mole	mol	amount of substance	The amount of substance of exactly $6.02214076 \times 10^{23}$ elementary entities. Thi number is the fixed numerical value of the Avogadro constant, N_A, when expressed in the unit mol^{-1} and is called the Avogadro number.
candela	cd	luminous intencity	The luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 5.4×10^{14} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian.

Table A3. SI derived units

Unit name	Unit symbol	Quantity name
hertz	Hz	frequency
radian	rad	angle
steradian	sr	solid angle
newton	N	force, weight
pascal	Pa	pressure, stress
joule	J	energy, work, heat
watt	W	power, radiant flux
coulomb	C	electric charge
volt	V	voltage, electromotive force
farad	F	electrical capacitance
ohm	Ω	electrical resistance, impedance
siemens	S	electrical conductance
weber	Wb	magnetic flux
tesla	T	magnetic field strength
henry	H	electrical inductance
degree Celsius	ºC	temperature relative to 273.15 K
lumen	lm	luminous flux
lux	lx	illuminance
becquerel	Bq	radioactivity
gray	Gy	absorbed dose
sievert	Sv	equivalent dose
katal	kat	catalytic activity

Table A4. Greek alphabet

Name	Capital	Lower-case	Name	Capital	Lower-case	Name	Capital	Lower-case
Alpha	A	α	Iota	I	ι	Rho	P	ρ
Beta	B	β	Kappa	K	κ	Sigma	Σ	σ
Gamma	Γ	γ	Lambda	Λ	λ	Tau	T	τ
Delta	Δ	δ	Mu	M	μ	Upsilon	ϒ	υ
Epsilon	E	ε	Nu	N	ν	Phi	Φ	ϕ
Zeta	Z	ζ	Xi	Ξ	ξ	Chi	X	χ
Eta	H	η	Omicron	O	o	Psi	Ψ	ψ
Theta	Θ	θ	Pi	Π	π	Omega	Ω	ω

Table A5. Physical constants

Quantity	Symbol	Value
Avogadro constant	N_A	6.0221415(10)×10^{23} mol^{-1}
Boltzmann constant	k	1.3806505(24)×10^{-23} J / K
Electric constant	ε_0	8.854187817×10^{-12} F×m^{-1}
Faraday constant	F	96485.3383(83) C×mol^{-1}
Fine-structure constant	α	7.297352568(24) ×10^{-3}
Gravitational constant	G	6.6742(10)×10^{-11} N×m^2 / kg^2
Magnetic constant	μ_0	4π×10^{-7} T×m /A (exact)
Molar gas constant	R	8.314472(15) J/(mol×K)
Planck constant	h	6.6260693(11)×10^{-34} J×s
Rydberg constant	R_H	1.0973731568525(73)×10^7 m^{-1}
Stefan-Boltzmann constant	σ	5.670400(40)×10^{-8} W×m^{-2}×K^{-4}
Wien displacement law constant	b	2.8977685(51)×10^{-3} m×K
Atomic mass unit	u	1.66053886(28)×10^{-27} kg
Electron mass	m_e	9.1093826(16)×10^{-31} kg
Neutron mass	m_n	1.67492728(29)×10^{-27} kg
Proton mass	m_p	1.67262171(29)×10^{-27} kg
Elementary charge	e	1.60217653(14)×10^{-19} C
Speed of light in vacuum	c	2.99792458×10^8 m /s
Bohr magnetron	μ_B	9.27400949(80)×10^{-24} J/T
Bohr radius	a_0	5.291772108(18)×10^{-11} m
Compton wavelength	λ_C	2.426310238(16)×10^{-12} m

Table A6. Astronomical data

Body	Mass, kg	Equatorial radius, m	Perihelion/ Aphelion, m	Sidereal period	Orbital speed, km/s
Sun	1.998×10^{30}	6.955×10^8	2.5×10^{20} (*)	2.3×10^8 y(*)	2.2×10^2 (*)
Moon	7.342×10^{22}	1.738×10^6	(3.63/4.05)×10^8	27.321661 d	1.002
Mercury	3.301×10^{23}	2.440×10^6	(4.60/6.98)×10^{10}	87.9691 d	47.362
Venus	4.867×10^{24}	6.052×10^6	(1.08/1.09)×10^{11}	224.698 d	35.02
Earth	5.973×10^{24}	6.378×10^6	(1.47/1.52)×10^{11}	365.256363004 d	29.783
Mars	6.417×10^{23}	3.396×10^6	(2.07/2.49)×10^{11}	686.971 d	24.007
Jupiter	1.898×10^{27}	7.149×10^7	(7.40/7.78)×10^{11}	11.862 y	13.07
Saturn	5.683×10^{26}	6.027×10^7	(1.35/1.51)×10^{12}	29.4571 y	9.68
Uranus	8.683×10^{25}	2.556×10^7	(2.75/3.00)×10^{12}	84.01 y	6.81
Neptune	1.024×10^{26}	2.476×10^7	(4.45/4.55)×10^{12}	164.79 y	5.4349

- (Milky Way)

Table A7. Periodic table of elements

Name	Symbol	Atomic number	Standard atomic weight	Name	Symbol	Atomic number	Standard atomic weight
1	2	3	4	5	6	7	8
Actinium	$_{89}$Ac	89	227	Hassium	$_{108}$Hs	108	270
Aluminium	$_{13}$Al	13	26.9815384	Helium	$_{2}$He	2	4.002602
Americium	$_{95}$Am	95	243	Holmium	$_{67}$Ho	67	164.930328
Antimony	$_{51}$Sb	51	121.760	Hydrogen	$_{1}$H	1	1.008
Argon	$_{18}$Ar	18	39.948	Indium	$_{49}$In	49	114.818
Arsenic	$_{33}$As	33	74.921595	Iodine	$_{53}$I	53	126.90447
Astatine	$_{85}$At	85	210	Iridium	$_{77}$Ir	77	192.217
Barium	$_{56}$Ba	56	137.327	Iron	$_{26}$Fe	26	55.845
Berkelium	$_{97}$Bk	97	247	Krypton	$_{36}$Kr	36	83.798
Beryllium	$_{4}$Be	4	9.0121831	Lanthanum	$_{57}$La	57	138.90547
Bismuth	$_{83}$Bi	83	208.98040	Lawrencium	$_{103}$Lr	103	266
Bohrium	$_{107}$Bh	107	270	Lead	$_{82}$Pb	82	207.2
Boron	$_{5}$B	5	10.81	Lithium	$_{3}$Li	3	6.94
Bromine	$_{35}$Br	35	79.904	Livermorium	$_{116}$Lv	116	293
Cadmium	$_{48}$Cd	48	112.414	Lutetium	$_{71}$Lu	71	174.9668
Calcium	$_{20}$Ca	20	40.078	Magnesium	$_{12}$Mg	12	24.305
Californium	$_{98}$Cf	98	251	Manganese	$_{25}$Mn	25	54.938043
Carbon	$_{6}$C	6	12.011	Meitnerium	$_{109}$Mt	109	278
Caesium	$_{55}$Cs	55	132.905452	Mendelevium	$_{101}$Md	101	258
Cerium	$_{58}$Ce	58	140.116	Mercury	$_{80}$Hg	80	200.592
Chlorine	$_{17}$Cl	17	35.45	Molybdenum	$_{42}$Mo	42	95.95
Chromium	$_{24}$Cr	24	51.9961	Moscovium	$_{115}$Mc	115	290
Cobalt	$_{27}$Co	27	58.933194	Neodymium	$_{60}$Nd	60	144.242
Copernicium	$_{112}$Cn	112	285	Neon	$_{10}$Ne	10	20.1797
Copper	$_{29}$Cu	29	63.546	Neptunium	$_{93}$Np	93	237
Curium	$_{96}$Cm	96	247	Nickel	$_{28}$Ni	28	58.6934
Darmstadtium	$_{110}$Ds	110	281	Nihonium	$_{113}$Nh	113	286
Dubnium	$_{105}$Db	105	268	Niobium	$_{41}$Nb	41	92.90637
Dysprosium	$_{66}$Dy	66	162.500	Nitrogen	$_{7}$N	7	14.007
Einsteinium	$_{99}$Es	99	252	Nobelium	$_{102}$No	102	259
Erbium	$_{68}$Er	68	167.259	Oganesson	$_{118}$Og	118	294
Europium	$_{63}$Eu	63	151.964	Osmium	$_{76}$Os	76	190.23
Fermium	$_{100}$Fm	100	257	Oxygen	$_{8}$O	8	15.999
Flerovium	$_{114}$Fl	114	289	Palladium	$_{46}$Pd	46	106.42
Fluorine	$_{9}$F	9	18.9984032	Phosphorus	$_{15}$P	15	30.973761998
Francium	$_{87}$Fr	87	223	Platinum	$_{78}$Pt	78	195.084
Gadolinium	$_{64}$Gd	64	157.25	Plutonium	$_{94}$Pu	94	244
Gallium	$_{31}$Ga	31	69.723	Polonium	$_{84}$Po	84	209
Germanium	$_{32}$Ge	32	72.630	Potassium	$_{19}$K	19	39.0983
Gold	$_{79}$Au	79	196.966570	Praseodymium	$_{59}$Pr	59	140.90766
Hafnium	$_{72}$Hf	72	178.49	Promethium	$_{61}$Pm	61	145

1	2	3	4	5	6	7	8
Protactinium	$_{91}$Pa	91	231.03588	Tantalum	$_{73}$Ta	73	180.94788
Radium	$_{88}$Ra	88	226	Technetium	$_{43}$Tc	43	98
Radon	$_{86}$Rn	86	222	Tellurium	$_{52}$Te	52	127.60
Rhenium	$_{75}$Re	75	186.207	Tennessine	$_{117}$Ts	117	294
Rhodium	$_{45}$Rh	45	102.90549	Terbium	$_{65}$Tb	65	158.925354
Roentgenuim	$_{111}$Rg	111	282	Thallium	$_{81}$Tl	81	204.38
Rubidium	$_{37}$Rb	37	85.4678	Thorium	$_{90}$Th	90	232.0377
Ruthenium	$_{44}$Ru	44	101.07	Thulium	$_{69}$Tm	69	168.934218
Rutherfordium	$_{104}$Rf	104	267	Tin	$_{50}$Sn	50	118.710
Samarium	$_{62}$Sm	62	150.36	Titanium	$_{22}$Ti	22	47.867
Scandium	$_{21}$Sc	21	44.955908	Tungsten	$_{74}$W	74	183.84
Seaborgium	$_{106}$Sg	106	269	Uranium	$_{92}$U	92	238.02891
Selenium	$_{34}$Se	34	78.971	Vanadium	$_{23}$V	23	50.9415
Silicon	$_{14}$Si	14	28.085	Xenon	$_{54}$Xe	54	131.293
Silver	$_{47}$Ag	47	107.8682	Ytterbium	$_{70}$Yb	70	173.045
Sodium	$_{11}$Na	11	22.98976928	Yttrium	$_{39}$Y	39	88.90584
Strontium	$_{38}$Sr	38	87.62	Zinc	$_{30}$Zn	30	65.38
Sulfur	$_{16}$S	16	32.06	Zirconium	$_{40}$Zr	40	91.224

Table A8. Relative permittivity of some materials at 20 °C under 1 kHz

Material	ε_r	Material	ε_r	Material	ε_r
Vacuum	1	Ammonia	17	Paraffin	2.1
Air	1.00058986	Methanol	30	Rochelle salt	10000
PTFE/Teflon	2.1	Ethylene glycol	37	Porcelain	5.7 – 6.3
Polyethylene/XLPE	3.4	Furfural	42	Ebonite	2.6
Polystyrene	2.4 - 2.7	Glycerol	47	Wood, dry	2 - 6
Carbon disulfide	2.6	Water	80.2	Turpentine	2.2
Mylar	3.1	Hydrofluoric acid	83.6	Steatite	6
Mica	3 - 6	Hydrazine	52.0	Slate	4
Silicon dioxide	3.9	Formamide	84	Shellac	3.5
Sapphire	8.9 – 11.1	Sulfuric acid	84	Paper, waxed	2.5
Concrete	4.5	Hydrogen peroxide	60	Nylon	4.0 – 5.0
Pyrex (glass)	4.7	Hydrocyanic acid	2.3	Nitrogen	1.00058
Neoprene	6.7	Titanium dioxide	86 - 173	Marbe	8
Rubber	7	Strontium titanate	310	Isoprene	2.1
Diamond	5.5 - 10	Wax	7.8	Ice (-2 °C)	3.2
Salt	5.9	Germanium	16	Granite	7 - 9
Graphite	10 - 15	Quartz	4.5	Caster oil	4.7
Silicon	11.68	Kerosene	2.0	Calcium	3.0
Silicon nitride	7 - 8	Silicon	12	Amber	2.8 – 2.9

Table A9. Relative permeability of some materials

Material	μ/μ_0	Material	μ/μ_0	Material	μ/μ_0
Air	1.00000037	Iron (99.8% pure)	5000	Permalloy	8000
Aluminum	1.000022	Nanoperm	80000	Platinum	1.000265
Bismuth	0.999834	Neodymium magnet	1.05	Sapphire	0.99999976
Carbon steel	100	Nickel	100 - 600	Teflon	1
Cobalt-Iron	18000	Mu-metal	20000	Vacuum	1
Cooper	0.999994	Electrical steel	4000	Water	0.999992
Nickel-Zinc	16 - 640	Manganese steel	640	Wood	1.00000043
Hydrogen	1	Concrete (dry)	1		

Table A10. Refractive indices of gases for λ = 589 nm, P = 101325 Pa, t = 0 $^\circ$C

Material	n	Material	n	Material	n
Acetone	1.001090	Chlorine	1.000773	Methyl alcohol	1.000586
Air	1.000292	Chloroform	1.001450	Methyl ether	1.000891
Ammonia	1.000376	Ethyl alcohol	1.000878	Nitric oxide	1.000297
Argon	1.000281	Ethyl ether	1.001533	Nitrogen	1.000298
Benzene	1.001762	Helium	1.000035	Nitrous oxide	1.000516
Bromine	1.001132	Hydrochloric acid	1.000447	Oxygen	1.000271
Carbon dioxide	1.000449	Hydrogen	1.000132	Pentate	1.0001711
Carbon disulphide	1.001481	Hydrogen sulphide	1.000634	Sulphur dioxide	1.000686
Carbon monoxide	1.000338	Methane	1.000444	Water vapour	1.000256

Table A11. Refractive indices of liquids for λ = 589 nm, t = 20 $^\circ$C

Liquid	n	Liquid	n
Water	1.333	Benzyl benzoate	1.568
Paraldehyde	1.405	Aniline	1.586
Carbon tetrachlorine	1.46	Quinoline	1.627
Glycerol	1.47	Methylene iodine	1.737
Liquid paraffin	1.48	Oil, paraffin	1.44
Toluene	1.497	Oil, olive	1.46
Benzene	1.501	Oil, turpentine	1.47
Ethyl salicylate	1.523	Oil, cedar	1.516
Chlorobenzene	1.525	Oil, cloves	1.532
Methyl salicylate	1.538	Oil, cinnamon	1.601
Ethyl cinnamate	1.559		

Table A12. Refractive indices of optical materials for λ = 589.3 nm, t = 20 °C

Material	n	Material	n
Diamond	2.417	Ruby	1.76
Garnet	1.74 – 1.89	Sugar	1.56
Gelatin	1.525	Mica	1.56 – 1.60
Rock salt	1.544	Quartz glass	1.458
Quartz	1.544	Glass	1.48 – 1.53
Corundum	1.769	Optical glass	1.47 – 2.04
Ice (0 – 4 °C)	1.310	Topaz	1.63
Organic glass	1.485 – 1.500	Amber	1.532
Polystyrene	1.592		

Table A13. Refractive indices of calcite (Iceland Spar), t = 20 °C

λ, nm	n		λ, nm	n	
	O-ray	*E-ray*		*O-ray*	*E-ray*
200	1.90284	1.57649	1422	1.63590	
303	1.71959	1.51365	1497	1.63457	1.47744
410	1.68014	1.49640	1609	1.63261	
508	1.66527	1.48956	1682	1.63127	
643	1.65504	1.48490	1749		1.47638
706	1.65207	1.48353	1761	1.62974	
801	1.64869	1.48216	1849	1.62800	
905	1.64578	1.48098	1909		1.47573
1042	1.64276	1.47985	1946	1.62602	
1159	1.64051	1.47910	2053	1.62372	
1229	1.63926	1.47870	2100		1.47492
1307	1.63789	1.47831	2172	1.62099	
1396	1.63637	1.47789	3324		1.47392

Table A14. Photopic spectral luminous efficiency function $V(\lambda)$

λ, nm	$V(\lambda)$	λ, nm	$V(\lambda)$	λ, nm	$V(\lambda)$	λ, nm	$V(\lambda)$	λ, nm	$V(\lambda)$
360	0.000004	450	0.038000	540	0.954000	620	0.381000	710	0.002091
370	0.000012	460	0.060000	550	0.994950	630	0.265000	720	0.001047
380	0.000039	470	0.090980	555	1.000000	640	0.175000	730	0.000520
390	0.000120	480	0.139020	560	0.995000	650	0.107000	740	0.000249
400	0.000396	490	0.208020	570	0.952000	660	0.061000	750	0.000120
410	0.001210	500	0.323000	580	0.870000	670	0.032000	760	0.000060
420	0.004000	510	0.503000	590	0.757000	680	0.017000	770	0.000030
430	0.011600	520	0.710000	600	0.631000	690	0.008210	780	0.000015
440	0.023000	530	0.862000	610	0.503000	700	0.004102	790	0.000007

Table A15. Electron configurations of elements, I.

El	1s	2s	2p	3s	3p	3d	4s	4p	4d	4f	5s	5p	5d	5f	5g	6s	6p	6d	6f	6g
H	1																			
He	2																			
Li	2	1																		
Be	2	2																		
B	2	2	1																	
C	2	2	2																	
N	2	2	3																	
O	2	2	4																	
F	2	2	5																	
Ne	2	2	6																	
Na	2	2	6	1																
Mg	2	2	6	2																
Al	2	2	6	2	1															
Si	2	2	6	2	2															
P	2	2	6	2	3															
S	2	2	6	2	4															
Cl	2	2	6	2	5															
Ar	2	2	6	2	6															
K	2	2	6	2	6		1													
Ca	2	2	6	2	6		2													
Sc	2	2	6	2	6	1	2													
Ti	2	2	6	2	6	2	2													
V	2	2	6	2	6	3	2													
Cr	2	2	6	2	6	5	1													
Mn	2	2	6	2	6	5	2													
Fe	2	2	6	2	6	6	2													
Co	2	2	6	2	6	7	2													
Ni	2	2	6	2	6	8	2													
Cu	2	2	6	2	6	10	1													
Zn	2	2	6	2	6	10	2													
Ga	2	2	6	2	6	10	2	1												
Ge	2	2	6	2	6	10	2	2												
As	2	2	6	2	6	10	2	3												
Se	2	2	6	2	6	10	2	4												
Br	2	2	6	2	6	10	2	5												
Kr	2	2	6	2	6	10	2	6												
Rb	2	2	6	2	6	10	2	6			1									
Sr	2	2	6	2	6	10	2	6			2									
Y	2	2	6	2	6	10	2	6	1		2									
Zr	2	2	6	2	6	10	2	6	2		2									
Nb	2	2	6	2	6	10	2	6	4		1									
Mo	2	2	6	2	6	10	2	6	5		1									
Tc	2	2	6	2	6	10	2	6	5		2									
Ru	2	2	6	2	6	10	2	6	7		1									
Rh	2	2	6	2	6	10	2	6	8		1									
Pd	2	2	6	2	6	10	2	6	10											
Ag	2	2	6	2	6	10	2	6	10		1									
Cd	2	2	6	2	6	10	2	6	10		2									
In	2	2	6	2	6	10	2	6	10		2	1								
Sn	2	2	6	2	6	10	2	6	10		2	2								
Sb	2	2	6	2	6	10	2	6	10		2	3								
Te	2	2	6	2	6	10	2	6	10		2	4								
I	2	2	6	2	6	10	2	6	10		2	5								
Xe	2	2	6	2	6	10	2	6	10		2	6								

Table A15. Electron configurations of elements, II.

El	K	L	M	4s	4p	4d	4f	5s	5p	5d	5f	5g	6s	6p	6d	6f	6g	6h	7s
Cs	2	8	18	2	6	10		2	6				1						
Ba	2	8	18	2	6	10		2	6				2						
La	2	8	18	2	6	10		2	6	1			2						
Ce	2	8	18	2	6	10	2	2	6				2						
Pr	2	8	18	2	6	10	3	2	6				2						
Nd	2	8	18	2	6	10	4	2	6				2						
Pm	2	8	18	2	6	10	5	2	6				2						
Sm	2	8	18	2	6	10	6	2	6				2						
Eu	2	8	18	2	6	10	7	2	6				2						
Gd	2	8	18	2	6	10	7	2	6	1			2						
Tb	2	8	18	2	6	10	9	2	6				2						
Dy	2	8	18	2	6	10	10	2	6				2						
Ho	2	8	18	2	6	10	11	2	6				2						
Er	2	8	18	2	6	10	12	2	6				2						
Tm	2	8	18	2	6	10	13	2	6				2						
Yb	2	8	18	2	6	10	14	2	6				2						
Lu	2	8	18	2	6	10	14	2	6	1			2						
Hf	2	8	18	2	6	10	14	2	6	2			2						
Ta	2	8	18	2	6	10	14	2	6	3			2						
W	2	8	18	2	6	10	14	2	6	4			2						
Re	2	8	18	2	6	10	14	2	6	5			2						
Os	2	8	18	2	6	10	14	2	6	6			2						
Ir	2	8	18	2	6	10	14	2	6	7			2						
Pt	2	8	18	2	6	10	14	2	6	9			1						
Au	2	8	18	2	6	10	14	2	6	10			1						
Hg	2	8	18	2	6	10	14	2	6	10			2						
Tl	2	8	18	2	6	10	14	2	6	10			2	1					
Pb	2	8	18	2	6	10	14	2	6	10			2	2					
Bi	2	8	18	2	6	10	14	2	6	10			2	3					
Po	2	8	18	2	6	10	14	2	6	10			2	4					
At	2	8	18	2	6	10	14	2	6	10			2	5					
Em	2	8	18	2	6	10	14	2	6	10			2	6					
Fr	2	8	18	2	6	10	14	2	6	10			2	6					1
Ra	2	8	18	2	6	10	14	2	6	10			2	6					2
Ac	2	8	18	2	6	10	14	2	6	10			2	6	1				2
Th	2	8	18	2	6	10	14	2	6	10			2	6	2				2
Pa	2	8	18	2	6	10	14	2	6	10	2		2	6	1				2
U	2	8	18	2	6	10	14	2	6	10	3		2	6	1				2
Np	2	8	18	2	6	10	14	2	6	10	4		2	6	1				2
Pu	2	8	18	2	6	10	14	2	6	10	6		2	6					2
Am	2	8	18	2	6	10	14	2	6	10	7		2	6					2
Cm	2	8	18	2	6	10	14	2	6	10	7		2	6	1				2
Bk	2	8	18	2	6	10	14	2	6	10	9		2	6					2
Cf	2	8	18	2	6	10	14	2	6	10	10		2	6					2
Es	2	8	18	2	6	10	14	2	6	10	11		2	6					2
Fm	2	8	18	2	6	10	14	2	6	10	12		2	6					2
Md	2	8	18	2	6	10	14	2	6	10	13		2	6					2
No	2	8	18	2	6	10	14	2	6	10	14		2	6					2
Lr	2	8	18	2	6	10	14	2	6	10	9		2	6	1				2

Table A16. List of radioactive isotopes by half-life

Isotope	$T_{1/2}$	Isotope	$T_{1/2}$	Isotope	$T_{1/2}$	Isotope	$T_{1/2}$
^{7}H	$2.3{,}10^{-23}$ s	^{19}B	$2.92{,}10^{-3}$ s	^{47}Ca	$3.92{,}10^{5}$ s	^{87}Rb	$1.55{,}10^{18}$ s
^{5}H	$8.0{,}10^{-23}$ s	^{22}C	$6.2{,}10^{-3}$ s	^{7}Be	$4.59{,}10^{6}$ s	^{144}Nd	$7.2{,}10^{22}$ s
^{10}N	$2.0{,}10^{-22}$ s	^{12}Be	$2.15{,}10^{-2}$ s	^{49}V	$2.9{,}10^{7}$ s	^{113}Cd	$2.4{,}10^{23}$ s
^{5}He	$7.6{,}10^{-22}$ s	^{24}O	$6.5{,}10^{-2}$ s	^{63}Ni	$3.16{,}10^{9}$ s	^{180}W	$5.68{,}10^{25}$ s
^{8}C	$2.0{,}10^{-21}$ s	^{20}N	$1.30{,}10^{-1}$ s	^{39}Ar	$8.5{,}10^{9}$ s	^{82}Se	$3.1{,}10^{27}$ s
^{9}He	$7.0{,}10^{-21}$ s	^{8}Li	$8.40{,}10^{-1}$ s	^{108}Ag	$1.32{,}10^{10}$ s	^{130}Te	$2.5{,}10^{28}$ s
^{18}B	$2.6{,}10^{-8}$ s	^{289}Fl	2.6, s	^{202}Pb	$1.66{,}10^{12}$ s	^{76}Ge	$5.7{,}10^{28}$ s
^{254}Rf	$2.3{,}10^{-5}$ s	^{10}C	19.29, s	^{60}Fe	$8.2{,}10^{13}$ s	^{128}Te	$6.94{,}10^{31}$ s
^{254}Fm	$2.3{,}10^{-5}$ s	^{262}Db	34, s	^{182}Hf	$2.8{,}10^{14}$ s		
^{264}Hs	$5.4{,}10^{-4}$ s	^{259}No	$3.5{,}10^{3}$ s	^{244}Pu	$2.5{,}10^{15}$ s		
^{241}Fm	$7.3{,}10^{-4}$ s	^{160}Er	$3.73{,}10^{4}$ s	^{40}K	$4.03{,}10^{16}$ s		

Table A17. Quarks

Name	Symbol	Charge, e	Mass, MeV/c^2	Spin
up	u	+2/3	2.2	1/2
down	d	–1/3	4.6	1/2
charm	c	+2/3	1280	1/2
strange	s	–1/3	96	1/2
top	t	+2/3	173,10	1/2
bottom	b	–1/3	4,18	1/2

Table A18. Leptons and bosons

Name	Symbol	Charge, e	Mass, MeV/c^2	Spin
		Leptons		
Electron	e^{-}	–1	0.511	1/2
Electron neutrino	ν_e	0	$< 2.2\times10^{-6}$	1/2
Muon	μ^{-}	–1	105.7	1/2
Muon neutrino	ν_μ	0	0.170	1/2
Tau	τ^{-}	–1	1776,86	1/2
Tau neutrino	ν_τ	0	<15.5	1/2
		Bosons		
Photon	γ	0	0	1
W boson	W^{-}	–1	80.385, GeV/c^2	1
Z boson	Z	0	91.1875, GeV/c^2	1
Gluon	g	0	0	1
Higgs boson	H^0	0	125.09, GeV/c^2	0

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