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PHYSICS COURSE
MECHANICS

TEXTBOOK

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МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ
ОДЕСЬКА ДЕРЖАВНА АКАДЕМІЯ БУДІВНИЦТВА ТА АРХІТЕКТУРИ

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КУРС ФІЗИКИ
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The textbook discusses the basic phenomena of kinematics, the dynamics of translational and rotational movements, oscillatory processes, and the basic relationships of classical relativistic mechanics. Each theoretical section ends with self-examination questions. The textbook contains material that can be used in practical classes, namely: examples of problem solving, problem conditions for independent solutions, as well as reference data needed to solve them.

This textbook is intended for vocational training of students in the specialty 192 Construction and Civil Engineering for the Bachelor degree.

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PREFACE

Study guide “Physics course. Mechanics" is intended for students of higher educational institutions of specialty 192 "Construction and Civil Engineering". This study guide aims to provide an up-to-date and comprehensive coverage of the core curriculum in physics specified in the current Odessa State Academy of Civil Engineering and Architecture syllabus.

The guide covers topics related to kinematics, dynamics of translational and rotational motion, oscillatory motion, and relativistic dynamics. It builds from concrete experiments to more abstract understanding. Elements of the study guide include the following:

- fundamental concepts of physics
- test questions
- problem-solving examples
- problems
- appendices.

The problem-solving examples at the end of each chapter are provided to clarify concepts and to guide students in the analytical approach to the solutions of problems. In order to unify conceptual, analytical and calculation skills within the learning process, the International System of Units is used in study guide.

Most of the chapters are relatively independent, but some necessary background is established in certain key chapters.

By the end of study guide “Physics course. Mechanics" students will be able to:

- apply principles and concepts of physics to explain various phenomena
- construct models and simulations to describe and explain natural phenomena
- use mathematics as a precise method for showing relationships
- solve problems by applying physics principles and laws
- select and use appropriate technological instruments to collect data,
- analyze data, check it for accuracy and construct reasonable conclusions
- use precise scientific language in oral and written communication.

Physics is the science that studies the simplest and, at the same time, the most general patterns of natural phenomena, the properties and structure of matter. The most simple forms of matter motion (mechanical, thermal, electromagnetic) are part of more complex movements (chemical and biological). Physics has common objects and research methods with other natural sciences, as a result of which the following areas of knowledge have emerged: physical chemistry, chemical physics, chemical thermodynamics, astrophysics, biophysics, geophysics.

Mathematics is the basis of modern physics. The mathematical apparatus is widely used in the processing and generalization of experimental results. The electromagnetic field theory, statistical theory, thermodynamics, the theory of relativity, as well as quantum mechanics could not be developed without mathematics.

Physics is the basis of modern scientific and technological progress. The successful development of such areas of technology as: mechanical transport, electrical engineering, electronics, heat engineering, automation and remote control, construction equipment, modern technology, semiconductor and computing technology is strongly dependent on knowledge of physical laws and phenomena.

Physics is of great importance in the development of all areas of the economy. This fact determines the place of the physics course in the curriculums of higher education, especially in the curriculums of higher technical educational institutions. Acquaintance with the main physical phenomena, their mechanisms, laws and practical application can be postulated as the goal of studying physics. Achieving this goal is the physical basis for the study of general technical and special disciplines. Proper understandings of the nature of physical phenomena are particularly important in the practice of engineering.

Course of general physics refers to the experimental knowledge, and one of its main tasks is to represent knowledge as a result of observation, experiment, reflection and generalization of the experience. Therefore, in general, the course statement must be inductive. However, this does not preclude the use of the deductive method of presentation.

The model nature of physical theories, various methods for determining physical quantities and concepts, features of measuring physical quantities, the correct choice of units of measurement and systems of units occupy a significant place in this study guide.

It is well known that theoretical knowledge is useless without the ability to use it to solve practical problems. Therefore, the acquisition of problem solving skills is an integral part of studying the course of general physics. Currently, there are a sufficient number of collections of physical problems, but, unfortunately, there are practically no manuals intended for training in methods of solving problems. The material located at the end of each chapter of study guide is intended to remove the indicated disadvantage. This material is divided into three blocks. The first block contains test questions on the theoretical information that is present in the chapter. Examples of solving typical problems are included in the second block. The third block contains a number of problems for independent solution. These tasks are accompanied only by short answers. It is worth noting that in the theoretical part, the descriptions of experiments and in the methods of solving problems, the SI system is mainly used, which is convenient from a practical point of view.

The appendices placed at the end of the textbook are, on the one hand, an illustrative addition to the laws and phenomena that are described in the physics course, and on the other hand, have a reference character necessary for successful problem solving.

CHAPTER 1. KINEMATICS AND DYNAMICS

1.1. Types of Mechanics

The most important assumptions of mechanics are the definition of measurement procedures, obtaining quantitative expressions of distances, directions in space and time intervals. Depending on the nature of the motion being studied and the necessary accuracy, various assumptions are made about the procedures and measurement possibilities. According to these assumptions, the basic concepts, laws, and methods for describing mechanical motion turn out to be different.

Classical mechanics is used if the accuracy of studying the motion allows us to neglect the following values compared to unity:

a) the value v^2/c^2 (where v is the average speed of motion, c is the speed of light in vacuum);

b) the value φ/c^2, (where φ is the average value of the gravitational field potential).

In addition (case c), in classical mechanics, the product of an allowable error in measuring the speed of movement Δv and an allowable error in measuring the coordinate Δr and mass of the body m are significantly larger than a certain constant h called the ***Planck constant***. Max Karl Ernst Ludwig Planck (23.04.1858 – 4.10.1947) introduced the concept of energy quanta and action quantum.

If the first of these conditions is violated, the mechanics of the ***special theory of relativity*** (relativistic mechanics), are used. This condition is associated with a restriction on the speed of bodies, the description of the motion of which is possible in the language of classical physics. The speed of motion of bodies should be small compared with the speed of light in vacuum: c = 299792458 m/s. In our immediate environment, the largest of the velocities of macroscopic bodies encountered is the speed of the orbital motion of the center of mass of the Earth around the Sun, which varies from 29.29 km/s to 30.29 km/s. For parameter v^2/c^2, a change interval of 9.54×10^{-9} to 1.02×10^{-8} is obtained.

However, in the universe there are objects whose speed is comparable to the speed of light in a vacuum. For example, GN-z11 is a high-redshift galaxy found in the constellation Ursa Major. GN-z11 is currently the oldest and most distant known galaxy in the observable universe. This galaxy has helio radial velocity ~ 295000 km/s. Therefore, for the parameter v^2/c^2 we get the value 0.969, and a relativistic approach is required to describe the motion of such celestial bodies.

As an impressive example, one can cite the characteristics of the motion of a particle of cosmic radiation with the maximum recorded energy. Researchers, struck by the magnitude of the energy of this microparticle (about 3×10^{20} eV ≈ 50 J), gave it a name "Oh-My-God particle". Assuming it was a proton, this particle traveled at 99.99999999999999999999951 % of the speed of light. The Large Proton Collider is capable of accelerating protons at 99.9999991 % of the speed of light. Naturally, a description of the motion of particles mentioned in the last two cases is not possible from the point of view of classical mechanics.

If the first and second conditions are violated, the ***general theory of relativity*** is used, namely, Einstein's theory of gravitation. Albert Einstein (14.03.1879 – 18.04.1955) predicted gravitational waves. The equations of the relativistic theory of gravity go over into the equations of classical mechanics and Newtonian theory of gravitation if the speeds acquired by bodies under the influence of gravity are small compared with the speed of light, i.e. while the gravitational energy of the body is a small part of the total energy (including the rest energy). In this case, the concept of a weak gravitational field is used. In the solar system, we are dealing only with weak gravity. Assuming that the gravitational potential is zero at an infinitely distant point, the gravitational potential on the surface of the Earth is $\varphi/c^2 \approx 7\times10^{-10}$, and on the surface of the Sun, the gravitational potential is $\varphi/c^2 \approx 2.12\times10^{-6}$. In both cases, we can assume that the ratio $\varphi/c^2 << 1$ is satisfied with sufficient accuracy. Therefore, in our time, all calculations in celestial mechanics and cosmic dynamics are based on the Newtonian theory of gravity, which uses the laws and relations of classical mechanics.

If the third condition is violated, the system under study is described by ***quantum mechanics***. This limit of applicability of the classical method of describing motion, associated with the particle-wave nature of matter, is mathematically expressed by uncertainty relations, first formulated by Heisenberg and underlying the modern quantum theory. Heisenberg's (Werner Karl Heisenberg (5.12.1901 – 1.02.1976)) article "On the Visual Contents of Quantum Theoretical Kinematics and Mechanics" with a detailed exposition of the uncertainty principle was received by the editors of Zeitschrift für Physik on March 23, 1927. In accordance with the uncertainty relations, the particle cannot simultaneously have exact values of the coordinate and the corresponding projection of the momentum, namely, the product of the uncertainties satisfies the following relation $\Delta x \times \Delta p_x \geq \hbar/2$, (where $\hbar$ is the reduced Planck constant $\hbar \approx 1.054571800\times10^{-34}$ J·s). For macroscopic bodies (i.e., bodies consisting of a number of particles equal to the Avogadro number $N_A \approx 6.02214076\times10^{-23}$ mol^{-1}), the existing possibilities for measuring coordinates and momentum are such that the uncertainty relations do not impose restrictions on the applicability of the classical way of describing motion in which the state of particles is specified by indicating its coordinates and momentum. This possibility is associated with the small (on the scale of the macroscopic world) Planck constant.

Moreover, in describing the motion of microparticles in many cases, classical mechanics can be used. For example, when electrons move in macroscopic vacuum electronic devices (such as accelerators, electro-optical converters, electron microscopes), the uncertainties in the coordinates and momentum of electrons, determined by the experimental conditions, are much larger than the limit values established by the uncertainty relations. The classical method does not describe the motion of an electron in an atom: if we take as the uncertainty of the coordinate a value of the order of the size of the atom, then the corresponding uncertainty in the value of the speed of the electron, calculated from the uncertainty relation, turns out to be greater than the speed of the electron itself.

The following theories have not been fully developed: ***relativistic quantum mechanics*** (violation of the first and third conditions) and ***relativistic quantum theory of gravitation*** (violation of all three conditions).

Non-classical mechanics satisfy the ***compliance principle***. This means that if in the fundamental law or in any other consequence of nonclassical mechanics, the values that are considered negligible in classical mechanics (v^2/c^2, φ/c^2, h) tend to zero, then as a limit we obtain the corresponding laws and consequences of classical mechanics.

Relativistic mechanics is used in calculations of elementary particle accelerators, and Einstein's theory of gravitation is used in some astrophysical studies. The laws of quantum mechanics form the basis of a variety of sciences that study the properties of matter and develop methods for their targeted change and use. Quantum mechanics is necessary in studying the structure of the atom and atomic nucleus, the structure and many important properties of solids and liquids, the properties of matter at low temperatures, and the issues of optics, spectroscopy, chemistry and materials science.

Relativistic and quantum representations are more general in comparison with non-relativistic and classical (non-quantum) laws. The laws of non-relativistic mechanics follow from relativistic in the limit of low speeds, i.e. at $v^2/c^2 \ll 1$. This transition is carried out if in relativistic equations it is accepted that $c \to \infty$. The laws of classical motions are the limiting case of quantum motions at $h \to 0$.

However, this does not mean that classical mechanics has lost its significance. In many cases, the actual changes introduced by the theory of relativity and quantum mechanics come down to small corrections to classical mechanics. These corrections are called, respectively, relativistic and quantum corrections. In the case of ordinary slow motions of macroscopic bodies, these corrections are so insignificant that, as a rule, they go far beyond the accuracy of the finest physical measurements.

In addition, even the simplest problems on the motion of macroscopic bodies, which classical mechanics can easily cope with, would lead to insurmountable mathematical difficulties in trying to find their exact solutions in relativistic and quantum mechanics.

Thus, classical mechanics has a very broad and practically important area of applicability. Within this area, it will never lose its scientific and practical significance. Refusal of classical mechanics is necessary only outside the field of its applicability, when it leads to incorrect or insufficiently accurate results.

The laws of classical mechanics are the theoretical basis of many technical sciences (resistance of materials, technical mechanics, hydraulics, technical hydrodynamics, etc.), as well as celestial mechanics, which studies the motion of various celestial bodies, including stars, their clusters and interstellar gas.

Three main sections are considered in classical mechanics: kinematics, dynamics, and statics.

Kinematics describes mechanical motion without considering the interaction of bodies and the force fields applied to them. Kinematics operates with concepts that characterize motion, makes basic assumptions about measuring quantities, establishes relationships between quantities describing motion, and classifies movements.

Dynamics consider the effects of the interaction of bodies and the action of force fields on the characteristics of their mechanical motion, find out the causes of body motion in a certain way. In addition, dynamics operates with concepts that describe the properties of bodies and systems. These concepts are essential for the mechanical movement of systems. The basic laws of dynamics establish a relationship between quantities, which are a quantitative measure of these properties, and the kinematic characteristics of motion. An important task of dynamics is the study of the fundamental and general consequences of the basic laws, namely, conservation laws.

An important section of dynamics in terms of practical applications is ***statics***. Statics studies the state of rest of bodies and systems that are under the influence of other bodies and force fields. Since the balance of bodies and systems is a special case of motion, the laws of statics are a natural consequence of the laws of dynamics.

1.2. Basic Concepts of Mechanics

A quantitative description of the mechanical motion of bodies is carried out using quantities characterizing space, time, as well as geometric and physical parameters of the bodies themselves. Consider the basic units of mechanical quantities.

Length l is defined as the geometric distance between two points in space. In the International System of Units (SI), a meter is taken as a unit of length (m): $[l] = \mathrm{m}$. According to the current definition, a meter (from the Greek noun "measure") is the length of the path travelled by light in a vacuum over a time interval of 1/299792458 s. The characteristic sizes of objects that are found in nature are shown in Table 1.

Table 1. Characteristic body sizes in nature

Specification	l, m
Diameter of the whole observable universe	8.8×10^{26}
Milky Way's diameter	$(1.42\div1.89)\times10^{21}$
Sun's equatorial radius	6.957×10^{8}
Earth's equatorial radius	6.378×10^{6}
Everest's summit	8.848×10^{3}
Height of the moai Hoa Hakananai'a	2.42×10^{0}
Visible wavelength range of electromagnetic radiation	$(3.6\div7.8)\times10^{-7}$
$_{12}$Mg covalent radius	$(1.41 \pm 0.07)\times10^{-10}$
Compton wavelength of the electron	2.426×10^{-12}

Derivatives of length are area S and volume V. They characterize the regions of spaces of two and three dimensions occupied by extended bodies. The units of measurement of area and volume in SI are: $[S]$ = m^2, $[V]$ = m^3.

The ***time*** t between two events at a given point in space is defined as the difference between the readings of a device called a clock, which is based on a strictly periodic and uniform physical process. The unit of time in SI is indicated as second (s): $[t]$ = s. The word “second” comes from the lat. secunda (in the expression secunda pars minuta "the next part of the minute"). By definition, a ***second*** is a time interval equal to 9192631770 periods of radiation corresponding to the transition between two ultrathin levels of the ground state of the cesium-133 atom, which is at rest at a temperature of 0 K in the absence of external fields. Typical time intervals are shown in Table 2.

Table 2. Characteristic time intervals

Specification	t, s
Age of the whole observable universe	4.34×10^{17}
Galactic year	$(7.10\div7.89)\times10^{15}$
Pluto’s orbital period	7.83×10^{9}
Earth’s orbital period	3.16×10^{7}
Day	8.64×10^{4}
Bat’s sonar signals period	$(2.5\div1.0)\times10^{-5}$
Ytterbium lattice clock period	1.93×10^{-15}

Body ***mass*** m is a scalar physical quantity that determines the inertial and gravitational properties of bodies when their speed is much less than the speed of light in a vacuum. The unit of mass in SI is a kilogram (kg): $[m]$ = kg. The word "kilogram" comes from the French word "kilogramme", which in turn was formed from the Greek words "χίλιοι" (chilioi), which means "thousand," and "γράμμα" (gram), which means "low weight".

The definition of a kilogram is based on fixing the numerical value of the Planck constant h (h = 6.62607015×10^{-34} J·s): $m = Cf_1 f_2 g^{-1} v^{-1} h$. In this formula, the mass m is linearly related to the Planck constant h. In turn, parameters C, f_1, f_2, g, v are determined using the Kibble balance. Bryan Peter Kibble (1938 – 28.04.2016) invented the moving-coil watt balance in 1975.

The characteristic body masses that are found in nature are shown in Table 3.

Table 3. Characteristic masses of objects

Specification	m, kg
Mass of the whole observable universe	4.5×10^{51}
Milky Way's mass	$(1.6\div3.0)\times10^{42}$
Sun's mass	1.99×10^{30}
Jupiter's mass	1.90×10^{27}
Halley's Comet mass	2.2×10^{14}
Rosetta Stone's mass	7.6×10^{2}
Nettle (Urtica dioica) pollen mass	7.7×10^{-13}
Water's molecule mass	2.99×10^{-23}
Electron rest mass	9.109×10^{-31}

Any motion of the body can be divided into two main types: translational and rotational motion. ***Translational motion*** means a movement in which any straight line (axis) associated with a moving body remains parallel to itself. A movement in which all points of a body move in circles whose centres lie on the same straight line (axis of rotation) is called a ***rotational movement***.

A macroscopic body whose size can be neglected when describing its motion is called a ***material point***. The question of whether a given body can be considered as a material point does not depend on the size of this body, but depends on the conditions of the problem being solved. For example, the radius of the Earth is much less than the distance from the Earth to the Sun and its orbital motion can be well described as the movement of a material point with a mass equal to the mass of the Earth and located in its centre. However, when considering the daily movement of the Earth around its own axis, replacing it with a material point does not make sense.

The mechanics of the material point is the basis of all mechanics. Any macroscopic body can be represented as a set of interacting material points with masses equal to the masses of its parts. The study of the motion of these parts is reduced to the study of the motion of material points.

A body or a set of motionless bodies relative to one another, with respect to which the spatial and temporal position of other bodies is determined, is called a ***reference frame***. From the point of view of kinematics, all reference frames are equivalent. Kinematics does not allow to indicate the advantages of one reference system relative to another. The choice of a reference frame for solving kinematics problems is determined by considerations of expediency (convenience).

The description of the space in which the movement of the material point is carried out is made by linking the spatial coordinate system with the reference system. The ***coordinate system*** is understood as a triple of linearly independent directed segments of straight lines (coordinate axes), leaving one point (reference point). In this case, the position of the material point M in space is determined by the radius vector $\vec{r}$ drawn from the origin O of the coordinates to this point, and the movement is represented as the vector sum of independent movements along the three spatial axes of the selected coordinate system. In the Cartesian system, we have

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, \tag{1.2.1}$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are ***unit vectors*** directed along the positive directions of the coordinate axes; x, y, z are the projection of the radius vector on these axes, which are expressed by numbers.

René Descartes (31.031596 – 11.02.1650) (latinized name: Cartesius) invented Cartesian coordinates in the 17th century. In a rectangular coordinate system

$$(\vec{i},\vec{j}) = (\vec{j},\vec{k}) = (\vec{k},\vec{i}) = 0 \quad \left|\vec{i}\right| = \left|\vec{j}\right| = \left|\vec{k}\right| = 1, \tag{1.2.2}$$

point M position is given by three numbers x, y, z:

$$M = M(x, y, z). \tag{1.2.3}$$

In a cylindrical coordinate system, the position of point M is specified by three other numbers ρ, φ, z

$$M = M(\rho, \varphi, z)$$

$$\rho = \sqrt{x^2 + z^2}, \quad \varphi = \arctan(y/x), \quad z = z,$$

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z, \quad 0 \le \varphi \le 2\pi \tag{1.2.4}$$

In a spherical coordinate system, we have

$$M = M(\rho, \theta, z)$$

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan\frac{\sqrt{x^2 + y^2}}{z}, \quad \varphi = \arctan\frac{y}{x}$$

$$x = r\sin\theta\cos\varphi, \quad y = r\cos\theta\sin\varphi, \quad z = r\cos\theta$$

$$0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi. \tag{1.2.5}$$

An unambiguous determination of the position of point M in space is made by assuming that the radius vector $\vec{r}$ depends on parameter t, called time, so that one value t corresponds to one value of function:

$$\vec{r} = \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}. \tag{1.2.6}$$

Equality (1.1.6) is called the ***kinematic equation of motion*** of a point M written in vector form.

The line along which the material point M moves is called the ***trajectory***. The distance between two given points 1 and 2, counted along the path, is called a ***path*** Δs. A rectilinear directional segment (vector) drawn from start point 1 to end point 2 is called ***displacement*** $\Delta\vec{r}_{12}$. The sum of two consecutive movements from point 1 to point 2 and from point 2 to point 3 is a movement from point 1 to point 3:

$$\Delta\vec{r}_{12} + \Delta\vec{r}_{23} = \Delta\vec{r}_{13}. \tag{1.2.7}$$

If the radius vector of the material point M at time t is $\vec{r}(t)$, and at time $t + \Delta t$ is $\vec{r}(t+\Delta t)$, then the displacement $\Delta\vec{r}$ of this point over time interval Δt is

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t). \tag{1.2.8}$$

Moving $\Delta\vec{r}$ is a function of time t:

$$\Delta\vec{r} = \Delta\vec{r}(t). \tag{1.2.9}$$

In the general case, the trajectory of a material point is a curved line. In addition to moving $\Delta\vec{r}$, the motion of a material point is characterized by speed and acceleration.

The ***average speed*** of $\langle\vec{v}\rangle$ over a period of time Δt is called the movement of $\Delta\vec{r}$ per unit of this time

$$\langle\vec{v}\rangle = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}. \tag{1.2.10}$$

Instantaneous speed is the limit to which the average speed $\langle\vec{v}\rangle$ tends when the time interval Δt tends to zero

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}. \tag{1.2.11}$$

The concept of instantaneous speed was introduced by William of Heytesbury (1313 – 1373) in his work written about 1335. Velocity $\vec{v}$ is directed along the tangent to the curved path, because infinitesimal (elementary) displacement $d\vec{r}$ coincides with an infinitely small element of the trajectory ds. In Cartesian coordinates, formula (1.2.11) is equivalent to the three equations

$$\upsilon_x = \frac{dx}{dt} = \dot{x}, \quad \upsilon_y = \frac{dy}{dt} = \dot{y}, \quad \upsilon_z = \frac{dz}{dt} = \dot{z}, \tag{1.2.12}$$

where v_x, v_y, v_z are projections of the vector $\vec{v}$ onto the coordinate axes with orts $\vec{i}$, $\vec{j}$, $\vec{k}$:

$$\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} = \dot{x}(t)\vec{i} + \dot{y}(t)\vec{j} + \dot{z}(t)\vec{k}. \tag{1.2.13}$$

The modulus of the vector $\vec{v}$ is

$$|\vec{v}| = v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}. \tag{1.2.14}$$

The transition from Cartesian rectangular coordinates to curvilinear coordinates is carried out according to the rules of differentiation of complex functions.

Let the radius vector $\vec{r}$ be a function of curvilinear coordinates:

$$\vec{r} = \vec{r}(q_1, q_2, q_3), \tag{1.2.15}$$

then

$$\vec{v} = \frac{d\vec{r}}{dt} = \sum_{i=1}^{3}\frac{\partial\vec{r}}{\partial q_i}\frac{\partial q_i}{\partial t} = \sum_{i=1}^{3}\frac{\partial\vec{r}}{\partial q_i}\dot{q}_i. \tag{1.2.16}$$

For spherical coordinates, relations $q_1 = r$, $q_2 = \varphi$, $q_3 = \theta$. In this case, for speed $\vec{v}$ we get

$$\vec{v} = v_r\vec{e}_r + v_\varphi\vec{e}_\varphi + v_\theta\vec{e}_\theta$$

$$\upsilon_r = \dot{r}, \quad \upsilon_\varphi = r\dot{\varphi}\sin\theta, \quad \upsilon_\theta = r\dot{\theta},$$

$$v = \sqrt{\dot{r}^2 + r^2\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right)}, \tag{1.2.17}$$

where $\vec{e}_r$, $\vec{e}_\varphi$, $\vec{e}_\theta$ are unit vectors of tangents to lines r, φ, θ.

The elementary movement of the body $\Delta\vec{r}$ over a period of time from t_1 to t_2 is determined by the formula

$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1) = \int_{t_1}^{t_2} d\vec{r}' = \int_{t_1}^{t_2}\vec{v}(t')dt'. \tag{1.2.18}$$

The path s covered by the material point is equal to the area of the curved trapezoid

$$s = \int_{t_1}^{t_2} v(t')dt'. \tag{1.2.19}$$

The average speed over a period of time $\Delta t = t_2 - t_1$ is

$$\langle \vec{v} \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \vec{v}(t')dt' = \frac{\Delta \vec{r}}{\Delta t}. \tag{1.2.20}$$

In the SI system, speed is measured in meters per second: $[v]$ = m/s.

The ***average acceleration*** $\langle \vec{a} \rangle$ over a period of time Δt is called the increment of speed $\Delta \vec{v} = \vec{v}(t + \Delta t) - \vec{v}(t)$ over a time interval Δt:

$$\langle \vec{a} \rangle = \frac{\Delta \vec{v}}{\Delta t}. \tag{1.2.21}$$

Instantaneous acceleration $\vec{a}$ is the limit of the average acceleration $\langle \vec{a} \rangle$ when Δt tends to zero:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \dot{\vec{v}}$$

$$\vec{a} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2} = \frac{d\dot{\vec{r}}}{dt} = \ddot{\vec{r}}. \tag{1.2.22}$$

In Cartesian coordinates, formula (1.1.22) is equivalent to the three equations

$$a_x = \dot{v}_x = \ddot{x}, \quad a_y = \dot{v}_y = \ddot{y}, \quad a_z = \dot{v}_z = \ddot{z}, \tag{1.2.23}$$

where a_x, a_y, a_z are projections of the acceleration vector $\vec{a}$ on the coordinate axes x, y, z with orts $\vec{i}$, $\vec{j}$, $\vec{k}$

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

$$\vec{a} = \dot{v}_x\vec{i} + \dot{v}_y\vec{j} + \dot{v}_z\vec{k}$$

$$\vec{a} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}. \tag{1.2.24}$$

Acceleration module is equal to

$$|\vec{a}| = a = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{\dot{v}_x^2 + \dot{v}_y^2 + \dot{v}_z^2} = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}. \tag{1.2.25}$$

The representation of acceleration in a spherical coordinate system has the form:

$$\vec{a} = a_r \vec{e}_r + a_\varphi \vec{e}_\varphi + a_\theta \vec{e}_\theta$$

$$a_r = \ddot{r} - r\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right)$$

$$a_\varphi = r\ddot{\varphi}\sin\theta + 2\dot{\varphi}r\left(\sin\theta + \dot{\theta}\cos\theta\right)$$

$$a_\theta = r\ddot{\theta} - r\dot{\varphi}^2 \sin\theta\cos\theta + 2\dot{r}\dot{\theta}$$

$$a = \sqrt{a_r^2 + a_\varphi^2 + a_\theta^2}\,. \qquad (1.2.26)$$

The end of the velocity vector when the material point moves describes a curve called the ***velocity hodograph***. The acceleration at a given point of the hodograph of speed is directed tangentially to the hodograph at this point:

$$\vec{a} = \frac{d\left(\upsilon\vec{\tau}\right)}{dt}, \qquad (1.2.27)$$

where $\vec{\tau}$ is unit vector tangent to the trajectory and directed along the vector $\vec{\upsilon}$.

Acceleration can be represented as:

$$\vec{a} = \vec{a}_\tau + \vec{a}_n$$

$$a = \sqrt{a_\tau^2 + a_n^2}$$

$$\vec{a}_\tau = \dot{\upsilon}\vec{\tau}, \quad \vec{a}_n = \upsilon\dot{\vec{\tau}}\,. \qquad (1.2.28)$$

Vector $\vec{a}_\tau$ is called ***tangential acceleration***, and vector $\vec{a}_n$ is called ***normal acceleration***. Tangential acceleration $\vec{a}_\tau$ characterizes the in the modulus of speed $\vec{\upsilon}$. Normal acceleration $\vec{a}_n$ characterizes the change in the velocity vector $\vec{\upsilon}$ in the direction of curvilinear acceleration. Orth $\vec{\tau}$ is directed along the tangent to the trajectory.

Consider the case when the body trajectory is in the same plane. Then, for normal acceleration of the body, we can write

$$\vec{a}_n = \frac{\upsilon^2}{R}\vec{n}, \qquad (1.2.29)$$

where $R = ds / d\varphi$ is ***radius of curvature*** of the trajectory at a given point, $\vec{n}$ is the unit vector for the normal to the trajectory, directed towards the direction of rotation of the vector $\vec{\tau}$ when the body moves (i.e., to the centre of curvature of the trajectory).

Vector $\vec{a}_\tau$ coincides in the direction with unit vector $\vec{\tau}$ at $\dot{v} > 0$ (the speed increases with time) and opposite to it at $\dot{v} < 0$ (speed decreases with time).

The final increment of speed $\Delta\vec{v}$ over time $\Delta t = t_2 - t_1$ is

$$\Delta\vec{v} = \vec{v}(t_2) - \vec{v}(t_1) = \int_{t_1}^{t_2} d\vec{v} = \int_{t_1}^{t_2} \vec{a}(t')dt'. \qquad (1.2.30)$$

The average acceleration over a period of time $\Delta t = t_2 - t_1$ is

$$\langle\vec{a}\rangle = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \vec{a}(t')dt' = \frac{\Delta\vec{v}}{\Delta t}. \qquad (1.2.31)$$

In the general case, a material point can participate simultaneously in several translational movements. Since speed and acceleration are vectors, they can be added according to the laws of vector addition

$$\vec{v} = \vec{v}_1 + \vec{v}_2, \quad |\vec{v}| = \sqrt{|\vec{v}_1|^2 + |\vec{v}_2|^2 + 2|\vec{v}_1| \times |\vec{v}_2| \times \cos\alpha},$$

$$\vec{a} = \vec{a}_1 + \vec{a}_2, \quad |\vec{a}| = \sqrt{|\vec{a}_1|^2 + |\vec{a}_2|^2 + 2|\vec{a}_1| \times |\vec{a}_2| \times \cos\beta}. \qquad (1.2.32)$$

Consider a special case $\alpha = \pi/2$, $\beta = \pi/2$, then

$$|\vec{v}| = \sqrt{|\vec{v}_1|^2 + |\vec{v}_2|^2}, \quad |\vec{a}| = \sqrt{|\vec{a}_1|^2 + |\vec{a}_2|^2}. \qquad (1.2.33)$$

1.3. Classification of Mechanical Movements

Mechanical movements are classified according to specific driving conditions. This classification is shown in Table 4.

Table 4. Classification of mechanical movements

	Uniform motion	*Uneven motion*
Rectilinear motion	$\vec{v} = const$, $v = const$	$\vec{v} \neq const$, $v \neq const$
	$a = 0$, $a_n = 0$, $a_\tau = 0$	$\vec{a} = \vec{a}_\tau$, $a_n = 0$, $\alpha_\tau \neq 0$
Curvilinear motion	$\vec{v} \neq const$, $v = const$	$\vec{v} \neq const$, $v \neq const$
	$\vec{a} = \vec{a}_n$, $a_n \neq 0$, $a_\tau = 0$	$\vec{a} = \vec{a}_n + \vec{a}_\tau$, $a_n \neq 0$, $a_\tau \neq 0$

A ***uniform movement*** is a movement in which a material point (body) travels the same path for equal arbitrarily small intervals of time. The module of the displacement vector $\Delta\vec{r}$ is equal to the path Δs in ***rectilinear motion***. The speed $\vec{v}$ is constant and equal to the average speed $\langle\vec{v}\rangle$ with ***uniform rectilinear movement***

$$\vec{v} = \langle\vec{v}\rangle = \frac{\Delta\vec{r}}{\Delta t} = const. \tag{1.3.1}$$

Equation for the trajectory of motion

$$\vec{r}(t) = \vec{r}(t_0) + \vec{v}\cdot(t - t_0), \tag{1.3.2}$$

where $\vec{r}(t_0)$ is the radius vector of the body at the initial time t_0.

The vector equation for $\vec{r}(t)$ is equivalent to three equations for coordinates $x(t)$, $y(t)$, $z(t)$

$$x(t) = x(t_0) + v_x\cdot(t - t_0)$$
$$y(t) = y(t_0) + v_y\cdot(t - t_0)$$
$$z(t) = z(t_0) + v_z\cdot(t - t_0), \tag{1.3.3}$$

where v_x, v_y, v_z are projections of the velocity vector $\vec{v}$ on the coordinate axis.

The path traveled by the body during t is determined by the formula

$$s = v\cdot t. \tag{1.3.4}$$

Translational motion is called ***uniformly accelerated motion*** if the acceleration over time remains constant. With this movement, the acceleration $\vec{a}$ is equal to the average acceleration $\langle\vec{a}\rangle$

$$\vec{a} = \langle\vec{a}\rangle = \frac{\Delta\vec{v}}{\Delta t} = const, \tag{1.3.5}$$

where $\Delta\vec{v}$ is speed increment over time Δt.

At $a > 0$, the movement is accelerated, and at $a < 0$, the movement is slowed down. The speed $\vec{v}(t)$ at time t is determined by the expression

$$\vec{v}(t) = \vec{v}(t_0) + \vec{a}\cdot(t - t_0). \tag{1.3.6}$$

The last formula is equivalent to the three equations for velocity components v_x, v_y, v_z

$$v_x(t) = v_x(t_0) + a_x\cdot(t - t_0)$$
$$v_y(t) = v_y(t_0) + a_y\cdot(t - t_0)$$
$$v_z(t) = v_z(t_0) + a_z\cdot(t - t_0), \tag{1.3.7}$$

where a_x, a_y, a_z are the projections of vector $\vec{a}$ on the coordinate axis.

For the case of motion with constant acceleration, the equation for the trajectory of motion has the form

$$\vec{r}(t)=\vec{r}(t_0)+\vec{v}(t_0)\cdot(t-t_0)+\frac{a(t-t_0)^2}{2}. \qquad (1.3.8)$$

The equations for the projections of vector $\vec{r}(t)$ on the coordinate axis in this case are of the form

$$x(t)=x(t_0)+v_x(t_0)\cdot(t-t_0)+\frac{a_x(t-t_0)^2}{2}$$

$$y(t)=y(t_0)+v_y(t_0)\cdot(t-t_0)+\frac{a_y(t-t_0)^2}{2}$$

$$z(t)=z(t_0)+v_z(t_0)\cdot(t-t_0)+\frac{a_z(t-t_0)^2}{2}. \qquad (1.3.9)$$

The path, travelled by the body, in time t is

$$s=v_0t+\frac{at^2}{2}, \qquad (1.3.10)$$

where v_0, is speed at the initial time t_0.

In case $t_0 = 0$, for the path, speed, average speed, travel time and acceleration, the following formulas can be written

$$s=\frac{v^2-v_0^2}{2a}=v_0t+\frac{at^2}{2}=\frac{v_0+v}{2}t,$$

$$v=\sqrt{v_0^2+2as}=v_0+at,$$

$$\langle v\rangle=\frac{v_0+v}{2}=v_0+\frac{at}{2}=\frac{s}{t},$$

$$t=\frac{2s}{v_0+v}=\sqrt{\frac{2s}{a}+\left(\frac{v_0}{a}\right)^2}-\frac{v_0}{a},$$

$$a=\frac{v^2-v_0^2}{2s}=2\left(\frac{s}{t^2}-\frac{v_0}{t}\right). \qquad (1.3.11)$$

Free fall is a special case of uniformly accelerated motion without initial speed $v_0 = 0$. The acceleration during this movement is equal to the acceleration of gravity $a = g = 9.81$ m/s^2. Consider the case where the daily rotation of the Earth, the dependence of the acceleration of gravity on the distance to the centre of the Earth, and air resistance can be neglected. Then we can write the following formulas

$$v = gt = \sqrt{2gh}, \quad h = \frac{gt^2}{2} = \frac{vt}{2} \tag{1.3.12}$$

where v is body fall rate over time t; h is height at which the body falls.

A body thrown vertically upward with an initial speed of v_0 moves equally slowly with an acceleration of $a = -g = -9.81$ m/s^2. The rise time to a height of h and the speed acquired during time t can be determined by the following formulas

$$h = \frac{v_0 + v}{2}t = v_0 t - \frac{gt^2}{2}, \quad v = v_0 - gt = \sqrt{v_0^2 - 2gh} \tag{1.3.13}$$

The last formulas are written without air resistance.

In turn, the time t_{max}, during which the body reaches a maximum height of $h_{max} = h$ at $v = 0$, and a height of h_{max} can be determined by formulas

$$t_{max} = \frac{v_0}{g}, \quad h_{max} = \frac{v_0^2}{2g}. \tag{1.3.14}$$

The motion of a body thrown horizontally at a speed of $\vec{v}_0$ is the vector sum of two independent motions: uniform horizontal motion with a constant speed v_0 along x axis and uniformly accelerated vertical motion (free fall along y axis with acceleration g). In this case, the movements of the body in the horizontal and vertical directions during the time t are equal, respectively

$$s = v_0 t = v_0\sqrt{\frac{2h}{g}}$$

$$y = h = \frac{gt^2}{2}$$

$$t = \frac{s}{v_0} = \sqrt{\frac{2h}{g}}. \tag{1.3.15}$$

The equation of the trajectory is parabola

$$y = \left(\frac{g}{2v_0^2}\right)x^2. \tag{1.3.16}$$

Other characteristics of motion include:
radius vector

$$\vec{r} = \vec{v}_0 t + \frac{\vec{g}t^2}{2}, \tag{1.3.17}$$

instantaneous velocity

$$\vec{v} = \vec{v}_0 + \vec{g}t, \tag{1.3.18}$$

angle between initial velocity $\vec{v}_0$ and instantaneous velocity $\vec{v}$

$$\tan\alpha = \frac{gt}{v_0}, \tag{1.3.19}$$

instantaneous speed module

$$v = \sqrt{v_0^2 + g^2 t^2}. \tag{1.3.20}$$

Formulas (1.3.17) – (1.3.20) also do not take into account air resistance and daily rotation of the Earth.

The motion of a body thrown at an angle α to the horizon with a speed of $\vec{v}_0$ is the vector sum of two independent motions: a uniform rectilinear motion with a speed of $v_0 \cos\alpha$ along the x axis and free fall along the y axis. Suppose that during time t, the movement of the body in the horizontal direction is s, and the lift height is h. Then, for such a movement, the following formulas

$$x = s = v_0 t \cos\alpha$$

$$y = h = v_0 t \sin\alpha - \frac{gt^2}{2}$$

$$t = \frac{x}{v_0 \cos\alpha}. \tag{1.3.21}$$

The equation of the trajectory of the body is parabola

$$y = x\tan\alpha - \frac{g}{2v_0^2\cos^2\alpha}x^2. \tag{1.3.22}$$

The horizontal v_x and vertical v_y components of the instantaneous velocity vector $\vec{v}$ are equal

$$v_x = \frac{dx}{dt} = v_0 \cos\alpha, \quad v_y = \frac{dy}{dt} = v_0 \sin\alpha - gt \tag{1.3.23}$$

The following characteristics of this motion can be written for the case when air resistance and the Earth's daily rotation can be neglected:
instantaneous speed module

$$v = \sqrt{{v_0}^2 - 2gh}, \tag{1.3.24}$$

body lift time to maximum height

$$t_{\max} = \frac{v_0 \sin\alpha}{g}, \tag{1.3.25}$$

total movement time

$$t_s = \frac{2v_0 \sin\alpha}{g}, \tag{1.3.26}$$

maximum lifting height

$$h_{\max} = \frac{v_0^2 \sin^2\alpha}{2g}, \tag{1.3.27}$$

maximum path

$$s_{\max} = \frac{v_0^2 \sin 2\alpha}{g}. \tag{1.3.28}$$

1.4. Movement of Material Point in a Circle

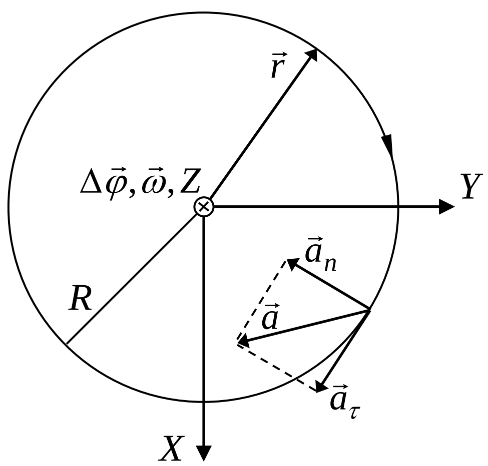

Figure 1.4. Kinematic characteristics of circular motion.

The motion of a material point in a circle is a special case of curvilinear motion along a path lying in one plane (i.e., along a plane curve). We will choose the XY plane as this plane (Figure 1.4). The radius of curvature in this case is constant and equal to the radius of the circle

$$R = \sqrt{x^2 + y^2} = const. \tag{1.4.1}$$

The radius vector of the moving material point is

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \tag{1.4.2}$$

or in polar coordinates

$$x(t)=R\cos\varphi(t),\quad y(t)=\sin\varphi(t), \tag{1.4.3}$$

where $\varphi(t)$ is the angle of rotation of the radius vector $\vec{r}(t)$ in the XY plane at a given point in time t.

The speed

$$v(t)=\frac{d\vec{r}(t)}{dt}, \tag{1.4.4}$$

with which a material point moves in a circle is called the ***linear velocity***.

Circular motion can also be characterized by angular velocity and angular acceleration. To determine these values, we define a rotation through a small angle $\Delta\varphi$ in the form of a vector $\Delta\vec{\varphi}$, whose module is equal to the angle of rotation ($|\Delta\vec{\varphi}|=\Delta\varphi$), and the direction coincides with the axis of rotation around which the rotation is made, and is determined by the rule of the right screw.

Angular velocity $\vec{\omega}$ is the limit to which the ratio of the small angle of rotation $\Delta\varphi$ to the time interval Δt, over which this rotation occurred, tends to the time interval to zero

$$\vec{\omega}=\lim_{\Delta t\to 0}\frac{\Delta\vec{\varphi}}{\Delta t}=\frac{d\vec{\varphi}}{dt}=\dot{\vec{\varphi}}. \tag{1.4.5}$$

Albert of Saxony (1320 – 07.08.1390) is the author of the concept of angular velocity of rotation. The direction of the angular velocity $\vec{\omega}$ coincides with the direction of the infinitesimal angular displacement $d\vec{\varphi}$. The relationship of linear velocity $\vec{v}$ with angular velocity $\vec{\omega}$ has the form

$$\vec{v}=[\vec{\omega},\vec{r}],\quad |\vec{v}|=v=\omega r\sin\alpha=\omega R, \tag{1.4.6}$$

where $\vec{r}$ is radius vector drawn from the origin; α is the angle between the vectors $\vec{\omega}$ and $\vec{r}$; R is the distance from the material point M to the axis of rotation (the radius of the circle along which point M moves)

Normal acceleration equals

$$\vec{a}_n=-\omega^2\vec{R}$$

$$|\vec{a}_n|=a_n=\frac{v^2}{R}=\omega^2R,\quad |\vec{R}|=R. \tag{1.4.7}$$

The minus sign in (1.4.7) reflects the fact that vectors $\vec{a}_n$ and $\vec{R}$ have opposite directions.

Angular acceleration $\vec{\beta}$ is the limit to which the ratio of the small increment of the angular velocity $\Delta\vec{\omega}$ tends due to a change in the speed of rotation of the material point around the axis or a small rotation of the axis of rotation in space, obtained in time Δt when Δt tends to zero

$$\vec{\beta} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\varphi}}{dt^2} = \dot{\vec{\omega}} = \ddot{\vec{\varphi}}. \tag{1.4.8}$$

The tangential acceleration module $\vec{a}_\tau$ and the angular acceleration module $\vec{\beta}$ are interconnected as follows

$$|\vec{a}_\tau| = a_\tau = \left| \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} \right| = \left| \lim_{\Delta t \to 0} R \frac{\Delta\vec{\omega}}{\Delta t} \right| = \beta R. \tag{1.4.9}$$

The elementary increment of the angle of rotation is

$$d\vec{\varphi} = \vec{\omega}(t)dt. \tag{1.4.10}$$

The rotation angle $\Delta\varphi$ (***angular displacement***) for a period of time from t_0 to t is determined by the formula

$$|\Delta\vec{\varphi}| = \Delta\varphi = \varphi(t) - \varphi(t_0) = \int_{t_0}^{t} \omega(t')dt', \tag{1.4.11}$$

where $\varphi(t_0)$ is the value of the angle of rotation at the initial moment of time t_0.

The elementary increment of the angular velocity is

$$d\vec{\omega} = \vec{\beta}(t)dt. \tag{1.4.12}$$

The increment of the angular velocity $\Delta\vec{\omega}$ for a time from t_0 to t is

$$\Delta\vec{\omega} = \vec{\omega}(t) - \vec{\omega}(t_0) = \int_{t_0}^{t} \vec{\beta}(t')dt', \tag{1.4.13}$$

where $\vec{\omega}(t_0)$ is the angular velocity at the initial moment of time t_0.

The average angular velocity $\langle\vec{\omega}\rangle$ and the average angular acceleration $\langle\vec{\beta}\rangle$ over a period of time $\Delta t = t - t_0$ are determined by formulas

$$\langle\vec{\omega}\rangle = \frac{1}{t-t_0}\int_{t_0}^{t}\vec{\omega}(t')dt' = \frac{\Delta\vec{\varphi}}{\Delta t}$$

$$\langle\vec{\beta}\rangle = \frac{1}{t-t_0}\int_{t_0}^{t}\vec{\beta}(t')dt' = \frac{\Delta\vec{\omega}}{\Delta t}. \qquad (1.4.14)$$

SI units of angular velocity and angular acceleration: $[w]$ = rad/s, $[\beta]$ = rad/s^2.

The period T of the rotation of the material point around the fixed axis of rotation is the time during which this point rotates through an angle of $\Delta\varphi$

$$\Delta\varphi = \int_{0}^{T}\omega(t)dt = 2\pi\ . \qquad (1.4.15)$$

The number ν of turns per unit of time is

$$\nu = \frac{1}{2\pi}\int_{0}^{1}\omega(t)dt\,. \qquad (1.4.16)$$

Uniform circular motion (or uniform rotation) is a movement in which for equal intervals of time there is a rotation at the same angle. With this movement, the angular velocity $\vec{\omega}$ is constant and equal to the average angular velocity $\langle\vec{\omega}\rangle$

$$\vec{\omega} = \langle\vec{\omega}\rangle = \frac{\Delta\vec{\varphi}}{\Delta t} = const\,. \qquad (1.4.17)$$

Uniform rotation is characterized by a period of T. At $\Delta\varphi = 2\pi$ we have $\Delta t = T$, and therefore

$$T = \frac{2\pi}{\omega}, \qquad (1.4.18)$$

the number ν of turns per unit of time is

$$\nu = \frac{\omega}{2\pi} = \frac{1}{T}, \qquad (1.4.19)$$

angular displacement

$$\varphi = 2\pi N, \qquad (1.4.20)$$

where N is number of turns.

Between the formulas that describe the translational motion and circular motion, there is an analogy, which is shown in table 5.

Table 5. Translational motion and circular motion

Translational motion	*Rotational motion*
Core quantities	
Path s	Angle φ (rad)
Velocity v	Angular velocity ω
Acceleration a	Angular acceleration β
Relationship formulas	
$s=\varphi R$, $v=\omega R$	
$a=a_\tau+a_n$, $a_\tau=\beta R$, $a_n=\omega^2 R$	
Uniform movement	
$s=vt$	$\varphi=\omega t$
Uniformly accelerated movement $(t_0=0)$	
$s=\dfrac{v^2-v_0^2}{2a}=v_0 t+\dfrac{at^2}{2}$	$\varphi=\dfrac{\omega^2-\omega_0^2}{2\beta}=\omega_0 t+\dfrac{\beta t^2}{2}$
$s=\dfrac{v_0+v}{2}t$	$\varphi=\dfrac{\omega_0+\omega}{2}t$
$v=\sqrt{v_0^2+2as}=v_0+at$	$\omega=\sqrt{\omega_0^2+2\beta\varphi}=\omega_0+\beta t$
$\langle v\rangle=\dfrac{v_0+v}{2}=v_0+\dfrac{at}{2}=\dfrac{s}{t}$	$\langle \omega\rangle=\dfrac{\omega_0+\omega}{2}=\omega_0+\dfrac{\beta t}{2}=\dfrac{\varphi}{t}$
$a=\dfrac{v^2-v_0^2}{2s}=2\left(\dfrac{s}{t^2}-\dfrac{v_0}{t}\right)$	$\beta=\dfrac{\omega^2-\omega_0^2}{2\varphi}=2\left(\dfrac{\varphi}{t^2}-\dfrac{\omega_0}{t}\right)$

1.5. Newton's Laws

Newton's laws were formulated by Isaak Newton (25.12.1642 – 20.03.1726) in his book Mathematical Principles of Natural Philosophy (1687).

Newton's first law is formulated as follows: a body (material point) is at rest or in a uniform and rectilinear motion, if it is not subject to external influences from other bodies. The property of the body to maintain its speed in the absence of interaction with other bodies is called ***inertia***. A measure of the inertia of the body (material point) in translational motion is a physical quantity called the ***inert mass*** of the body. The presence of mass in the body is confirmed by experiments, which show that the same effect gives different bodies different accelerations in magnitude. Mass is one of the main quantities characterizing the mechanical movement of the body. The mass of a body contained in a unit volume is called its ***density***.

The density ρ of the body at a given point is equal to the limit to which the ratio of the mass Δm of the body element tends in the vicinity of this point to the volume ΔV of this element with unlimited decrease

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}. \tag{1.5.1}$$

Density ρ is a function of three spatial coordinates x, y, z, characterizing a point within a volume V defined by the size of the body: $\rho = \rho(x, y, z) = \rho(\vec{r})$.

Therefore, body mass can be determined by formula

$$m = \int_V \rho dV = \int_V \rho(x, y, z) dxdydz, \tag{1.5.2}$$

where the integral is taken over the entire volume V of the body; dx, dy, dz are linear sizes of elementary volume dV, counted along the coordinate axes x, y, z.

Average density is

$$\langle \rho \rangle = \frac{1}{V} \int_V \rho dV = \frac{m}{V}. \tag{1.5.3}$$

A body is called ***homogeneous body*** if

$$\rho = \langle \rho \rangle = \frac{m}{V}. \tag{1.5.4}$$

SI unit of density: $[\rho]$ = kg/m^3.

A consequence of Newton's first law is that any change in the state of motion of the body is due to the action of forces on it.

Newton's first law is not absolute. This law is not valid in all reference frames. The reference frame in which this law holds is called the ***inertial reference frame***. The equivalent definition of an inertial reference frame has the following form: a system with respect to which space is homogeneous (i.e., the law of conservation of momentum is fulfilled in it) and isotropic (i.e., the law of conservation of angular momentum is valid), and time is homogeneous (i.e., the law of conservation of energy is valid), called the inertial reference frame.

The existence of an inertial reference frame in classical mechanics is a postulate generalizing a large number of experimental data. The physical content of Newton's first law, therefore, consists in asserting that there is at least one inertial reference frame.

Newton's second law establishes a relationship between the force $\vec{F}$ acting on the body and the acceleration $\vec{a}$ acquired by the body under the influence of this force: acceleration is directly proportional to the acting force and inversely proportional to the mass of the body

$$\vec{a} = \frac{\vec{F}}{m}. \tag{1.5.5}$$

Other forms of equation (1.5.5):

$$m\vec{a} = \vec{F},$$

$$m\frac{d\vec{v}}{dt} = \vec{F}, \quad m\dot{\vec{v}} = \vec{F},$$

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}, \quad m\ddot{\vec{r}} = \vec{F}. \tag{1.5.6}$$

SI unit of force: $[F]$ = N.

According to Newton's second law, force $\vec{F}$ in the general case is a function of the radius vector $\vec{r}$, velocity $\vec{v}$ and time t

$$\vec{F} = \vec{F}(\vec{r}, \vec{v}, t). \tag{1.5.7}$$

For two material points 1 and 2 interacting with a force of $\vec{F}$, we obtain

$$\vec{F} = \vec{F}(\vec{r}_2 - \vec{r}_1, \vec{v}_2 - \vec{v}_1, t), \tag{1.5.8}$$

where $\vec{r}_1$, $\vec{r}_2$ and $\vec{v}_1$, $\vec{v}_2$ are the coordinates and speeds of the material points 1 and 2.

Equation (1.5.6) is called the ***equation of body motion*** (equation material point motion) or the basic law of the dynamics of the material point. This equation is a second-order differential equation with respect to time t. Solutions of this equation with boundary conditions determining the position of the material point at the initial time t_0, and its speed $\dot{\vec{r}}(t_0) = \vec{v}(t_0)$, make it possible to determine the position $\vec{r}(t)$ of the point and its velocity $\vec{v}(t)$ at any subsequent moment $t > t_0$ in time.

Differential equation (1.5.6) is equivalent to such equations:

1) Cartesian coordinates

$$m\ddot{x} = F_x, \quad m\ddot{y} = F_y, \quad m\ddot{z} = F_z \tag{1.5.9}$$

where F_x, F_y, F_z are projections of the force vector $\vec{F}$ on the coordinate axes with orts $\vec{i}$, $\vec{j}$, $\vec{k}$.

2) spherical coordinates

$$m\left(\ddot{r} - r\dot{\varphi}^2 \sin^2\theta - r\dot{\theta}^2\right) = F_r$$

$$m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\sin\theta + 2r\dot{\varphi}\dot{\theta}\cos\theta = F_\varphi$$

$$m\left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2 \sin\theta\cos\theta\right) = F_\theta, \tag{1.5.10}$$

where F_r, F_φ, F_θ are projections of vector $\vec{F}$ on coordinate axes with orts $\vec{e}_r$, $\vec{e}_\varphi$, $\vec{e}_\theta$.

In the general case, when body mass m is a function of time (motion with variable mass), the basic law of dynamics has the form

$$\frac{d\vec{p}}{dt} = \vec{F}, \tag{1.5.11}$$

where value

$$\vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt} \tag{1.5.12}$$

is called the ***momentum*** of a material point by a mass m moving at a speed of $\vec{v} = \dfrac{d\vec{r}}{dt}$.

The law of dynamics (1.5.6) can be written in integral form

$$\Delta\vec{p} = \vec{p}(t) - \vec{p}(t_0) = \int_{t_0}^{t} \vec{F}(t')dt', \tag{1.5.13}$$

where the value $\Delta\vec{p}$ characterizes the change in momentum over a period of time from t_0 to t, and is called the ***momentum of the force***.

In the case of a body moving under the influence of a force $\vec{F}$ which is independent of time, the momentum of the force is

$$\Delta\vec{p} = \vec{F}\Delta t, \quad \Delta t = t - t_0. \tag{1.5.14}$$

Then the momentum $\vec{p}(t)$ is determined by the formula

$$\vec{p}(t) = \vec{p}(t_0) + \vec{F}(t - t_0), \tag{1.5.15}$$

where $\vec{p}(t_0)$ is the momentum of the body at the initial moment of time.

In the case of inertia, the following relations can be written

$$\vec{F} = 0, \quad \vec{p} = \text{const}. \tag{1.5.16}$$

When moving under the action of a variable force, the change in momentum $\Delta\vec{p}$ over a period of time $\Delta t = t - t_0$ is expressed by the relation

$$\Delta\vec{p} = \vec{p}(t) - \vec{p}(t_0) = <\vec{F}> (t - t_0), \tag{1.5.17}$$

where

$$\langle \vec{F} \rangle = \frac{1}{t - t_0} \int_{t_0}^{t} \vec{F}(t')dt' = \frac{\Delta\vec{p}}{\Delta t} \tag{1.5.18}$$

is the average value of the force vector $\vec{F}$ over a period of time Δt.

Newton's third law is formulated as follows: <u>the forces with which two bodies act on each other are equal in magnitude and opposite in direction</u>. We denote by symbol $\vec{F}_{12}$ the force with which body 1 acts on body 2, and by symbol $\vec{F}_{21}$ we denote the force with which body 2 acts on body 1. In this case

$$\vec{F}_{12} = -\vec{F}_{21}. \tag{1.5.19}$$

From experience it follows that the forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are directed along one straight line connecting the bodies 1 and 2.

Newton's third law implicitly contains the assumption of an infinitely large velocity of propagation of the disturbance in the field through which the bodies interact. In classical mechanics, considering the motion of bodies with velocities v much lower than the speed of light in vacuum ($v << c$), the above assumption is fulfilled with very high accuracy.

1.6. Forces

The ***absolute movement*** of a point is its movement with respect to some inertial reference frame. The ***relative motion*** of a point is its movement with respect to the moving reference frame. The portable movement is the absolute movement of that point of the moving frame of reference through which the moving point passes at the moment in time. The choice of an absolute and relative reference frame is conditional. The relationship between the radius vectors $\vec{r}$ and $\vec{r}'$ of the moving point, drawn, respectively, from the beginning O of the fixed reference system (x, y, z) and the beginning O' of the moving reference system (x', y', z') has the form

$$\vec{r} = \vec{r}_0 + \vec{r}' = \vec{r}_0 + \left(x'\vec{i}' + y'\vec{j}' + z'\vec{k}'\right). \tag{1.6.1}$$

We get the expression for absolute speed

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{v}_0 + \left(\dot{x}'\vec{i}' + \dot{y}'\vec{j}' + \dot{z}'\vec{k}'\right) + \left(x'\dot{\vec{i}}' + y'\dot{\vec{j}}' + z'\dot{\vec{k}}'\right) =$$

$$= \vec{v}_0 + \vec{v}' + x'\left[\vec{\omega}, \vec{i}'\right] + y'\left[\vec{\omega}, \vec{j}'\right] + z'\left[\vec{\omega}, \vec{k}'\right] =$$

$$= \vec{v}_0 + \vec{v}' + \left[\vec{\omega}, \left(x'\vec{i}' + y'\vec{j}' + z'\vec{k}'\right)\right] = \vec{v}_0 + \vec{v}' + \left[\vec{\omega}, \vec{r}'\right], \tag{1.6.2}$$

where $\vec{v}_0$ is the translational speed of the moving system; $\vec{v}'$ is the speed of the point relative to the moving system (relative speed); $\vec{v}_e = \vec{v}_0 + \left[\vec{\omega}, \vec{r}'\right]$ is ***portable speed***.

The absolute acceleration is

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{a}_0 + \left(\ddot{x}'\vec{i}' + \ddot{y}'\vec{j}' + \ddot{z}'\vec{k}' + \dot{x}'\dot{\vec{i}}' + \dot{y}'\dot{\vec{j}}' + \dot{z}'\dot{\vec{k}}'\right) +$$

$$+ \left(\dot{x}'\dot{\vec{i}}' + \dot{y}'\dot{\vec{j}}' + \dot{z}'\dot{\vec{k}}' + \dot{x}'\ddot{\vec{i}}' + \dot{y}'\ddot{\vec{j}}' + \dot{z}'\ddot{\vec{k}}'\right) =$$

$$= \vec{a}_0 + \left(\vec{a}' + \left[\vec{\omega}, \vec{v}'\right]\right) + \left(\left[\vec{\omega}, \vec{v}'\right] + \left[\vec{\varepsilon}, \vec{r}'\right] + \left[\vec{\omega}, \left[\vec{\omega}, \vec{r}'\right]\right]\right) =$$

$$= \vec{a}_0 + \vec{a}' + 2\left[\vec{\omega}, \vec{\upsilon}'\right] + \left[\vec{\varepsilon}, \vec{r}'\right] + \left[\vec{\omega}, \left[\vec{\omega}, \vec{r}'\right]\right], \tag{1.6.3}$$

where $\vec{a}_0$ is the acceleration of the translational movement of the moving system;

$\vec{a}'$ is relative acceleration;

$\left[\vec{\omega}, \left[\vec{\omega}, \vec{r}'\right]\right]$ is ***centrifugal acceleration***;

$\vec{a}_e = \vec{a}_0 + \left[\vec{\varepsilon}, \vec{r}'\right] + \left[\vec{\omega}, \left[\vec{\omega}, \vec{r}'\right]\right]$ is ***portable acceleration***;

$\vec{a}_C = 2\left[\vec{\omega}, \vec{v}'\right]$ is ***Coriolis acceleration***.

Gaspard-Gustave de Coriolis (21.05.1792 – 19.09.1843) in 1835 introduced the concept of a special type of inertia force, which is now called the ***Coriolis force***.

Non-inertial reference frames are reference frames that move with acceleration relative to inertial reference frames. In non-inertial reference frames there are accelerations that are associated with forces of the same nature that are known in inertial reference frames. For convenience, it is assumed that in non-inertial reference systems, as well as in inertial reference systems, accelerations are caused only by forces. However, along with the "ordinary" forces, there are forces of a special nature, which are called the ***forces of inertia***.

Newton's second law in non-inertial reference frames has the form

$$m\vec{a}' = \vec{F} + \vec{F}_i, \tag{1.6.4}$$

where $\vec{F}_i$ are inertia forces.

Then for the inertia forces we get

$$\vec{F}_i = m\vec{a}' - \vec{F} = m\vec{a}' - m\vec{a} = m(\vec{a}' - \vec{a}) =$$

$$= -m\left(\vec{a}_0 + 2[\vec{\omega}, \vec{v}'] + [\vec{\varepsilon}, \vec{r}'] + [\vec{\omega}, [\vec{\omega}, \vec{r}']]\right). \tag{1.6.5}$$

Therefore, the inertia forces are due to the difference between relative and absolute accelerations. The names of the inertia forces are associated with the names of the corresponding accelerations (see the notation for the formula (1.6.3)).

When solid bodies come into contact, an interaction arises between them, which prevents their relative movement. The forces of this interaction are called ***friction forces***. Friction forces act along the surfaces of the contacting bodies (i.e., tangent to the rubbing surfaces) and are directed in the direction opposite to the movement of these bodies relative to one another. Friction between the surfaces of two solids in the absence of any layer is called ***dry friction***. Three types of dry friction are distinguished: ***rest friction***, ***sliding friction*** and ***rolling friction***.

With an increase in the external force $\vec{F}_{ex,r}$, which tends to give relative motion to the bodies at rest, the friction force of rest increases from zero to a certain maximum value $\vec{F}_m$. The bodies begin to move relative to each other under condition $\left|\vec{F}_{ex,r}\right| > \left|\vec{F}_m\right|$. We denote by symbol $\vec{F}_{\min, r}$ the minimum external force at which the bodies begin to move, and by symbol $\vec{F}_N$ we denote the modulus of the normal pressure force pressing one body to another. Then, by definition, the ***rest friction force*** $\vec{F}_{f,r}$ is

$$\vec{F}_{f,r} = -\vec{F}_{\min, r}, \quad \left|\vec{F}_{f,r}\right| = \mu_r \left|\vec{F}_N\right|, \tag{1.6.6}$$

where μ_r called the ***coefficient of rest friction***.

Force $\vec{N}'$, equal in magnitude to force $\vec{F}_N$ and directed opposite to it, is called the ***normal reaction force***

$$\vec{N}' = -\vec{F}_N . \tag{1.6.7}$$

The force of rest friction increases with an increase in the force of normal pressure and does not depend on the area of contacting surfaces. The rest friction coefficient depends on the substance of the contacting bodies and the state of their surfaces.

The force that arises between bodies in contact and moving relative to each other is called the ***sliding friction force***. The sliding friction force is less than the rest friction force. This force depends on the kind and condition of the rubbing surfaces, as well as on the speed of their mutual movement. By definition, the sliding friction force $\vec{F}_{f,s}$ is equal in magnitude and opposite in direction to that external force $\vec{F}_{ex,s}$, at which the contacting bodies move uniformly one relative to the other

$$\vec{F}_{f,s} = -\vec{F}_{ex,s}, \quad \left|\vec{F}_{f,s}\right| = \mu_s \left|\vec{N}'\right|. \tag{1.6.8}$$

The value μ_s is called the ***coefficient of sliding friction***. In general, $\mu_s < \mu$. The sliding friction coefficient depends on the speed v of the mutual movement of the bodies: $\mu_s = \mu_s(v)$. For the case when the speed v is small, a relation of $\mu_s \approx \mu_r$ takes place. Formula (1.6.8) is called the ***Amontons' first law***. Guillaume Amontons (08.31.1663 – 10.10.1705) in 1699 and Charles-Augustin de Coulomb (06.14.1736 – 08.23.1806) in 1785 investigated the physical processes associated with sliding friction.

The force acting from the supporting surface on the rolling body is called the ***rolling friction force***. The rolling friction force $\vec{F}_{f,rol}$ is much less than the sliding friction force

$$\left|\vec{F}_{f,rol}\right| << \left|\vec{F}_{f,s}\right|. \tag{1.6.9}$$

The rolling friction force $\vec{F}_{f,rol}$ is equal in magnitude and opposite in direction to that external force $\vec{F}_{ex,rol}$ at which the body rolls along the support uniformly

$$\vec{F}_{f,rol} = -\vec{F}_{ex,rol}, \quad \left|\vec{F}_{f,rol}\right| = \mu_{rol} \left|\vec{N}\right|, \tag{1.6.10}$$

where μ_{rol} called ***rolling friction coefficient*** ($\mu_{rol} << \mu_s$). The coefficient μ_{rol} depends on the radius of the rolling body.

All real solids under the influence of external forces change their linear dimensions and volume. Such changes are called ***solid deformation***. Deformations are conditionally divided into elastic and plastic deformations. ***Elastic deformations*** disappear after the termination of the action of external forces. ***Plastic deformation*** is stored in the body after the termination of external forces. Deformation leads to the appearance in the body of forces that impede this deformation. Such forces are called ***elastic forces***.

Bodies are called ***isotropic bodies*** if their elastic properties are the same in all directions. An anisotropic body is characterized by a difference in elastic properties in different directions.

Elastic deformations are characterized by ***stress***

$$\vec{\sigma}_n = \frac{d\vec{F}_{el}}{dS}, \tag{1.6.11}$$

where $d\vec{F}_{el}$ is the elastic force with which two parts of the body interact on an infinitely small area dS.

To uniquely determine the mechanical stress $\vec{\sigma}_n$ in a solid at point O of area S, it is sufficient to set the stresses $\vec{\sigma}_x$, $\vec{\sigma}_y$ and $\vec{\sigma}_z$ on three mutually perpendicular platforms passing through point O:

$$\vec{\sigma}_n = \vec{\sigma}_x n_x + \vec{\sigma}_y n_y + \vec{\sigma}_z n_z, \tag{1.6.12}$$

where n_x, n_y и n_z are projections of the external normal $\vec{n}$ to the area S on the axis of the Cartesian coordinate system with the origin at point O.

The set of nine quantities, which are projections of vectors $\vec{\sigma}_x$, $\vec{\sigma}_y$, and $\vec{\sigma}_z$ on axis x, y, z, is called the ***tensor of elastic strains***

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{1.6.13}$$

Elastic deformations are called small if ***Hooke’s law*** is valid for them: the stress σ at elastic deformation of the body is proportional to the relative deformation ε:

$$\sigma = K\varepsilon, \tag{1.6.14}$$

where K is the modulus of elasticity.

The law (1.6.14) is named after Robert Hooke (28.07.1635 – 3.03.1703).

In the case of linear tension or compression of the rod, the relative deformation is

$$\varepsilon = \frac{\Delta l}{l_0}, \qquad (1.6.15)$$

where l_0 is the length of the undeformed rod; $l_0 + \Delta l$ is the length of the deformed rod.

The elastic modulus $K = E$ is called ***Young's modulus***. Young's modulus is named after Thomas Young (13.06.1773 – 10.05.1829).

Consider the case of comprehensive (volumetric) extension or contraction of the body. Then the stress, relative strain, and elastic modulus are

$$\sigma = K\varepsilon, \quad \varepsilon = \frac{\Delta V}{V_0}, \quad K = \frac{E}{3(1-2\mu)}, \qquad (1.6.16)$$

where V_0 is the volume of the non deformed body; $V_0 + \Delta V$ is the volume of the deformed body.

The value μ is called the ***Poisson's ratio*** (named after Siméon Poisson (21.06. 1781 – 25.04.1840)).

$$\frac{\Delta d}{d_0} = -\mu \frac{\Delta l}{l_0}, \qquad (1.6.17)$$

where d_0, l_0 are the transverse size and length of the deformed body; $d_0 + \Delta d$ and $l_0 + \Delta l$ is the transverse size and length of the deformed body.

Young's modulus E and Poisson's ratio μ fully characterize the elastic properties of isotropic bodies.

An example of uniform deformation is mechanical shear. A deformation in which all layers of a solid body parallel to a plane (shear plane) move in the same direction is called ***shear***. The direction of the shift is parallel to the plane of the shear. According to Hooke's law, the mechanical stress at shear is

$$\tau = \frac{F}{S} = G\gamma, \qquad (1.6.18)$$

where S is the surface area perpendicular to which an external force F acts; γ is the angle between the surfaces before and after deformation. The quantity G is called the ***shear modulus***. The shear modulus is related to Young and Poisson moduli by

$$G = \frac{E}{2(1+\mu)}. \qquad (1.6.19)$$

An example of heterogeneous deformation is torsion and bending. ***Torsion*** is a deformation of a rigid body in which, under the action of an external force, a relative rotation of parallel sections of the body around a certain axis occurs. External torsional force creates torque M. According to Hooke's law, the following equation can be written

$$M = f\varphi, \tag{1.6.20}$$

where f is the ***torsion module***; φ is the angle of rotation of one section relative to another closely spaced section.

The torsion modulus f depends on the physical properties of the body and its geometric dimensions. For a solid wire with a radius of r and a length of l, a relation of

$$f = \frac{\pi G}{2l} r^4 \tag{1.6.21}$$

takes place.

Any two bodies having masses are attracted to each other. Gravity is a universal property. The gravitational force can only be the forces of attraction. The ***law of gravity*** is formulated as follows: the force with which two material points are attracted to each other is directly proportional to the masses m_1 and m_2 of these points and inversely proportional to the square of the distance r_{12} between them

$$F_{12} = G_N \frac{m_1 m_2}{r_{12}^2}, \tag{1.6.22}$$

where G_N = 6.67430(15) $\frac{\mathrm{m}^3}{\mathrm{kg} \times \mathrm{s}^2}$ is a gravitational constant.

Gravitational force is directed along a straight line passing through interacting material points. Consider the radius vector $\vec{r}_{12}$, directed from the first material point with radius vector $\vec{r}_1$ to the second with radius vector $\vec{r}_2$. The modulus of vector $\vec{r}_{12}$ is equal to the distance between the material points. In this case, the law of gravity can be written in vector form

$$\vec{F}_{12}(\vec{r}_{12}) = G_\mathrm{N} \frac{m_1 m_2}{r_{12}^3} \vec{r}_{12}, \quad \vec{r}_{12} = \vec{r}_2 - \vec{r}\,. \tag{1.6.23}$$

In the general case, to determine the gravitational force of the interaction of two extended bodies, these bodies are mentally divided into elementary masses Δm. Each of these masses can be considered a material point. Then the gravitational force between element i of one body and element k of another is

$$\Delta\vec{F}_{ik} = G_N \frac{\Delta m_i \Delta m_k}{r_{ik}^3} \vec{r}_{ik}, \quad \vec{r}_{ik} = \vec{r}_k - \vec{r}_i\,. \tag{1.6.24}$$

For the case of continuous mass distribution, we can write

$$\vec{F}_{12} = G_N \iint_{V_1, V_2} \frac{\rho_1(\vec{r}_1)\rho_2(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1)dV_1 dV_2,$$

$$\vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k},$$

$$|\vec{r}_2 - \vec{r}_1|^3 = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{3/2}, \qquad (1.6.25)$$

where $dV_1 = dx_1 dy_1 dz_1$, $dV_2 = dx_2 dy_2 dz_2$.

The law of universal gravitation in the form of (1.6.22) is applicable to describe the interaction not only of material points, but also for two bodies, which are homogeneous balls.

The motion of bodies in the Earth's gravitational field describes ***Galileo's law***: all bodies under the influence of the Earth's gravitational force fall with the same acceleration relative to the Earth's surface, equal to $\vec{g}$. Consequently, on every body of mass m the force P acts

$$\vec{P} = m\vec{g}. \qquad (1.6.26)$$

This force is called ***gravity***.

The force $\vec{N}'$, with which the body acts on the suspension or support, is called the ***weight*** of the body. If the body moves with acceleration $\vec{a}$, then its weight is determined by the formula

$$\vec{N}' = m(\vec{g} - \vec{a}). \qquad (1.6.27)$$

The state of weightlessness (body weight is equal to zero) corresponds to equality $N = 0$, those $\vec{a} = \vec{g}$. Forces $\vec{P}$ and $\vec{N}'$, equal at $\vec{a} = 0$, are different forces. The force $\vec{P}$ is applied to the body, and the force $\vec{N}$ is applied to the support or suspension, which limit the movement of the body in the gravitational field of the Earth.

The acceleration of gravity $\vec{g}$, as well as gravity $\vec{P}$ depend on the latitude of the terrain φ and altitude h above sea level, time of day and other factors. Approximately the acceleration of gravity can be calculated by empirical formula

$$|\vec{g}| \approx 9.780318 \cdot (1 + 0.005302 \cdot \sin^2\varphi - 0.0000006 \cdot \sin^2 2\varphi) - 0.00003086 \cdot h. \qquad (1.6.28)$$

Test questions

1. Formulate the principle of compliance.
2. Which section of mechanics includes the law of conservation of momentum?
3. Give a definition of a second.
4. Can a rectangular Cartesian coordinate system include the x-axis and y-axis, which are superimposed by relation $y = 2x – 5$?
5. Calculate the displacement of the centre of mass of the Earth that it completed in half a year.
6. Describe the conditions under which the directions of instantaneous and average speeds coincide during the entire movement.
7. Under what conditions is tangential acceleration modulo almost equal to full acceleration?
8. The normal acceleration is zero at each point on the path. What is the radius of curvature in this case?
9. Write down the equation for the trajectory of uniform rectilinear motion.
10. Is it possible to change slow motion to accelerated motion, if the acceleration of the body does not depend on time?
11. What shape does the graph of dependence $r = f(t)$ have for a body thrown at a certain speed at an angle to the horizontal?
12. Can the concept of free fall be applied at distances comparable to the radius of the Earth?
13. Is it possible to replace the concept of linear velocity with the concept of average speed when a material point moves in a circle?
14. Estimate the angular velocity of a soccer ball that has rolled across the field, a distance of 10 m in a time equal to 2.5 s.
15. Write down a formula that describes the relationship between linear and angular velocities.
16. Estimate the normal acceleration of the material point, which is resting relative to the surface of the Earth.
17. Formulate Newton's first law.
18. Write down Newton's second law for the body that moves by inertia.
19. Indicate the unit of measure for the coefficient of sliding friction.
20. Calculate the gravitational force of interaction between two protons at a distance of 1 cm.

Problem-solving examples

Problem 1.1

Problem description. The kinematic equation of motion of a material point in a straight line (x axis) has the form $x = A + Bt + Ct^3$, where $A = 4$ m, $B = 2$ m/s, $C = -0.5$ m/s^3. For time $t_1 = 2$ s calculate: 1) t_1 coordinate of the point; 2) instantaneous speed v_1; 3) instantaneous acceleration a_1.

Known quantities: $A = 4$ m, $B = 2$ m/s, $C = -0.5$ m/s^3, $t_1 = 2$ s

Quantities to be calculated: x_1, v_1, a_1.

Problem solution. The coordinate of the point for which the kinematic equation of motion is known can be found by substituting the set value of time t_1 instead of time t in the equation of motion

$$x_1 = A + Bt_1 + Ct_1^3. \qquad \text{(P.1.1.1)}$$

We substitute the values A, B, C, t_1 into the formula (P.1.1.1) and calculate the coordinates

$$x_1 = x\big|_{t=t_1} = 4 + 2\times 2 - 0.5\times 2^3 = 4\,\text{m} .$$

We differentiate the x coordinate with respect to time

$$\frac{dx}{dt} = v = B + 3Ct^2. \qquad \text{(P.1.1.2)}$$

We substitute the values B, C, t_1 into the formula (P.1.1.2). In this case, the velocity value is

$$v_1 = v\big|_{t=t_1} = 2 - 3\times 0.5\times 2^2 = -4\,\text{m/s}.$$

A minus sign indicates that at time $t_1 = 2$ s, the point moves in the negative direction of the coordinate axis.

Instantaneous acceleration at an arbitrary point in time can be found if we take the second-order derivative of the x coordinate in time or the first-order derivative of speed in time

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = 6Ct. \qquad \text{(P.1.1.3)}$$

We substitute the values C, t_1 into the formula (P.1.1.3). Then the instantaneous acceleration at time t_1 is

$$a_1 = a\big|_{t=t_1} = -6 \times 0.5 \times 2 = -6\,\text{m/s}^2.$$

A minus sign indicates that the direction of the acceleration vector coincides with the negative direction of the coordinate axis. Under the conditions of this problem, this is true for any moment in time.

Answer. The coordinate of the point is $x_1 = 4$ m. Instantaneous speed equals $v_1 = -4$ m/s. Instantaneous point acceleration equals $a_1 = -6$ m/s^2.

Problem 1.2

Problem description Calculate the centripetal acceleration of points on the earth's surface: at the equator (point A), at latitude 45° (point B) and at the pole (point C). The reason for this centripetal acceleration is the daily rotation of the Earth.

Known quantities: T = 24 h = 8.64×10^4 s, φ =45° ≈ 0.79 rad.

Quantities to be calculated: a_c.

Problem solution. A point on the Earth's surface at the equator makes one complete turn with the Earth in a day. Therefore, the linear velocity of point A is

$$v_e = \frac{l_e}{T} = \frac{2\pi R}{T}, \qquad \text{(P.1.2.1)}$$

where l_e is the circumference of the Earth's equator; R is the radius of the Earth ($R \approx 6.63\times10^6$ m).

The centripetal acceleration of point A at the equator is

$$a_{c,e} = \frac{v_e^2}{R}. \qquad \text{(P.1.2.2)}$$

We substitute the formula (P.1.2.1) into the formula (P.1.2.2). Then

$$a_{c,e} = \frac{4\pi^2 R^2}{RT^2} = \frac{4\pi^2 R}{T^2}. \qquad \text{(P.1.2.3)}$$

Substituting the numerical values, we obtain

$$a_{c,e} = \frac{4\times 3.14^2 \times 6.36\times 10^6}{\left(8.64\times 10^4\right)^2} \approx 3.36\times 10^{-2}\ \text{m/s}^2 .$$

The linear velocity of point B on the Earth's surface at latitude φ is

$$v = \frac{l}{T} = \frac{2\pi r}{T}, \qquad \text{(P.1.2.4)}$$

where r is the radius of the circle described by point B.

For radius r, we can write the following relation

$$r = R\cos\varphi . \qquad \text{(P.1.2.5)}$$

By definition, the centripetal acceleration of point B is

$$a_{c,\varphi} = \frac{v^2}{r}, \qquad \text{(P.1.2.6)}$$

or, taking into account (P.1.2.4) and (P.1.2.5)

$$a_{c,\varphi} = \frac{(2\pi r)^2}{T^2 r} = \frac{4\pi^2 r}{T^2} = \frac{4\pi^2 R}{T^2}\cos\varphi . \qquad \text{(P.1.2.7)}$$

We take into account that $4\pi^2 R/T^2 = a_{c,e}$. In this case, formula (P.1.2.7) can be rewritten as follows

$$a_{c,\varphi} = a_{c,e}\cos\varphi . \qquad \text{(P.1.2.8)}$$

Substitute the numerical values in the formula (P.1.2.8)

$$a_{c,\varphi} = 3.36\times 10^{-2} \times 0.707 \approx 2.38\times 10^{-2}\ \text{m/s}^2 .$$

The linear velocity of point C of the earth's surface at the pole is zero. Therefore, the centripetal acceleration at the pole is zero: $a_{c,p} = 0$.

Answer. Centripetal accelerations at the equator, at latitude φ =45° and at the pole, are: $a_{c,e} \approx 3.36\times 10^{-2}$ m/s^2, $a_{c,\varphi} \approx 2.38\times 10^{-2}$ m/s^2, $a_{c,p} = 0$ respectively.

Problem 1.3

Problem description A 5000 kg freight elevator serves a 900 m deep mine. When the elevator is at the bottom of the shaft, a thrust force of 60 kN begins to act vertically upwards. At a distance of 150 m after the start of the lift, the traction force changes so that over 600 m the movement of the elevator becomes uniform. Finally, the traction force changes once again so that the elevator stops, reaching the top of the shaft. The friction force along the entire path is constant and equal to 5 kN. Consider the movement of the elevator in these areas and determine the total time of movement of the elevator.

Known quantities: $m = 5\times10^3$ kg, $h = 9\times10^2$ m, $F_f = 5$ kN $= 5\times10^3$ N, $F_T = 50$ kN $= 6\times10^4$ N, $h_1 = 1.5\times10^2$ m, $h_2 = 6\times10^2$ m.

Quantities to be calculated: t.

Problem solution. Consider the movement of the elevator in each section. In the first section of length h_1, the elevator moves uniformly accelerated at zero initial speed. In the second section of length h_2, the elevator moves evenly. In the third section of length

$$h_3 = h - \left(h_1 + h_2\right) \qquad \text{(P.1.3.1)}$$

the elevator moves equally slowly and stops at the end of the section.

To describe the movement of the elevator in the first section, we direct the y axis vertically upward and select the beginning of the axis at the bottom of the shaft.

We write the equation of motion of the elevator with acceleration a_1

$$y_1 = \frac{a_1 t^2}{2}. \qquad \text{(P.1.3.2)}$$

For the case when $t = t_1$, and $y_1 = h_1$, equation (P.1.3.1) takes form $h_1 = a_1 t_1^2 / 2$, whence follows

$$t_1 = \sqrt{\frac{2h_1}{a_1}}. \qquad \text{(P.1.3.3)}$$

We write the second Newton's law for the chosen direction of the y axis

$$F_T - mg - F_f = ma_1, \qquad \text{(P.1.3.4)}$$

where F_T is traction, F_f is the force of friction, g is the acceleration of gravity, m is the mass of the elevator.

Rewrite the equation (P.1.3.4)

$$a_1 = \frac{F_T - mg - F_f}{m}. \qquad \text{(P.1.3.5)}$$

Solving equations (P.1.3.3) and (P.1.3.5) together, we obtain

$$t_1 = \sqrt{\frac{2h_1 m}{F_T - mg - F_f}}. \tag{P.1.3.6}$$

The elevator speed at the end of the first section is $v_1 = a_1 t_1$ or, taking into account equations (P.1.3.5) and (P.1.3.6)

$$v_1 = \frac{F_T - mg - F_f}{m}\sqrt{\frac{2h_1 m}{F_T - mg - F_f}} = \sqrt{\frac{2h_1\left(F_T - mg - F_f\right)}{m}}. \tag{P.1.3.7}$$

In the second section, the elevator moves uniformly at the same speed that it received at the end of the first section $v_2 = v_1$.

Choose the beginning of the y axis for the second section at a height of h_1 from the bottom of the shaft. We write the equation of motion of the elevator

$$y_2 = v_2 t. \tag{P.1.3.8}$$

If $t = t_2$, then $y_2 = h_2$ and equation (P.1.3.8) takes form $h_2 = v_2 t_2$, whence follows $t_2 = h_2/v_2$, or, given that $v_2 = v_1$ we get

$$t_2 = h_2\sqrt{\frac{m}{2h_1\left(F_T - mg - F_f\right)}}. \tag{P.1.3.9}$$

We find the movement time in the third section using the concept of average speed: $< v_3 >= \left(v_2 + 0\right)/2 = v_2/2$, or, given that $v_2 = v_1$ we get

$$< v_3 >= \sqrt{\frac{h_1\left(F_T - mg - F_f\right)}{2m}}. \tag{P.1.3.10}$$

Then the elevator travel time in the third section is

$$t_3 = \frac{h_3}{< v_3 >}. \tag{P.1.3.11}$$

We transform the formula (P.1.3.11) taking into account the formulas (P.1.3.1) and (P.1.3.10)

$$t_3 = \left[h - \left(h_1 + h_2\right)\right]\sqrt{\frac{2m}{h_1\left(F_T - mg - F_f\right)}}. \tag{P.1.3.12}$$

The total lift time is

$$t = t_1 + t_2 + t_3 = (2h - h_2)\sqrt{\frac{m}{2h_1\left(F_T - mg - F_f\right)}} = \qquad \text{(P.1.3.13)}$$

$$= \left(2\times 9\times 10^2 - 6\times 10^2\right)\sqrt{\frac{5\times 10^3}{2\times 1.5\times 10^2\times(60-50-5)\times 10^3}} \approx 69\,\mathrm{s}$$

<u>Answer</u>. The total lift time is t = 69 s.

Problems

Problem A

<u>*Problem description*</u>. The material point begins its uniformly accelerated movement from a state of rest. A material point passes a path equal to 90 m in the fifth second. Identify the displacement of the material point in the seventh second.

<u>*Answer*</u>. s = 1.3 m.

Problem B

<u>*Problem description*</u>. From a balloon descending at a constant speed of 2 m/s, a load was thrown vertically upward at a speed of 18 m/s relative to the ground. Determine the distance between the ball and the load at the moment when the load reaches the highest point of its lift.

<u>*Answer*</u>. s = 20 m.

Problem C

<u>*Problem description*</u>. The dependence of the angle of rotation of the radius of the wheel on time is given by equation $\varphi = 4 + 5t - t^3$. Find at the end of the first second of rotation the angular velocity of the wheel, as well as the linear velocity and the full acceleration of the point lying on the wheel rim. The radius of the wheel is 2 cm.

<u>*Answer*</u>. ω = 2 rad/s, v = 0.4 m/s, a = 1.44 m/s^2.

Problem D

Problem description. A vehicle with a mass of 1 t rises along a highway with a slope of 30° under the influence of a traction force of 7 kN. The coefficient of friction between the tires of the car and the surface of the highway is 0.1. Find car acceleration.

Answer. a = 1.2 m/s^2.

Problem E

Problem description. A ball weighing 200 g, tied with a thread to the suspension. The trajectory of a ball moving at a constant speed is a circle. Determine the speed of the ball and the period of its rotation around the circle if the length of the thread is 1 m and the angle of deviation of the thread from the vertical is 60°.

Answer. v = 3.8 m/s, t = 1.4 s.

CHAPTER 2. CONSERVATION LAWS IN MECHANICS

2.1. Law of Momentum Conservation

Newton's second law can be written in the following form

$$m\frac{d\vec{v}}{dt} = \vec{F}. \tag{2.1.1}$$

We take into account that mass m in classical mechanics can be a constant value. Therefore, the mass can be introduced under the sign of the derivative and the formula (2.1.1) can be written as follows

$$\frac{d(m\vec{v})}{dt} = \vec{F}. \tag{2.1.2}$$

The vector value

$$\vec{p} = m\vec{v} \tag{2.1.3}$$

is called the momentum of the material point.

Taking into account formulas (2.1.1) and (2.1.3), Newton's second law can be written as follows

$$\frac{d\vec{p}}{dt} = \vec{F}. \tag{2.1.4}$$

Consequently, Newton's second law can be formulated as: the time derivative of the momentum of a material point is equal to the resultant of all forces acting on this point.

Equation (2.1.4) has a wider scope. The theory of relativity claims that body mass is a function of speed: with increasing speed, the mass grows. In this case, the form of Newton's second law, in which mass and acceleration are explicitly present, becomes inapplicable. However, Newton's second law in the form of (2.1.4) can also be applied in the relativistic case.

Multiply the left and right sides of equation (2.1.4) by dt:

$$d\vec{p} = F\,dt. \tag{2.1.5}$$

Integration (2.1.5) results in an increment of the momentum over a period of time $\tau = t_2 - t_1$

$$\vec{p}_2 - \vec{p}_1 = \int_{t_1}^{t_2} \vec{F}dt. \tag{2.1.6}$$

In the particular case, if $\vec{F} = const$, then the increment of the pulse for the time interval τ is $\Delta\vec{p} = \vec{F}\tau$.

Consider a system consisting of N material points (bodies). The bodies that are part of this system can interact both among themselves and with bodies that do not belong to this system. In accordance with this, the forces acting on the bodies of the system can be divided into internal and external forces. The forces with which all other bodies belonging to the system act on this body are called ***internal forces***. The forces with which bodies that do not belong to a given system act on a given body are called ***external forces***.

A system of bodies is called a ***closed system*** if external forces are absent.

The momentum $\vec{p}$ of a system of bodies is the vector sum of the momenta of the bodies that make up the system

$$\vec{p} = \sum_{i=1}^{n} \vec{p}_i \,. \qquad (2.1.7)$$

The ***centre of inertia*** of a system is a point whose position in space is defined by a radius vector

$$\vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_N\vec{r}_N}{m_1 + m_2 + ... + m_N} = \frac{\sum m_i\vec{r}_i}{\sum m_i} = \frac{\sum m_i\vec{r}_i}{m}, \qquad (2.1.8)$$

where m_i is body mass with an index of i, $\vec{r}_i$ is a radius-vector that determines the position of this body in space, m is the mass of the system.

The Cartesian coordinates of the centre of inertia are equal to the projections of the vector $\vec{r}_c$ onto the coordinate axes

$$x_c = \frac{\sum m_i x_i}{m}, \quad y_c = \frac{\sum m_i y_i}{m}, \quad z_c = \frac{\sum m_i z_i}{m}. \qquad (2.1.9)$$

The speed of the centre of inertia can be obtained by differentiating the vector $\vec{r}_c$ with respect to time

$$\vec{v}_c = \dot{\vec{r}}_c = \frac{\sum m_i\dot{\vec{r}}_i}{m} = \frac{\sum m_i\vec{v}_i}{m}. \qquad (2.1.10)$$

We write the relations $\vec{p}_i = m_i\vec{v}_i$ and $\vec{p} = \sum \vec{p}_i$, then

$$\vec{p} = m\vec{v}_c \,. \qquad (2.1.11)$$

Thus, the momentum of the system is equal to the product of the mass of the system and the speed of its centre of inertia.

Consider the case when the system consists of three bodies. Each of the internal forces, for example, the force $\vec{F}_{12}$ with which body 2 acts on body 1, corresponds to the force $\vec{F}_{21}$ with which body 1 acts on body 2. According to Newton's third law $\vec{F}_{12} = -\vec{F}_{21}$. Let us denote by symbols $\vec{F}_1'$, $\vec{F}_2'$, $\vec{F}_3'$ the results of all the forces with which the external bodies act on the bodies 1, 2 and 3 respectively.

We write down Newton's second law for each of these bodies

$$\frac{d\vec{p}_1}{dt} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_1',$$

$$\frac{d\vec{p}_2}{dt} = \vec{F}_{21} + \vec{F}_{23} + \vec{F}_2',$$

$$\frac{d\vec{p}_3}{dt} = \vec{F}_{31} + \vec{F}_{32} + \vec{F}_3'. \quad (2.1.12)$$

Summarize the three equations (2.1.12). The sum of the internal forces will be zero, therefore

$$\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3\right) = \frac{d}{dt}\vec{p} = \vec{F}_1' + \vec{F}_2' + \vec{F}_3'. \quad (2.1.13)$$

In the absence of external forces, equation (2.1.13) will have the form

$$\frac{d\vec{p}}{dt} = 0 \quad (2.1.14)$$

or

$$\vec{p} = const. \quad (2.1.15)$$

Therefore, for a closed system, momentum $\vec{p}$ is a constant.

We generalize the result to a system consisting of an arbitrary number n of bodies. In this case, equation (2.1.12) will have the form

$$\frac{d\vec{p}_i}{dt} = \sum_{k \neq i} \vec{F}_{ik} + \vec{F}_i', \; i = 1, 2, \ldots, n. \quad (2.1.16)$$

Let us summarize by index i all the equations (2.1.16), taking into account the fact that $\vec{F}_{ik} = -\vec{F}_{ki}$. As a result, we obtain the relation

$$\frac{d\vec{p}}{dt} = \sum_{i=1}^{n} \vec{F}_i'. \quad (2.1.17)$$

Consequently, the time derivative of the momentum vector of the system is equal to the vector sum of all external forces applied to the bodies of the system.

For a closed system, the right-hand side of equation (2.1.17) is equal to zero, as a result of which the vector $\vec{p}$ does not depend on time. This statement is the content of the law of momentum conservation, which is formulated as follows: the momentum of a closed system of material points remains constant.

It should be noted that the momentum remains constant for a system subject to external influences, provided that the external forces acting on the bodies of the system give a zero. If even the sum of the external forces is not equal to zero, but the projection of this sum to a certain direction is zero, then the momentum component in this direction will be constant. Indeed, projecting all the values of equation (2.1.17) in an arbitrary direction x and taking into account that

$$\left(\frac{d\vec{p}}{dt}\right)_x = \frac{d}{dt} p_x , \tag{2.1.18}$$

we get

$$\frac{d}{dt} p_x = \sum_{i=1}^{N} F'_{x,i} . \tag{2.1.19}$$

It follows from (2.1.19) that for the case $\sum_{i=1}^{n} F'_{x,i} = 0$ the following relation holds

$$p_x = const . \tag{2.1.20}$$

In accordance with equation (2.1.11), from the law of momentum conservation it follows that the centre of inertia of a closed system of bodies either moves linearly and uniformly, or remains stationary.

2.2. Law of Angular Momentum Conservation

The ***angular momentum*** of a particle with mass m relative to point O is called the pseudo-vector $\vec{L}$, equal to the vector product of vectors $\vec{r}$ and $\vec{p}$:

$$\vec{L} = [\vec{r}, \vec{p}] = [\vec{r}, (m\vec{v})], \tag{2.2.1}$$

where $\vec{r}$ is the radius vector of a particle drawn from a point O, $\vec{p} = m\vec{v}$ is the momentum of the particle, $\vec{v}$ is the speed of the particle.

The angular momentum of a system of particles relative to a point O is the vector sum of angular momenta of the particles $\vec{L}_i$ that make up the system. For a system of N particles, we can write the relation

$$\vec{L} = \sum_{i=1}^{n} \vec{L}_i = \sum_{i=1}^{n} [\vec{r}_i, \vec{p}_i]. \tag{2.2.2}$$

The moment of force $\vec{F}$ relative to the point O from which the radius vector $\vec{r}$ is drawn to the point of application of force is called a pseudovector $\vec{N}$ equal to the vector product of vectors $\vec{r}$ and $\vec{F}$

$$\vec{N} = \left[\vec{r}, \vec{F}\right] . \tag{2.2.3}$$

The modulus of the moment of force is

$$\left|\vec{N}\right| = N = |\vec{r}| \cdot \left|\vec{F}\right| \cdot \sin\alpha = rF \sin\alpha = l \cdot F , \tag{2.2.4}$$

where $l = r \cdot \sin\alpha$ is a ***arm of force*** $\vec{F}$ relative to the point O.

From the equation of motion for N particles, the equations of motion for momenta $\vec{L}_i$ follow

$$\frac{d\vec{L}_i}{dt} = \vec{N}_i + \sum_k \vec{N}_{ik} , \quad i = 1, 2, ..., n , \tag{2.2.5}$$

where $\vec{N}_i = \left[\vec{r}_i, \vec{F}_i\right]$ is the moment of external force $\vec{F}_i$ relative to the point O acting on the particle with index i, $\vec{r}_i$ is the radius vector of this particle drawn from point O, $\vec{N}_{ik} = \left[\vec{r}_i, \vec{F}_{ik}\right]$ is a moment of inner strength $\vec{F}_{ik}$.

From the fact that the resultant internal forces are equal to zero, it follows that the total moment of internal forces is also equal to zero

$$\sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} \vec{N}_{ik} = 0 . \tag{2.2.6}$$

Using these relations, we can write the following equation for the angular momentum of a system of n particles relative to a point O

$$\frac{d\vec{L}}{dt} = \vec{N} , \tag{2.2.7}$$

where $\vec{N} = \sum_i \vec{N}_i$ is the geometric sum of the moments of external forces relative to point O.

The moments of external forces for a closed system of bodies are equal to zero: $\sum_i \vec{N}_i = 0$. Therefore, in this case, the law of conservation of angular momentum is valid

$$\frac{d\vec{L}}{dt} = 0, \quad \vec{L} = const. \tag{2.2.8}$$

The law of conservation of angular momentum is also valid for open systems in the case when the sum of the moments of all external forces is zero: $\sum_i \vec{N}_i = 0$.

Formulas (2.2.1) and (2.2.3) determine the angular momentum $\vec{L}$ of a particle and the momentum of force $\vec{N}$ relative to point O. Values $\vec{L}$ and $\vec{N}$ are vectors. The projections of these vectors onto some z axis passing through point O are called the angular momentum and the momentum of force relative to z axis. The angular momentum and momentum of force relative to the z axis are indicated by symbols L_z and N_z, respectively. Values L_z and N_z are related by the equation of motion

$$\frac{dL_z}{dt} = N_z. \tag{2.2.9}$$

All three axes x, y, z are equal, therefore, two more analogs of formula (2.2.9) can be written

$$\frac{dL_x}{dt} = N_x$$

$$\frac{dL_y}{dt} = N_y. \tag{2.2.10}$$

The shoulder of a force $\vec{F}$ relative to z axis is the shortest distance between z axis and the line of action of force $\vec{F}$. The moment of force $\vec{F}$ relative to the axis is equal to the product of the component of the force perpendicular to this axis by the corresponding arm of the force. The angular momentum relative to the axis is equal to the product of the momentum $\vec{p}$ component perpendicular to the axis by the same arm of force. The signs of these products are determined by the directions and modules of the vectors $\vec{r}$, $\vec{p}$ and $\vec{F}$.

The moment of force $\vec{N}$ and angular momentum $\vec{L}$ of a particle depend on the position of the point relative to which they are determined. Suppose that $\vec{L}$ and $\vec{N}$ are angular momentum and force relative to some point O, and $\vec{L}'$ and $\vec{N}'$ are angular momentum and force relative to another point O'. In this case, we can write the following relation

$$\vec{L} = \vec{L}' - \left[\vec{R}, \vec{p}\right]$$

$$\vec{N} = \vec{N}' - \left[\vec{R}, \vec{F}\right]$$

$$R = \vec{r}' - \vec{r}\,, \tag{2.2.11}$$

where $\vec{r}'$ and $\vec{r}$ are the radius vectors of the same particle relative to the points O' and O.

2.3. Mechanical Work and Power

A force does work on a material point if this force moves the point a certain distance. The elementary work dA of force $d\vec{F}$, which can be considered constant (independent of displacement), on elementary displacement $d\vec{r}$, is called the scalar product of force $\vec{F}$ and displacement $d\vec{r}$

$$dA = \vec{F}d\vec{r} = \left|\vec{F}\right| \cdot \left|d\vec{r}\right| \cdot \cos\alpha = F\cos\alpha \cdot ds = F_r ds\,. \tag{2.3.1}$$

where α is the angle between the vectors $\vec{F}$ and $d\vec{r}$, $ds = \left|d\vec{r}\right|$ is the length of the elementary displacement $d\vec{r}$, $F_r = F\cos\alpha$ is the projection of the force $\vec{F}$ on the direction of movement $d\vec{r}$.

For Cartesian coordinates, we can write

$$dA = F_x dx + F_y dy + F_z dz\,, \tag{2.3.2}$$

where dx, dy and dz are coordinate increments the radius of the vector $\vec{r}$, F_x, F_y and F_z are the projections of the force on the coordinate axes.

The work A_{12} of force $\vec{F}$ along a curvilinear trajectory L from point s_1 to point s_2 is equal to the sum of all elementary work dA performed by the force $\vec{F}$ on this segment of the trajectory, and is determined by the curvilinear integral

$$A_{12} = \int_L \vec{F}d\vec{r} = \int_{s_1}^{s_2} F\cos\alpha\, ds = \int_{s_1}^{s_2} F_r ds\,. \tag{2.3.3}$$

Other formulas for mechanical work may be considered

$$A_{12} = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz$$

$$A_{12} = \int_{p_1}^{p_2} \frac{d\vec{p}}{dt}\vec{r} = \int_{p_1}^{p_2} \vec{v}\,d\vec{p}$$

$$A_{12} = m\int_{v_2}^{v_1} \vec{v}d\vec{v}\,, \qquad (2.3.4)$$

where x_1, y_1, z_1 and x_2, y_2, z_2 are Cartesian coordinates of points s_1 and s_2, p_1 and p_2 are initial and final momenta of the material point, v_1 and v_2 are initial and final speeds of the material point.

Work A_{12} is a scalar quantity. The work A_{12} is the area along the curve $F_r(s)$ between points s_1 and s_2.

For the case when the force $\vec{F}$ remains constant ($\left|\vec{F}\right| = F$) when moving the material point along the trajectory from the point s_1 with radius vector $\vec{r}_1$ to the point s_2 with radius vector $\vec{r}_2$, the work is equal to the scalar product of the force $\vec{F}$ and displacement $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$:

$$A_{12} = \vec{F}\Delta\vec{r} = \left|\vec{F}\right| \cdot \left|\Delta\vec{r}\right| \cos\alpha = F \cdot \left|\Delta\vec{r}\right| \cos\alpha\,, \qquad (2.3.5)$$

where α is the angle between the vectors $\vec{F}$ and $\Delta\vec{r}$.

SI unit of work is the joule: $[A]$ = J.

The work A of the resulting force $\vec{F}$ is equal to the sum of the work of all the forces $\vec{F}_i$ acting on the material point

$$A = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \int \vec{F}_i d\vec{r} = \int \left(\sum_{i=1}^{n} \vec{F}_i \right) d\vec{r} = \int \vec{F} d\vec{r}\,. \qquad (2.3.6)$$

The ***average power*** $< P >$ is the work ΔA per unit of time Δt spent on this work

$$< P > = \frac{\Delta A}{\Delta t}. \qquad (2.3.7)$$

Instantaneous power is the limit to which the average power $< P >$ tends when the time interval Δt tends to zero

$$P = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \dot{A}. \qquad (2.3.8)$$

The SI unit of power is a watt: $[P]$ = W.

The following formula can be used to calculate power

$$P = \left(\vec{F}\vec{v}\right) = \left|\vec{F}\right| \cdot \left|\vec{v}\right| \cos\alpha\,, \qquad (2.3.9)$$

where $\vec{v} = \dfrac{d\vec{r}}{dt}$ is the instantaneous velocity of the particle, which is affected by the instantaneous (constant when moving by $d\vec{r}$) force $\vec{F}$, α is the angle between the vectors $\vec{F}$ and $\vec{v}$.

If the work is proportional to time, then the power is constant. For the case of uniform accelerated motion ($\vec{F} = const$), we can write the following relation

$$< P >= \vec{F} < \vec{v} >, \tag{2.3.10}$$

where $< \vec{v} >$ is the average particle velocity.

Efficiency η is the ratio of useful work A_u performed by the forces when moving the body to the total work A_f of the forces applied to the body

$$\eta = \frac{A_u}{A_f}. \tag{2.3.11}$$

The efficiency can also be defined as the ratio of useful power P_u to the total power P_f

$$\eta = \frac{P_u}{P_f}. \tag{2.3.12}$$

The efficiency is often expressed as a percentage

$$\eta = \frac{A_u}{A_f} \times 100\%, \;\; \eta = \frac{P_u}{P_f} \times 100\%. \tag{2.3.13}$$

If the body is involved in various processes related to the transfer or conversion of energy, then the overall efficiency is equal to the product of the efficiency of each of the processes

$$\eta = \prod_i \eta_i . \tag{2.3.14}$$

2.4. Law of Energy Conservation

Kinetic energy is equal to

$$W_k = \frac{mv^2}{2} = \frac{p^2}{2m}, \tag{2.4.1}$$

where m is the mass of the material point, v and $p = mv$ is the speed and momentum of the material point.

It follows from Newton's second law that

$$d\left(\frac{mv^2}{2}\right)=\left(\vec{F}d\vec{r}\right), \qquad (2.4.2)$$

where $\vec{F}$ is the resultant of external forces acting on a material point.

Consider the case when the mechanical system is closed. In this case, the relation $\vec{F}=0$ is valid and the law of conservation of kinetic energy is satisfied

$$W_k = const\,. \qquad (2.4.3)$$

The work of all external forces acting on a material point is equal to the increment of the kinetic energy of this point

$$A_{12}=\frac{mv_2^2}{2}-\frac{mv_1^2}{2}=W_{k,2}-W_{k,1}, \qquad (2.4.4)$$

where v_1 and v_2 are the initial and final speeds of the material point, $W_{k,1}$ and $W_{k,2}$ are the initial and final kinetic energies of the material point, A_{12} is work when moving a material point from position 1 to position 2.

Formula (2.3.18) allows you to determine the kinetic energy of a material point as a measure of its mechanical motion. This measure is determined by the work that a material point can perform when braking to a complete stop.

The kinetic energy W_k of a system of n material points is the sum of the kinetic energies of these material points

$$W_k=\sum_{i=1}^{n}W_{k,i}=\sum_{i=1}^{n}\frac{m_i v_i^2}{2}, \qquad (2.4.5)$$

where m_i and v_i is the mass and velocity of the material point with index i.

The work of all external forces acting on the system of material points is equal to the increment of the kinetic energy of this system.

The unit of energy in the SI system is the joule: $[W_k]$ = J.

We state ***Koenig's theorem***. <u>The kinetic energy of the system of material points is equal to the sum of the kinetic energy of the motion of its center of mass with a speed of V and the kinetic energies of the motion of material points with speeds of v_i relative to the translationally motion coordinate system with the origin at the center of mass of the system</u>

$$W=\sum_{i=1}^{n}\frac{m_i v_i^2}{2}+\frac{1}{2}mV^2. \qquad (2.4.6)$$

Conservative forces are those whose work does not depend on the transition of the system of material points from the initial to the final position. Gravity and all central forces are conservative forces. A ***central force*** is a force that is directed to the same point in space, called a ***power centre***, and depends only on the distance to this centre. All other (except conservative forces) forces are called ***non-conservative forces***. Dissipative and gyroscopic forces are non-conservative.

For the case when only conservative forces act on the system of material points, the concept of ***potential energy*** can be introduced. A force field is called a potential force field if it can be described by a function $W_p = W_p(x, y, z, t) = W_p(\vec{r}, t)$. Function W_p depends on coordinates and time, and the partial derivatives of this function with respect to coordinates determine the projections F_x, F_y and F_z of the force $\vec{F}$ on the coordinate axes

$$F_x = -\frac{\partial W_p}{\partial x},\ F_y = -\frac{\partial W_p}{\partial y},\ F_z = -\frac{\partial W_p}{\partial z}. \tag{2.4.7}$$

The function W_p is called potential. In the case of a potential force field, we can write the following expression for the force

$$\vec{F} = -\nabla W_p = -grad\ W_p. \tag{2.4.8}$$

The value ∇ is called the ***Hamilton operator*** or the gradient. The Hamilton operator is named after William Rowan Hamilton (4.08.1805 – 2.09.1865). The representation of the potential gradient in Cartesian coordinates has the form

$$\nabla W_p = grad\ W_p = \frac{\partial W_p}{\partial x}\vec{i} + \frac{\partial W_p}{\partial y}\vec{j} + \frac{\partial W_p}{\partial z}\vec{k}, \tag{2.4.9}$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors of Cartesian coordinates.

The function W_p is also called potential energy. We write the following relation for potential energy when moving a material point from position 1 to position 2:

$$W_p = -\int_1^2 \vec{F} d\vec{r}. \tag{2.4.10}$$

Consequently, the work of conservative forces is equal to the loss of potential energy.

The unit of measurement of potential energy in the SI system is the joule: $[W_p]$ = J.

Examples of potential energies.

The potential energy of a material point with mass m in a uniform gravitational field at an altitude of h above sea level is

$$W_p = mgh, \tag{2.4.11}$$

where the sea level potential is taken equal to zero.

The potential energy of gravitational attraction of two material points with masses m_1 and m_2, which are at a distance r from each other, is

$$W_p = -G_N \frac{m_1 m_2}{r}, \tag{2.4.12}$$

where the potential of the gravitational field at infinity is zero.

The potential energy of a stretched spring with a coefficient of elasticity k when stretched equal to x is

$$W_p = \frac{1}{2} k x^2, \tag{2.4.13}$$

where the potential energy of the undeformed spring is taken equal to zero.

If only conservative forces act in the system, then the following relation is valid

$$A_{12} = \int_1^2 dW_k = -\int_1^2 dW_p, \qquad W_{k,2} - W_{k,1} = W_{p,1} - W_{p,2}. \tag{2.4.14}$$

Formula (2.3.25) can be written as

$$W_{k,1} + W_{p,1} = W_{k,2} + W_{p,2}. \tag{2.4.15}$$

From the equation (2.3.26) follows the ***law of conservation of mechanical energy***. The formulation of this law has the form: <u>if only conservative forces act in the system of material points, then for this system mechanical energy</u> $W_m = W_k + W_p$ <u>is conserved and does not depend on time:</u> $W_m = const$. ***Mechanical energy*** is defined as the sum of the kinetic and potential energies.

Consider the case when, in addition to conservative forces, a non-conservative force acts on a material point, the resultant of which is $\vec{F}'$. The work A'_{12} of non-conservative forces when moving a material point from position 1 to position 2 is to increment the total energy of the material point

$$A'_{12} = \int_1^2 \vec{F}' d\vec{r} = W_2 - W_1. \tag{2.4.16}$$

Condition $W > W_p$ defines the range of permissible coordinates of the material point. If the total energy of a material point is positive $W > 0$, then its motion is ***infinite motion***, i.e. point can go to infinity. If the total energy is negative $W < 0$, then the motion of the material point is ***finite motion***, i.e. this point can only move in a limited area of space.

The total energy W of a system of n non-interacting particles is equal to the sum of all kinetic W_k and potential W_p energies of individual particles $i = 1,2,\ldots, n$. If only external conservative forces act on these particles, then the total energy remains constant

$$W = \sum_{i=1}^{n} \left(W_k + W_p \right) = const\,. \tag{2.4.17}$$

For the total energy of a system of n non-interacting particles, the following relation holds

$$W = W_k + W_{p,ex}, \tag{2.4.18}$$

where $W_k = \sum_{i=1}^{n} W_{k,i}$ is the kinetic energy of the system, $W_{p,ex} = \sum_{i=1}^{n} W_{p,i}$ is the potential energy of particles in an external field of forces.

The force acting on the particle with index i from the side of the external field is determined by the relation

$$\vec{F}_i = -\nabla_i W_{p,ex} = -\left(\frac{\partial W_{p,ex}}{\partial x_i} \vec{i} + \frac{\partial W_{p,ex}}{\partial y_i} \vec{j} + \frac{\partial W_{p,ex}}{\partial z_i} \vec{k} \right), \tag{2.4.19}$$

where x_i, y_i, z_i are the coordinates of the particle with index i.

For a system of n material points with radius vectors $\vec{r}_i$ that are under the action of forces $\vec{F}_i$, the virial theorem is valid

$$< W_k > = -\frac{1}{2} < \sum_{i=1}^{n} \vec{F}_i \vec{r}_i >, \tag{2.4.20}$$

where $W_k = \sum_{i=1}^{n} \frac{m_i v_i^2}{2}$ is the total kinetic energy of the system.

The angle brackets in the formula (2.4.20) mean the time average determined by the characteristic relation

$$< f(t) > = \frac{1}{\tau} \int_0^{\tau} f(t') dt', \tag{2.4.21}$$

where τ is the time interval over which averaging is performed.

2.5. Laws of Conservation and Symmetry

The most general formulation of the law of motion of mechanical systems is given by the ***Hamilton principle*** (or principle of least action). According to this principle, every mechanical system is characterized by a specific function

$$L(q;\dot{q};t) = L(q_1, q_2, ..., q_s;\ \dot{q}_1,\ \dot{q}_2, ..., \dot{q}_s; t), \tag{2.5.1}$$

where $q_1, q_2, ..., q_s$ these are generalized coordinates (any quantities characterizing the position of the system in space), $\dot{q}_1,\ \dot{q}_2, ..., \dot{q}_s$ are generalized speeds, s is the number of degrees of freedom of the system.

The function $L(q;\dot{q};t)$ is called the ***Lagrange function***. Lagrangian mechanics is introduced by Joseph-Louis Lagrange (25.01.1736 – 10.04.1813).

The simultaneous assignment of all generalized coordinates and all generalized speeds completely allows you to determine the mechanical state of the system and determine its further movement. Suppose that at times t_1 and t_2 the system occupies certain positions, which are characterized by a certain set of generalized coordinates $q^{(1)}$ and $q^{(2)}$. Then the mechanical system moves between these positions so that the integral

$$S = \int_{t_1}^{t_2} L(q,\ \dot{q}, t)dt \tag{2.5.2}$$

has the smallest (for sufficiently small sections of the trajectory) possible value. The above statement is called the ***principle of least action***.

A consequence of the principle of least action is the system of ***Lagrange equations*** or ***Euler equations*** (Euler equations are named after Leonhard Euler (15.04.1707 – 18.09.1783) if we consider the relations of the calculus of variations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = 0$$

$$\dots$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0. \tag{2.5.3}$$

If the Lagrange function of this mechanical system is known, then the Lagrange equations establish a relationship between accelerations, velocities and coordinates, i.e. represent the equations of motion of the system.

The movement of the system is accompanied by a change over time of the quantities $2s$ (q_i and $\dot{q}_i$), which determine its state. However, there are functions of these quantities that maintain constant values when the mechanical system moves. These functions are called ***integrals of motion***. The number of independent integrals of motion for closed mechanical systems with s degrees of freedom is $2s - 1$. Among these $2s - 1$ integrals of motion there are those whose constancy is connected with the fundamental properties of space and time. These integrals of motion possess the ***property of additivity***. The values of the additive integrals of motion for systems consisting of parts whose interaction can be neglected are equal to the sum of the values for each of the parts separately. Consider the conservation laws associated with these integrals of motion.

The law of conservation of energy is associated with the uniformity of time. The ***uniformity of time*** means that if at any two points in time all the bodies of a closed system are placed under exactly the same conditions, then starting from these moments, all phenomena in a mechanical system will proceed in exactly the same way.

Due to the homogeneity of time, the Lagrange function is clearly independent of time. We write the total derivative of the Lagrange function with time

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \,. \tag{2.5.4}$$

We rewrite equation (2.5.3)

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\,. \tag{2.5.5}$$

Then equation (2.5.4) will have the form

$$\frac{dL}{dt} = \sum_i \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right). \tag{2.5.6}$$

or

$$\frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0\,. \tag{2.5.7}$$

It follows that the value of $W = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$ remains unchanged when the closed-loop system moves. Value W is called ***energy***. The law of conservation of energy was obtained as a consequence of the uniformity of time. This law is valid not only in closed systems, but also in systems that are in a constant external field.

The law of conservation of momentum is associated with the homogeneity of space. The ***homogeneity of space*** means that if you transfer a closed system of bodies from one place in space to another, while placing all the bodies in it in the same conditions in which they were in their former position, then this will not affect the course of all subsequent phenomena. Due to this homogeneity, the mechanical properties of a closed system do not change with any parallel transfer of the system as a whole in space. In this regard, we consider an infinitesimal transfer to segment $\vec{\varepsilon}$ and require that the Lagrange function remain unchanged. Parallel transfer means a transformation, during which all points of the system move to the same segment $\vec{r}_i' = \vec{r}_i + \vec{\varepsilon}$. The change in the Lagrange function is

$$\delta L = \sum_i \frac{\partial L}{\partial \vec{r}_i} \delta \vec{r}_i = \vec{\varepsilon} \sum_i \frac{\partial L}{\partial \vec{r}_i} = 0. \tag{2.5.8}$$

In view of arbitrariness $\vec{\varepsilon}$, requirement $\delta L = 0$ is equivalent to requirement

$$\sum_i \frac{\partial L}{\partial \vec{r}_i} = 0. \tag{2.5.9}$$

We use the Lagrange formulas

$$\sum_i \frac{\partial L}{\partial \vec{r}_i} = \sum_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \upsilon}\right) = \frac{d}{dt}(p) = 0$$

$$p = const. \tag{2.5.10}$$

Transformations (2.5.10) lead to the law of conservation of momentum for closed systems. Therefore, the law of conservation of momentum is a consequence of the homogeneity of space.

The law of conservation of angular momentum is related to the ***isotropy of space***. The isotropy of space means that if you turn the closed system of bodies in space at any angle, while placing all the bodies in this system under the same conditions that they were before turning, this will not affect the behaviour of the system at subsequent times.

We consider the infinitely small rotation of the system and require that the Lagrange function does not change. We introduce a vector $\delta\vec{\varphi}$ of infinitely small rotation, the absolute value of which is equal to the angle $\delta\varphi$ of rotation, and the direction coincides with the axis of rotation. We find the increment of the radius vector $\vec{r}$ drawn from the common origin to any of the material points of the rotated system. The linear displacement modulus of the end of the radius vector is

$$|\delta\vec{r}| = |\vec{r}| \cdot |\delta\varphi| \cdot \sin\theta. \tag{2.5.11}$$

We rewrite equation (2.5.11) taking into account the direction, in addition, we write the corresponding relation for speed

$$\delta\vec{r} = [\delta\vec{\varphi}, \vec{r}], \quad \delta\vec{v} = [\delta\vec{\varphi}, \vec{v}]. \tag{2.5.12}$$

The variation of the Lagrange function during rotation must be equal to zero

$$\delta L = \sum_i \left(\frac{\partial L}{\partial \vec{r}_i} \delta\vec{r}_i + \frac{\partial L}{\partial \vec{v}_i} \delta\vec{v}_i \right) = 0. \tag{2.5.13}$$

We take into account that

$$\frac{\partial L}{\partial \vec{r}_i} = \dot{\vec{p}}_i \text{ and } \frac{\partial L}{\partial \vec{v}_i} = \vec{p}_i. \tag{2.5.14}$$

Then equations (2.5.12) and (2.5.13) can be rewritten in the form

$$\sum_i \left(\dot{\vec{p}}_i [\delta\vec{\varphi}, \vec{r}_i] + \vec{p}_i [\delta\vec{\varphi}, \vec{v}_i] \right) = 0 \tag{2.5.15}$$

or

$$\delta\vec{\varphi} \sum_i \left([\vec{r}_i, \dot{\vec{p}}_i] + [\vec{\upsilon}_i, \vec{p}_i] \right) = \delta\vec{\varphi} \frac{d}{dt} \sum_i [\vec{r}_i, \vec{p}_i] = 0$$

and

$$\sum_i [\vec{r}_i, \vec{p}_i] = const. \tag{2.5.15}$$

We have found that the law of conservation of angular momentum (relation (2.5.15)) is a consequence of the isotropy of space.

Summing up the above, it can be argued that any closed system has only seven additive integrals of motion: energy, three components of the momentum and three components of the angular momentum.

Test questions

1. What form does the graph $p = f(v)$ have in classical non relativistic mechanics?
2. Give the general form of Newton's second law.
3. Can the force acting on the body be in one case external, and in another case internal?
4. Formulate the conditions under which for non-closed systems you can use the law of conservation of momentum.
5. Calculate the angular momentum module for case $\vec{r} \parallel \vec{v}$.
6. Write down the formula for the moment of force relative to the point.
7. Indicate the reason why the total moment of internal forces is zero.

8. Compare the concepts: shoulder strength relative to a point and shoulder strength relative to an axis.
9. Write down the formula for elementary work in case $\vec{F} \perp d\vec{r}$.
10. Give the units of work and power.
11. Is the power constant if the following relationship is maintained for work and time $A \sim t^2$?
12. Is relation $\eta > 1$ true for the case when the body is involved in several processes?
13. What form does the dependence graph $W_k = f(v)$ have in classical non relativistic mechanics?
14. Write down the relationship between the work of all external forces acting on the system and the increment of the kinetic energy of the system.
15. Formulate Koenig's theorem.
16. Indicate the explicit form of the dependence of the potential energy of attraction of two material points on the distance between them.
17. Formulate the law of conservation of mechanical energy.
18. What parameter affects the number of Lagrange equations describing the behavior of a given mechanical system?
19. Calculate the number of independent integrals of motion for a system with six degrees of freedom.
20. Formulate the property of homogeneity of time.

Problem-solving examples

Problem 2.1

<u>*Problem description*</u>. Two balls, whose masses are m_1 = 2 kg and m_2 = 1.5 kg, move towards each other at speeds of v_1 = 6 m/s and v_2 = 2 m/s. Determine the following values: 1) the speed u of the balls after the impact; 2) kinetic energy of the balls before (T_1) and after (T_2) the impact; 3) a fraction ω of the kinetic energy of the balls, converted into internal energy. The impact is seen as direct and inelastic.

<u>*Known quantities*</u>: m_1 = 2 kg, m_2 = 1.5 kg, v_1 = 6 m/s, v_2 = 2 m/s.

<u>*Quantities to be calculated*</u>: u, T_1, T_2, ω.

<u>*Problem solution*</u>. Inelastic balls do not restore their original shape after impact. Consequently, after an impact there are no forces repelling the balls from each other,

and after the impact I will move the balls together at the same speed u. We define this speed according to the law of conservation of momentum. Since the balls move in one straight line, the law of conservation of momentum can be written in scalar form

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) u . \tag{P.2.1.1}$$

Express speed u from equation (2.1.1)

$$u = \frac{m_1 \upsilon_1 + m_2 \upsilon_2}{m_1 + m_2} . \tag{P.2.1.2}$$

The direction of the velocity of the first ball is taken as positive. In this case, the speed of the second ball, which moves towards the first, should be taken with a negative sign. We substitute the numerical values in the formula (P.2.1.2)

$$u = \frac{2 \times 6 + 1.5 \times 2}{2 + 1.5} = 4.28\,\mathrm{m/s} . \tag{P.2.1.3}$$

The kinetic energies of the balls before and after the impact are determined by the formulas

$$T_1 = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} , \tag{P.2.1.4}$$

$$T_2 = \frac{(m_1 + m_2) u^2}{2} . \tag{P.2.1.5}$$

We substitute numerical values in formulas (P.2.1.4) and (P.2.1.5)

$$T_1 = \frac{2 \times 6^2}{2} + \frac{1.5 \times 2^2}{2} = 39\,\mathrm{J} ,$$

$$T_2 = \frac{(2 + 1.5) \times 4.28^2}{2} = 32\,\mathrm{J} .$$

A comparison of the kinetic energies of the balls before and after the impact shows that, as a result of the inelastic impact of the balls, their kinetic energy decreased. Consequently, there was an increase in the internal energy of the balls. The fraction of kinetic energy, which has turned into internal energy, is determined from equation

$$\omega = \frac{T_1 - T_2}{T_1} . \tag{P.2.1.6}$$

We substitute the numerical values in the formula (P.2.1.6)

$$\omega = \frac{39-32}{39} \approx 0.18.$$

Answer. The speed of the balls and their kinetic energy before and after the collision are u = 4.28 m/s, T_1 = 39 J, T_2 = 32 J. The fraction of kinetic energy that has turned into internal energy is $\omega \approx 0.18$.

Problem 2.2

Problem description. Sleds moving horizontally on ice at a speed of 5 m/s leave the road. Determine the distance travelled by the sled along the road if the length of their runners is 1 m, and the coefficient of friction of the runners on the road surface is 0.5. Neglect the friction of the sled on ice.

Known quantities: μ = 0.5, v = 5 m/s, l = 1 m.

Quantities to be calculated: L.

Problem solution. Divide the distance travelled by the sled on the rough surface of the road into two sections. In the first section, the length of which is equal to the length l of the runners, the friction force is variable. In this section, there is a gradual increase in the normal reaction force of the road surface from zero to a value equal to the gravity of the sled.

In the second section with length l_1, when the runners leave completely from the ice, the friction force is constant. We calculate the work of the friction force in these two sections

$$A_f = A_{f,1} + A_{f,2}. \tag{P.2.2.1}$$

Suppose that the sled passed a small portion of the first stretch of length x. The friction force acting on the length of two runners equal to 1 m is $\mu mg/(2l)$. The friction force acting on the length $2x$ of two runners is equal to $F_f = \mu mgx/l$. The work of the friction force in the first and second sections is

$$A_{f,1} = \int_0^l F_f \cos\alpha dx = -\int_0^l \frac{\mu mg}{l} x dx = -\frac{\mu mgl}{2}, \tag{P.2.2.2}$$

$$A_{f,2} = -\mu mgl_1. \tag{P.2.2.3}$$

The full work of the friction force in two sections is

$$A_f = -\mu mg\left(\frac{l}{2} + l_1\right). \tag{P.2.2.4}$$

The work of the friction force, on the other hand, is equal to the change in the kinetic energy of the sled

$$A_f = \Delta W_k = W_{k,2} - W_{k,1}. \qquad \text{(P.2.2.5)}$$

According to the condition of the problem $W_{k,2} = 0$, $W_{k,1} = mv_0^2/2$. Then for work, we can write the following expression

$$A_f = -W_{k,1} = -mv_0^2/2. \qquad \text{(P.2.2.6)}$$

Having completed the transformation of formulas (P.2.2.4) and (P.2.2.6), we obtain

$$-\mu mg\left(\frac{l}{2} + l_1\right) = -\frac{mv_0^2}{2}, \qquad \text{(P.2.2.7)}$$

then

$$l_1 = \frac{v_0^2 - \mu gl}{2\mu g}. \qquad \text{(P.2.2.8)}$$

All distance travelled by sled is equal

$$L = l + l_1 = l + \frac{\upsilon_0^2 - \mu gl}{2\mu g} = \frac{\upsilon_0^2 + \mu gl}{2\mu g}. \qquad \text{(P.2.2.9)}$$

We substitute the numerical values in the formula (P.2.2.9)

$$L = \frac{5^2 + 0.5 \times 9.8 \times 1}{2 \times 0.5 \times 9.8} \approx 3.1\text{m}.$$

<u>Answer</u>. The distance travelled by the sled is $L \approx 3.1$ m.

Problem 2.3

<u>Problem description</u>. Determine the distance at which the rocket will move away from the surface of the Earth. The initial velocity of the rocket is 9 km/s.

<u>Known quantities</u>: v = 9 km/s = 9×10^3 m/s.

<u>Quantities to be calculated</u>: h.

<u>Problem solution</u>. A variable force of gravity acts on a rocket with mass m

$$F = \frac{GmM}{r^2}, \qquad \text{(P.2.3.1)}$$

where G is the gravitational constant, m is the mass of the rocket, M is the mass of Earth, r is the distance between the rocket and Earth.

When climbing to a height of h, force F will do the work

$$A = \int_R^{R+h} F \cos\alpha dr , \tag{P.2.3.2}$$

where $\alpha = \pi$ rad is the angle between the directions of force and displacement, R is the radius of Earth.

Substituting the formula (2.3.1) into the formula (2.3.2), we obtain

$$A = -\int_R^{R+h} G\frac{mM}{r^2}dr = GmM\left(\frac{1}{R+h} - \frac{1}{R}\right) = -\frac{GmMh}{R(R+h)}. \tag{P.2.3.3}$$

Work is equal to the change in the kinetic energy of the rocket $A_f = \Delta W_k = W_{k,2} - W_{k,1}$. For height h, the following relation is true $W_{k,2} = 0$. In this case, we can write the following formula

$$A = -W_{k,1} = -\frac{mv^2}{2}. \tag{P.2.3.4}$$

We equate the right-hand sides of formulas (P.2.3.3) and (P.2.3.4)

$$\frac{GmMh}{R(R+h)} = \frac{mv^2}{2}. \tag{P.2.3.5}$$

We transform the formula (P.2.3.5)

$$\frac{RhGM}{R^2(R+h)} = \frac{v^2}{2}. \tag{P.2.3.6}$$

Since $GM / R^2 = g_0$ is the acceleration of gravity near the surface of the Earth, equality (P.2.3.6) can be written in the form $Rhg_0 / (R+h) = v^2 / 2$, and

$$h = \frac{v^2 R}{2Rg_0 - v^2}. \tag{P.2.3.7}$$

We substitute the numerical values in the formula (2.3.7)

$$h = \frac{(9\times10^3)^2 \times 6.37\times10^6}{2\times6.37\times10^6\times9.8 - (9\times10^3)^2} \approx 1.17\times10^7 \text{ m}.$$

Answer. The distance at which the rocket moves away from the Earth's surface is $h \approx 1.17\times10^7$ m.

Problems

Problem A
Problem description. A load whose mass is *5* kg falls from a certain height and reaches the surface of the earth in *2.5* s. Find the job done by the load.

Answer. $A = 1.5\times10^3$ J.

Problem B
Problem description. The slope of the highway section is 1 m for every 20 m of the path. Going downhill with the engine turned off, the car moves evenly at a speed of 60 km/h. Determine the power of the engine of the car, rising on the same slope at the same speed. The mass of the car is 1.5 t.

Answer. $N = 2.5\times10^4$ W.

Problem C
Problem description. A satellite with a mass of 12 t rotates in a circular orbit around the Earth, possessing kinetic energy 5.4×10^{10} J. Find the speed of the satellite and the altitude at which it is moving.

Answer. $v = 3\times10^3$ m/s, $h = 3.8\times10^7$ m.

Problem D
Problem description. A bullet flying horizontally gets into a ball suspended on a light rigid rod and gets stuck in it. The mass of the bullet is 1000 times less than the mass of the ball. The distance from the point of suspension of the rod to the centre of the ball is 1 m. Find the speed of the bullet, if it is known that the rod with the ball deviated from the bullet by an angle equal to 10°.

Answer. $v = 570$ m/s

Problem E
Problem description. At the top of a smooth hemisphere with a radius of 0.5 m there is a puck with a mass of 10 g. The puck began to slide along the sphere under the action of a horizontally directed short-term impulse of force 2×10^{-2} N·s. At what height from the base of the hemisphere does the puck come off its surface?

Answer. $h = 0.47$ m.

CHAPTER 3. ROTATIONAL MOTION

3.1. Rigid Body Kinematics

An ***rigid body*** in mechanics is an idealized system of material points, all distances between which remain unchanged when this system as a whole moves in space and time. Real bodies can be approximately regarded as absolutely rigid body if the deformations arising under the action of external forces are small and are not considered when solving the problem of motion of a rigid body.

To unambiguously determine the position of an absolutely rigid body in space, it is enough to set the position of any three of its points that do not lie on one straight line. The distance between the points of a solid does not change when the body moves. Therefore, of the nine coordinates characterizing the position of these three points, only six coordinates are independent. In view of the foregoing, it can be argued that a rigid body, the movement of which is not subject to any restrictions (constraints), is a mechanical system with six degrees of freedom: $s = 6$. The presence of bonds reduces the number of degrees of freedom. Let's look at some examples. Suppose a body has one fixed point. In this case, the body can rotate around this point, and this body has three degrees of freedom In this case, the body can rotate around this point, and this body has three degrees of freedom $s = 3$. A body that can rotate around a fixed axis has one degree of freedom $s = 1$. A rotating body, which can move along a fixed axis, has two degrees of freedom $s = 2$.

Any arbitrary motion of a rigid body under the action of external forces can be represented as the sum of the translational and rotational independent movements.

Translational motion is characterized by the same values of speeds and accelerations of all points of the body. Therefore, such a movement of the body is completely described by the movement of one of its points. It seems convenient to choose the centre of mass of the body as such a point. All points of a rotating body move in circles. The centres of these circles lie on the same line, which is called the ***axis of rotation***. This motion of the body is completely described by setting the position in space of the axis of rotation and the angular velocity of the body at each moment in time. The rotational movement of the body is flat. A plane motion is a motion in which the trajectories of all points are located in parallel planes. Plane motion is characterized by three degrees of freedom.

To describe the motion of a rigid body, two coordinate systems are introduced: a fixed (inertial) system x, y, z and moving coordinate system x', y', z', which is tightly bound to a solid body. The centre O' of the moving coordinate system is conveniently combined with the center of mass of the body. The position of the body in space relative to the system x, y, z is determined by the position of the moving system x', y', z'. Suppose that the position of an arbitrary point M in the system x, y, z is characterized by a radius vector $\vec{r}$. Position of a point M in the system x', y', z' is characterized by a radius vector $\vec{r}\,'$. Radius vector $\vec{R}$ indicates the position of the moving system x', y', z' initial point O'. In this case

$$\vec{r} = \vec{R} + \vec{r}\,'$$
$$\vec{v} = \vec{v}_{O'} + [\vec{\omega}, \vec{r}\,'], \qquad (3.1.1)$$

where $\vec{v} = \dot{\vec{r}}$ is the speed of a point M in a fixed coordinate system x, y, z;

$\vec{v}_{O'} = \dot{\vec{R}}$ is the speed of translational motion of a solid body, equal to the speed of motion of its centre of mass;

O' is the centre of mass of the body;

$\vec{\omega} = \dot{\vec{\varphi}}$ is the angular velocity of the point M.

Since the angular velocity $\vec{\omega}$ does not depend on the position of the point O' to which the motion is assigned, the quantity $\vec{\omega}$ is called the angular velocity of rotation of the solid.

Formula (3.1.1) shows that if the velocity vectors $\vec{v}_{O'}$ and $\vec{\omega}$ at a given time are mutually perpendicular for any choice of point O', then the velocities $\vec{v}$ of all points of the body lie in one plane (plane motion) perpendicular to vector $\vec{\omega}$. The value of the translational velocity of plane motion depends on the position of the axis of rotation passing through the point O' and perpendicular to the plane of motion.

The axis of rotation, for which the translational speed is zero, is called the instantaneous speed of rotation. The speed of all points at a given time can be represented as the speed of rotational motion around the instantaneous axis. Velocities of all points of the body that are located on the instantaneous axis are zero at a given time. Over time, the position of the instantaneous axis changes both relative to the body and relative to the fixed coordinate system.

The plane motion of a solid can be represented as a series of successive elementary rotations around instantaneous axes. In the case of non-plane motion, when the velocity vectors $\vec{v}_{O'}$ and $\vec{\omega}$ are not mutually perpendicular, elementary movements of the body cannot be represented as a rotation around the instantaneous axis. In this case, the motion of a solid at each moment of time is the sum of two independent motions, namely rotation around a certain axis and translational motion along this axis.

The motion of a rigid body can be described by Euler's theorem. A solid body having one fixed point can be moved from one position to any other by turning at a certain angle around a fixed axis passing through this point. This theorem holds for infinitesimal and finite displacements. The speed $\vec{v}$ of a rigid body in accordance with Euler's theorem can be represented as the sum of two speeds

$$\vec{v} = [\vec{\omega}_0, \vec{r}\,] + [\vec{\omega}\,', \vec{r}\,], \qquad (3.1.2)$$

where $\vec{r}$ is the radius vector of the moving point of the body relative to the fixed point, $\vec{\omega}_0$ is the angular velocity of rotation about an axis that is stationary in the coordinate system x, y, z;

The angular velocity $\vec{\omega}_0$ during body movement varies only in magnitude, but not in direction. The angular velocity $\vec{\omega}'$ can vary both in magnitude and in direction.

The orientation of the coordinate axes of the system x', y', z', associated with the moving solid, relative to the stationary system x, y, z is described using Euler angles ψ, θ, φ. Suppose that at the initial moment the axes of the systems x, y, z and x', y', z' coincide. Then the Euler angles correspond to the following successive rotations of a rigid body. The first rotation occurs at an angle φ around the z axis. The second rotation takes place at an angle θ around the new position of the x axis (line of nodes). The third rotation occurs at an angle ψ around the z' axis. The limits of change of angles are

$$0 \le \varphi \le 2\pi, \quad 0 \le \theta \le \pi, \quad 0 \le \psi \le 2\pi. \tag{3.1.3}$$

Angle ψ is called the ***angle of precession***. Angle θ is called the ***angle of nutation***. Angle φ is called the ***angle of pure rotation***.

The projections $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$ of the angular velocity vector $\vec{\omega}$ of the body on the axis of the moving coordinate system are associated with angles ψ, θ, φ and their time derivatives $\dot{\psi}$, $\dot{\theta}$, $\dot{\varphi}$ by the following relationships

$$\omega_{x'} = \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi$$

$$\omega_{y'} = -\dot{\theta}\sin\psi + \varphi\sin\theta\cos\psi$$

$$\omega_{z'} = \dot{\psi} + \dot{\varphi}\cos\theta. \tag{3.1.4}$$

The radius vector $\vec{r}$ of the material point of the body has the form

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x'\vec{i}\,' + y'\vec{j}\,' + z'\vec{k}\,', \tag{3.1.5}$$

where x, y, z are the coordinates of the vector $\vec{r}$ in a fixed coordinate system with orts $\vec{i}$, $\vec{j}$, $\vec{k}$; x', y', z' are the coordinates of the vector $\vec{r}$ in a moving coordinate system with orts $\vec{i}\,'$, $\vec{j}\,'$, $\vec{k}\,'$.

A moving coordinate system is connected to a solid.

Coordinates x, y, z and x', y', z' are interconnected by ratios

$$x = R_{11}x' + R_{12}y' + R_{13}z'$$

$$y = R_{21}x' + R_{22}y' + R_{23}z'$$

$$z = R_{31}x' + R_{32}y' + R_{33}z'$$

$$x' = R_{11}x + R_{21}y + R_{31}z$$
$$y' = R_{21}x + R_{22}y + R_{32}z$$
$$z' = R_{13}x + R_{23}y + R_{33}z\,, \qquad (3.1.6)$$

where R_{ij} are elements of the transformation matrix for which the following conditions are satisfied

$$\sum_{j=1}^{3} R_{ij}R_{jk} = \delta_{ik}\,, \quad \delta_{ik} = \begin{cases} 1, \ i = k \\ 0, \ i \neq k \end{cases}. \qquad (3.1.7)$$

3.2. Equations of Motion

The equations of motion of a solid body, regarded as a system of rigidly connected material points, have the form

$$m\frac{d\vec{v}_c}{dt} = \vec{F}\,, \quad \frac{d\vec{L}}{dt} = \vec{N}\,. \qquad (3.2.1)$$

The first equality in (3.2.1) describes the translational motion of the centre of inertia of a solid with a velocity of $\vec{v}_c$. A solid body has a mass of m and an external force $\vec{F}$ acts on it.

The second equality in (3.2.1) is the equation of angular momentum $\vec{L}$ of a solid. The angular momentum $\vec{L}$ changes with time under the action of the total moment $\vec{N}$ of external forces. The angular momentum $\vec{L}$ and the moment of external forces $\vec{N}$ can be considered with respect to an arbitrary fixed point or relative to the center of inertia of an absolutely rigid body.

Equation (3.2.1) contains only external forces. Internal forces do not affect the movement of an absolutely rigid body. Consider the case when the resultant of all external forces and the total moment of external forces are equal to zero $\vec{F} = 0$, $\vec{N} = 0$. In this case, an absolutely solid body can be in two states. The first condition is characterized by the balance of the body. The second state is characterized by the movement of an absolutely rigid body in space in a certain way. The centre of inertia of a solid will move rectilinearly and uniformly with arbitrary speed $\vec{v}_c$. At the same time, an absolutely rigid body will rotate. The angular momentum $\vec{L}$ of a solid will not depend on time.

In equilibrium, $\vec{F} = 0$ and the moment of these external forces does not depend on the choice of the point with respect to which it is determined. The arbitrariness in choosing a reference point of the moment of forces $\vec{N}$ greatly facilitates the solution of a number of practical problems in solid mechanics

3.3. Inertia Moment and Inertia Tensor

The ***moment of inertia*** of a rigid body relative to a given axis of rotation is the value I equal to the sum of the casts of the masses of the material points that make up this body per square of the distances of these points to the axis of rotation

$$I = \sum_i m_i r_i^2 \ . \tag{3.3.1}$$

The moment of inertia is an additive quantity. Consider a body whose mass distribution density is the same. In this case, the angular momentum $\vec{L}$ of a body rotating with an angular velocity $\vec{\omega}$ around its axis of symmetry is related to the moment of inertia by formula

$$\vec{L} = I\vec{\omega}. \tag{3.3.2}$$

From (3.2.1) and (3.3.2) the basic equation for the dynamics of the rotational motion of a rigid body around a fixed axis follows

$$I\frac{d\vec{\omega}}{dt} = \vec{N} \ . \tag{3.3.3}$$

In the case of a solid body of any shape that has an arbitrary mass distribution and rotates around a given z axis, the following relation can be indicated

$$I\beta_z = N_z, \qquad L_z = I\omega, \tag{3.3.4}$$

where $\beta_z = \dot{\omega}$ is the projection of angular acceleration on the z axis of rotation; N_z, L_z are projections of the moment of external forces and angular momentum on this axis; I is the moment of inertia about the z axis.

The moment of inertia I in the mechanics of rotational motion plays a role similar to the role of mass in the mechanics of translational motion. Each body has a certain moment of inertia, regardless of whether it rotates or not. For a body whose mass distribution at each point is characterized by density $\rho = \rho(x,y,z)$, the moment of inertia is

$$I = \int r^2 dm$$

$$dm = \rho(x,y,z)dV, \quad dV = dxdydz, \tag{3.3.5}$$

where dm is an element of body mass; dV is an element of volume; r is the distance from the given axis to the volume element, which in the general case is a function of the coordinates of the points of the body: $r = r(x,y,z)$.

Moments of inertia of the body can be calculated from the known body shape and mass distribution. Consider special cases.

The moment of inertia I_y about the y axis of a plane figure, for example, a curvilinear trapezoid, is determined by formula

$$I_y = \int_a^b x^2 \rho(x) \cdot [f_2(x) - f_1(x)] dx, \tag{3.3.6}$$

where $\rho(x)$ is linear density; $f_1(x)$, $f_2(x)$ are functions that define the shape of a figure in a plane (x,y).

Moments of inertia I_x and I_y relative to the x axes and y axes, a plane body occupying the region S with a mass distributed with a density of $\rho(x,y)$, are equal to

$$I_x = \int_S \rho(x,y) y^2 dS, \quad I_y = \int_S \rho(x,y) x^2 dS, \tag{3.3.7}$$

where $dS = dxdy$ is an element of area.

The moment of inertia relative to the origin is

$$I_0 = \int_S (x^2 + y^2) \rho(x,y) dS. \tag{3.3.8}$$

Moments of inertia I_x, I_y, I_z relative to the x, y, z axes of the body occupying the volume V, with the density $\rho(x,y,z)$ of mass distribution, are expressed by the relations

$$I_x = \int_V (y^2 + z^2) \rho(x,y,z) dV$$

$$I_y = \int_V (z^2 + x^2) \rho(x,y,z) dV$$

$$I_z = \int_V (x^2 + y^2) \rho(x,y,z) dV, \tag{3.3.9}$$

where $dV = dxdydz$ is an element of volume.

Consider a homogeneous disk of radius R and thickness b. Let z axis pass through the center of mass perpendicular to the plane of the disk, and x axis and y axis are located in the plane of the disk. Then moments of inertia relative to the axes x, y, z are equal

$$I_z = \rho \int r^2 dV = \rho \int_0^R r^2 b 2\pi r dr = 2\pi b r \frac{R^4}{4} = \frac{mR^2}{2}$$

$$I_x = I_y = \frac{mR^2}{4}, \quad I_z = I_x + I_y, \tag{3.3.10}$$

where $m = \rho V = \pi \rho b R^2$ is the mass of the disk.

Consider a homogeneous cylinder of length h with a base equal to R. Z axis passes through the centre of mass perpendicular to the plane of the base. In this case, the moment of inertia of the cylinder about the z axis is

$$I_z = \rho\int_0^h dz\int_0^{2\pi} d\varphi\int_0^R r^3 dr = \rho h\frac{R^4}{4}2\pi = \frac{mR^2}{2}, \qquad (3.3.11)$$

where $m = \rho\pi R^2 h$ is the mass of the cylinder.

Consider a homogeneous cone with a height of h and a base with a radius of R. Let the z axis pass through the center of mass perpendicular to the plane of the base. In this case, the moment of inertia of the cone relative to the z axis is

$$I_z = \rho\int_0^h dz\int_0^{2\pi} d\varphi\int_0^{Rz/h} r^3 dr = \rho\frac{\pi R^4 h}{10} = \frac{3}{10}mR^2, \qquad (3.3.12)$$

where $m = \dfrac{\rho\pi R^2 h}{3}$ is the mass of the cone.

Here, the equation of the cone $z^2 = \dfrac{h^2}{R^2}\left(x^2 + y^2\right)$ and the relation $x^2 + y^2 = r^2$ are taken into account.

Consider a homogeneous rod of mass m and length l. Let the z axis pass through the center of mass perpendicular to the rod. In this case, the moment of inertia of the rod relative to the z axis is

$$I_z = \frac{ml^2}{12}. \qquad (3.3.13)$$

Consider a homogeneous rectangular plate with a length of a and a width of b. Let the z axis be directed perpendicular to the plate and pass through its center of mass. Axes x and y are located in the plane of the plate. In this case, the moments of inertia relative to the axes x, y, z are equal to

$$I_x = \frac{mb^2}{12},\ \ I_y = \frac{ma^2}{12},\ \ I_z = \frac{m}{12}\left(a^2 + b^2\right). \qquad (3.3.14)$$

Consider a hollow ball with infinitely thin walls. The radius of the ball is R. In this case, the moments of inertia of the hollow ball relative to the axes x, y, z are

$$I = I_x = I_y = I_z = \frac{2}{3}mR^2, \quad I_x + I_y + I_z = 2mR^2. \qquad (3.3.15)$$

Consider a homogeneous solid ball. In this case, the moments of inertia of the continuous ball relative to the axes x, y, z are equal to

$$I = I_x = I_y = I_z = \rho\int_0^R r^2 4\pi r^2 dr = \frac{2}{5}mR^2, \qquad (3.3.16)$$

where $m = \rho \frac{4}{3}\pi R^3$ is the mass of a continuous ball.

Formula (3.3.16) assumes that the moment of inertia of the spherical layer relative to the diameter, i.e. a hollow ball with infinitely thin walls of radius r and mass dm is $dI = \frac{2}{3} r^2 dm = \frac{2mr^4 dr}{R^3}$.

Consider a homogeneous ellipse with axes a and b. In this case, the moments of inertia of the continuous ball relative to the axes x, y, z are

$$I_x = \frac{1}{4} ma^2,\ I_y = \frac{1}{4} mb^2,\ I_z = \frac{m}{4}\left(a^2 + b^2\right), \tag{3.3.17}$$

where m is the mass of the ellipse.

Consider a homogeneous triaxial ellipsoid with semiaxes a, b, c, directed along axes x, y, z, which coincide with the main axes of inertia. In this case, the moments of inertia of a continuous ball with respect to the axes x, y, z are equal

$$I_x = \frac{m}{5}\left(c^2 + b^2\right),\ I_y = \frac{m}{5}\left(a^2 + c^2\right),\ I_z = \frac{m}{5}\left(a^2 + b^2\right), \tag{3.3.18}$$

where m is the mass of an ellipsoid.

The calculation of the moments of inertia can be simplified using the ***Huygens-Steiner theorem***. This theorem is named after Christiaan Huygens (14.04.1629 – 8.07.1695) and Jakob Steiner (18.03.1796 – 1.04.1863). The moment of inertia I about any given axis K is determined by the formula

$$I_K = I_C + ma^2, \tag{3.3.19}$$

where I_C is the moment of inertia about the axis C, passing through the centre of mass of the body and parallel to the given axis K; m is body mass; a is the distance between axles K and C.

The Huygens-Steiner theorem reduces the calculation of the moment of inertia about an arbitrary axis to the calculation of the moment of inertia about an axis passing through the centre of mass of the body.

If the body is homogeneous and has an axis of symmetry, then the relationship between the angular momentum $\vec{L}$ of this body and the angular velocity $\vec{\omega}$ of rotation of the body around the axis of symmetry has the form according to which the directions of the vectors $\vec{\omega}$ and $\vec{L}$ coincide. In the general case of bodies of arbitrary shape and with an arbitrary distribution of masses, the directions of vectors $\vec{\omega}$ and $\vec{L}$ do not coincide, and the relation between these quantities has the form

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z, \qquad (3.3.20)$$

where L_x, L_y, L_z and ω_x, ω_y, ω_z are projections of vectors $\vec{L}$ and $\vec{\omega}$ on the coordinate axis, respectively.

The proportionality coefficients between L and ω are components of the ***inertia tensor***

$$I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}. \qquad (3.3.21)$$

The inertia tensor I characterizes the inert properties of the body during its rotation. The components of the inertia tensor I in the case of a continuous mass distribution are determined by the expressions

$$I_{xx} = \int\left(y^2 + z^2\right)dm, \quad I_{xy} = I_{yx} = -\int xydm$$

$$I_{yy} = \int\left(z^2 + x^2\right)dm, \quad I_{yz} = I_{zy} = -\int yzdm$$

$$I_{zz} = \int\left(x^2 + y^2\right)dm, \quad I_{zx} = I_{xz} = -\int zxdm, \qquad (3.3.22)$$

where $dm = \rho(x,y,z)dV$, $dV = dxdydz$ is an element of volume.

Diagonal components I_{xx}, I_{yy}, I_{zz} are called ***axial moments of inertia***, and off-diagonal elements I_{xy}, I_{yx}, I_{xz}, I_{zx}, I_{yz}, I_{zy} are called ***centrifugal moments of inertia***. For any point of a solid body, there is a Cartesian coordinate system in which the centre of inertia is diagonal

$$I = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}. \qquad (3.3.23)$$

The axes of such a coordinate system are called ***main axes of inertia***, and the diagonal components I_x, I_y, I_z are called ***main moments of inertia***.

In the general case, the principal axes for different points of a solid have a different direction, and the principal moments have different meanings. The calculation of the main moments of inertia of the body is reduced to the solution of the so-called ***secular equation***

$$\begin{vmatrix} I_{xx}-I & I_{xy} & I_{zx} \\ I_{xy} & I_{yy}-I & I_{yz} \\ I_{zx} & I_{yz} & I_{zz}-I \end{vmatrix} = 0. \tag{3.3.24}$$

The secular equation is a cubic equation for value I. The three roots of the secular equation determine the main moments of inertia.

In the case when the body in the absence of external forces rotates around a certain axis, the position of which does not change in space with time, then this axis is called the ***free axis of the body***. For any body, there are three mutually perpendicular axes passing through the centre of mass of the body, which are its free axes. These axes are also the main axes of inertia of the body. Moments of inertia about these axes are the main moments of inertia of the body.

If the body rotates around one of its main axes, for example, z axis, то $\omega_y \neq 0$ and $\omega_x = \omega_y = 0$ and

$$L_x = L_y = 0, \quad L_z = I_z \omega_z, \quad \vec{L} = I\vec{\omega}. \tag{3.3.25}$$

Finding the main axes of inertia of the body is greatly simplified if the body has symmetry. Consider, for example, a round flat plate of finite thickness. In this case, one main axis is perpendicular to the plane of the plate, and the other axis is located in the middle plane of the plate and passes through its centre. The third major axis is perpendicular to the first two axes. Consider the ball. One major axis passes through the centre of the ball. The two remaining main axes are oriented arbitrarily and lie in a plane perpendicular to the first axis.

In the case when the main moments of inertia are equal, the body is called a spherical top. A body is called a symmetric top if the following relations $I_x = I_y \neq I_z$.

3.4. Work and Energy of Moving Solids

Internal forces do not do work when the body rotates. The elementary work of dA external forces when the body rotates through an angle of $d\varphi$ during dt is determined by the formula

$$dA = \left(\vec{\omega}, \vec{N}\right) dt = \omega N_z dt = N_z d\varphi, \tag{3.4.1}$$

where ω is the angular velocity; $\vec{N}$ is the total moment of external forces; $N_z = N\cos(\alpha)$ is the projection of the vector $\vec{N}$ on the z axis of rotation; α is the angle between the vectors $\vec{N}$ and $\vec{\omega}$.

The work A of external forces when turning the body at a finite angle $\Delta\varphi = \varphi_2 - \varphi_1$ is determined by the expression

$$A = \int_{\varphi_1}^{\varphi_2} N_z d\varphi, \tag{3.4.2}$$

where φ_1 and φ_2 are the values of the angle in the initial and final positions of the body.

The projection of the moment of forces on the axis of rotation N_z in the general case depends on the angle of rotation $N_z = N_z(\varphi)$.

The instantaneous power P in the case of rotational motion is found by the formula

$$P = N_z \frac{d\varphi}{dt} = N_z \dot{\varphi} = N_z \omega. \tag{3.4.3}$$

A body rotating around a fixed z axis has kinetic energy

$$W_k = \frac{1}{2} I_z \omega^2, \tag{3.4.4}$$

where I_z is the moment of inertia about the axis of rotation; ω is the angular velocity.

The change in kinetic energy W_k with a change in angular velocity from ω_1 to ω_2 is determined by the formula

$$\Delta W_k = \frac{1}{2} I_z \left(\omega_2^2 - \omega_1^2 \right). \tag{3.4.5}$$

If the body rotates in an arbitrary way relative to a fixed point coinciding with its centre of mass, then its kinetic energy is

$$W_k = \frac{1}{2} \sum_{i,k=x,y,z} I_{ik} \omega_i \omega_k , \tag{3.4.6}$$

where the summation is performed independently over the three values of the axes x, y, z of the Cartesian coordinate system with the origin at the center of mass of the body; I_{ik} is a component of the inertia tensor; ω_i and ω_k are the angular velocities of rotation around the axes i and k, respectively.

If the axes of the Cartesian coordinate system coincide with the main axes of inertia of the body, then

$$W_k = \frac{1}{2} \left(I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 \right), \tag{3.4.7}$$

where I_x, I_y, I_z are the main moments of inertia of the body.

Consider the case of plane motion of a body. Suppose that a body point O moves with a speed of $\vec{v}_0$, and the body itself rotates with an angular speed of $\vec{\omega}$ around an axis passing through this point. Kinetic energy in this case has the form

$$W_k = \frac{1}{2}mv_0^2 + m\left(\vec{R}_c, [\vec{v}_0, \vec{\omega}]\right) + \frac{1}{2}I_0\omega^2, \tag{3.4.8}$$

where m is body mass; $\vec{R}_c$ is the radius vector of the centre of mass of the body; I_0 is the moment of inertia of the body about the axis passing through the point O.

Let us choose the center of mass of the body as point O. In this case, the kinetic energy of this body during planar motion will be equal to the sum of the kinetic energy of the translational motion of the center of mass with a speed equal to the velocity of the center of mass $\vec{V}_c$ and the energy of rotational motion with an angular velocity $\vec{\omega}$ around an axis passing through the center of mass of the body with a moment of inertia I_c about this axis

$$W_k = \frac{1}{2}mV_c^2 + \frac{1}{2}I_c\omega^2. \tag{3.4.9}$$

In the case of an arbitrary movement of the body, we have

$$W_k = \frac{1}{2}mV_c^2 + \frac{1}{2}\sum_{i,k=x,y,z} I_{ik}\omega_i\omega_k. \tag{3.4.10}$$

3.5. Motion of Rigid Body Fixed at Point

The consideration of plane motion is simplified by the fact that the angular velocity vector maintains a constant direction in space, perpendicular to the plane of motion, and does not change its direction relative to the body. When a solid moves about one fixed point, these simplifying circumstances disappear. In this case, the angular velocity vector in the general case can change its direction in space.

It is convenient to consider this type of motion in a coordinate system rigidly connected with the body. The origin of the coordinate system is placed at the point of fixation of the body. The equations of motion in this case are called the Euler equations.

The equation of motion of the centre of mass in this case has the form

$$m\frac{d\vec{v}_0}{dt} = m\frac{d}{dt}[\vec{\omega}, \vec{r}_0] = \vec{F}, \tag{3.5.1}$$

where $\vec{r}_0$ is the radius vector of the centre of mass of the body drawn from the point of attachment; $\vec{F}$ is a resultant force that includes bond reactions.

The axes of the coordinate system associated with the body can be conveniently directed along the main axes of inertia. In this case, the inertia tensor reduces to its three main values I_1, I_2, I_3, and the angular momenta are

$$L_1 = I_1\omega_1, \qquad L_2 = I_2\omega_2, \qquad L_3 = I_3\omega_3, \tag{3.5.2}$$

where ω_1, ω_2, ω_3 are the components of the angular velocity relative to the coordinate axes moving with the body.

Consideration of the equation of the rotational motion of a body around one fixed point requires preliminary recording of the total time derivative of vector $\vec{A}$

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + \left[\vec{\omega}, \vec{A}\right], \tag{3.5.3}$$

where

$$\frac{\partial \vec{A}}{\partial t} = \frac{dA'_x}{dt}\vec{i}\,' + \frac{dA'_y}{dt}\vec{j}\,' + \frac{dA'_z}{dt}\vec{k}\,'. \tag{3.5.4}$$

We apply formulas (3.5.3) and (3.5.4) for the angular momentum $\vec{L}$

$$\frac{\partial \vec{L}}{\partial t} + \left[\vec{\omega}, \vec{L}\right] = \vec{N}. \tag{3.5.5}$$

Taking into account that $L_x = I_x\,\omega_x$, $L_y = I_y\,\omega_y$, $L_z = I_z\,\omega_z$, we rewrite equation (3.5.5) for a moving coordinate system

$$I_x\frac{d\omega_x}{dt} + \left(I_z - I_y\right)\omega_z\omega_y = N_x$$

$$I_y\frac{d\omega_y}{dt} + \left(I_x - I_z\right)\omega_x\omega_z = N_y$$

$$I_z\frac{d\omega_z}{dt} + \left(I_y - I_x\right)\omega_y\omega_x = N_z. \tag{3.5.6}$$

Equations (3.5.6) are called Euler equations. These equations fundamentally allow us to determine the motion of a body fixed at one point, although in practice the solution can be very complex and difficult to fulfil.

Consider a body that is not affected by external forces, therefore, the moments of forces N_x, N_y and N_z are equal to zero. Let us direct the axes of the coordinate system rigidly connected with the body along the central principal axes. Therefore, quantities I_x, I_y, I_z are the central main moments of the inertia of the body. Let us find out the types of free movement of the body in this case.

A consequence of formulas (3.5.6) is the assertion that free rotation of a rigid body is possible only around free axes. Moments of inertia about these axes are generally different. It can be proved that the rotation of the body will be stable only relative to the central main axis with the maximum or minimum moment of inertia

Rotation around a central major axis with an average moment of inertia is not stable. This circumstance can be clearly demonstrated in the following experiment. The central principal axes of a rectangular parallelepiped are three mutually perpendicular axes passing through its geometric centre parallel to its sides. The parallelepiped has the largest and smallest moments of inertia with respect to axes parallel to its longest and shortest sides. If you toss it with simultaneous rotation around one of these axes, the movement will be stable while maintaining the direction of the axis of rotation. The rotation of the box around an axis parallel to the middle side does not lead to a stable rotation, and the body begins to tumble randomly.

Consider a body that has axial symmetry about some axis. In this case, one of the central principal axes coincides with the axis of symmetry, and the other two are perpendicular to it. We direct x axis along the axis of symmetry, y axis and z axis direct along two other central principal axes. From the conditions of symmetry it follows that $I_x = I_1$, $I_y = I_z = I_2$. Equations (3.5.6) in this case have the form

$$I_1 \frac{d\omega_x}{dt} = 0$$

$$I_2 \frac{d\omega_y}{dt} + (I_1 - I_2)\omega_z\omega_x = 0$$

$$I_2 \frac{d\omega_z}{dt} + (I_2 - I_1)\omega_x\omega_y = 0. \qquad (3.5.7)$$

The consequence of equations (3.5.7) is the possibility of motion, at which $\omega_x = \omega_1 = \text{const}$ and $\omega_y = \omega_z = 0$. Therefore, rotation around the axis of symmetry of the body is possible at a constant speed. However, this is not the only possibility. We write the second and third equations (3.5.7) under the condition $\omega_x = \omega_1 = \text{const}$ as follows

$$\frac{d\omega_y}{dt} + \gamma\omega_z = 0$$

$$\frac{d\omega_z}{dt} - \gamma\omega_y = 0, \qquad (3.5.8)$$

where $\gamma = \dfrac{(I_1 - I_2)\omega_1}{I_2}$.

Equations (3.5.8) have solutions

$$\omega_y = A\cos(\gamma t)$$

$$\omega_z = A\sin(\gamma t). \qquad (3.5.9)$$

The angular velocity vector $\omega_\perp = \omega_y \vec{j} + \omega_z \vec{k}$ lying in the plane $(y,\, z)$ rotates around the origin of the reference frame with a circular frequency of γ.

The total angular velocity is

$$\vec{\omega} = \omega_1 \vec{i} + \vec{\omega}_\perp . \qquad (3.5.10)$$

This total vector moves around x axis along the surface of the cone with an angle of α at the apex. Angle α satisfies a ratio of $\tan\alpha = \omega_\perp / \omega_1$. In this case, the angular velocity of rotation of the body does not coincide in direction with the x axis of symmetry of the body. The axis of symmetry, in turn, does not remain stationary in space.

We describe the full movement of the body. The plane in which the angular velocity vectors $\vec{\omega}$ and the axis of symmetry are located rotates with angular velocity γ around vector $\vec{L}$. The relative position of the vector $\vec{\omega}$ and the axis of symmetry does not change. The motion of the axis of symmetry of the body around the motionless vector of the total angular momentum $\vec{L}$ is called ***nutation***. The angular velocity of rotation γ is called the ***nutation velocity***.

The amplitude of nutation depends on the initial conditions of motion. However, the nutation frequency is determined only by the moments of inertia and the angular velocity of rotation around the axis of symmetry. A body can rotate without nutation, if its angular velocity is directed along the axis of symmetry.

The moments of inertia of a homogeneous ball are equal to each other $I_x = I_y = I_z$, hence $\gamma = 0$. There is no nutation in a homogeneous ball. However, experiments indicate the presence of nutation at the Earth. This proves that the Earth cannot be considered as a uniform ball. Measurements of the moments of inertia of the Earth showed that $(I_1 - I_2)/I_2 \approx 300$. This means that the nutation period of the earth's axis should be approximately 300 days. Therefore, within 300 days the axis of rotation should make one turn on the surface of the cone around the axis of symmetry of the Earth.

However, the observed motion of the Earth is much more complicated. This movement is irregular, as it is affected by earthquakes and seasonal changes occurring on the surface of the Earth. In fact, the nutation period is approximately 440 days, which is apparently due to the Earth’s absolute stiffness. The maximum distance of a point on the earth's surface through which the axis of rotation passes from a point through which the axis of symmetry passes at the north pole does not exceed 5 m.

An axially symmetric body, brought into very fast rotation around its axis of symmetry, is called a ***gyroscope***. Suppose that the gyroscope is fixed at the point of the centre of mass, but its axis can rotate freely in any direction. This fastening is carried out using a gimbal, providing a free change in the orientation of the axis of the gyroscope in three mutually perpendicular directions.

Let the moment of external forces be applied to the gyroscope. The gyroscope rotates around its axis with a large angular velocity. The nutation of a gyroscope along its axis of rotation under the action of external forces is very small. Therefore, we can assume that the axis of rotation all the time coincides with the axis of symmetry of the gyroscope. The axis of rotation coincides with the central main axis of inertia of the gyroscope so it is chosen as to be stable. Around this axis is free steady rotation. This direction of the axis of stable rotation is maintained. For example, if, having taken up the base of a gimbal, to arbitrarily change its direction in space, then the hinges will rotate in such a way that the axis maintains a constant direction. Therefore, if the gimbal is mounted on any body, for example, on a rocket, then with an arbitrary movement of the rocket, the axis maintains a constant direction in space relative to the system of fixed stars. This circumstance makes the gyroscope the most important navigation tool for missile and aircraft flights.

The angular momentum vector $\vec{L}$ approximately coincides with the angular velocity vector $\vec{\omega}$ directed along the central main axis of the gyroscope. Strictly speaking, these vectors do not coincide. However, the deviations of the directions of the winds are so small that they can be neglected. The movement of the gyroscope is conveniently described by equation $\frac{d\vec{L}}{dt} = \vec{N}$, since a change in the vector $\vec{L}$ describes the movement of its axis, namely, rotation through an angle of $d\varphi$ over time dt. The direction of movement of the axis of the gyroscope can be determined according to relation $d\vec{L} = \vec{N}dt = \left[d\vec{\varphi}, \vec{L}\right]$ using a known value of $\vec{N}$.

The angular velocity of the gyroscope precession can be easily calculated

$$\Omega = \frac{d\varphi}{dt} = \frac{N}{L} = \frac{N}{I\omega}. \tag{3.5.11}$$

A characteristic feature of precession is that it does not have inertia. The precession movement ceases at the moment the termination of the moment of external forces.

Consider a gyroscope whose axis is fixed at one point and suspended by a thread at its end. In this case, the axis is not horizontal, but at an angle of α to the vertical. The motion of such a gyroscope can be described by the equations

$$N = mgl\sin\alpha$$

$$dL = L\sin\alpha \cdot d\varphi = mgl\sin\alpha dt$$

$$\Omega = \frac{d\varphi}{dt} = \frac{mgl}{L}, \tag{3.5.12}$$

where g is the acceleration of gravity; l is the length of the thread; m is the mass of the gyroscope.

The angular velocity Ω does not depend on the angle α of inclination of the axis to the vertical. This is due to the fact that when the angle is changed, both the angular momentum and the distance in the horizontal plane from the axis of rotation to the end of the vector $\vec{L}$ change. The independence of the precession rate of such a gyroscope from the angle of inclination of its axis gave reason to call it a ***gyroscopic pendulum***. The period of motion of such a pendulum is

$$T = \frac{2\pi}{\Omega} = \frac{2\pi I\omega}{mgl}. \tag{3.5.13}$$

The period of the gyroscopic pendulum at sufficiently large moments of inertia I and angular velocity of rotation ω, as well as a small suspension length l, can be quite large and can be minutes or even hours. A mathematical pendulum with such a large period would have a very large length. The length of a mathematical pendulum whose oscillation period is equal to the precession period of a gyroscopic pendulum is called the ***reduced length of the gyroscopic pendulum***.

Test questions

1. Under what conditions can real bodies be considered as absolutely solid?
2. Compare the number of degrees of freedom of the system for two cases when parts of the system interact and do not interact with each other.
3. Describe the plane motion.
4. Does the angular velocity depend on the position of the point to which the motion is related?
5. What form does the dependence $v = f(t)$ have for the points lying on the instantaneous axis of rotation?
6. Formulate Euler's theorem.
7. Describe the Euler angles and indicate the limits of their change.
8. Write down the equation of motion of a solid.
9. Indicate the ratio between the moment of inertia and the angular momentum of a homogeneous body.
10. Write down the basic equation of the dynamics of rotational motion.
11. Calculate the moment of inertia of the steel disk with a radius of 30 cm and a thickness of 0.2 cm.
12. Formulate the Huygens – Steiner theorem.
13. Give a comparative description of the axial and centrifugal moments of inertia.
14. What characteristics of a solid determine the roots of the secular equation?
15. Write down the formula for the work of external forces when turning the body at a finite angle.

16. Describe the dependence of the kinetic energy of a rotating body on its angular velocity.
17. Formulate the Euler equations in Cartesian coordinates, which describe the motion of a body fixed at one point.
18. Indicate the axes with respect to which free rotation of the solid is possible.
19. Indicate the factors on which the amplitude and frequency of nutation depend.
20. Write down the formula for the period of motion of the gyroscopic pendulum.

Problem-solving examples

Problem 3.1

Problem description. The physical pendulum is a rod with a length of l = 0.5 m and a mass of m_1 = 1 kg. A disk of mass $m_2 = 0.5m_1$ is attached to one of the ends of the rod. Determine the moment of inertia of such a pendulum about the axis Oz, passing through a point O on the rod perpendicular to the plane of the drawing.

Known quantities: l = 0.5 m, m_1 = 1 kg, $m_2 = 0.5m_1$.

Quantities to be calculated: I_z.

Problem solution. The total moment of inertia of the pendulum is equal to the sum of the moments of inertia of the rod I_{z1} and disk I_{z2}

$$I_z = I_{z1} + I_{z2}. \qquad \text{(P.3.1.1)}$$

To calculate the moments of inertia I_{z1} and I_{z2} we use the Steiner theorem

$$I = I_C + ma^2. \qquad \text{(P.3.1.2)}$$

Express the moment of inertia of the rod through its length l

$$I_{z1} = \frac{1}{12} m_1 l^2 + m_1 a_1^2, \qquad \text{(P.3.1.3)}$$

where a_1 is the distance between the axis Oz and the parallel axis passing through the center of mass C_1 of the rod; m_1 is the mass of the rod.

We write the relation for the quantity a_1

$$a_1 = \frac{l}{2} - \frac{l}{3} = \frac{l}{6}. \qquad \text{(P.3.1.4)}$$

Therefore, the moment of inertia of the rod is

$$I_{z1} = \frac{m_1 l^2}{12} + m_1\left(\frac{l}{6}\right)^2 = \left(\frac{1}{12} + \frac{1}{36}\right) m_1 l^2 \approx 0.111\, m_1 l^2. \qquad \text{(P.3.1.5)}$$

The moment of inertia of the disk, taking into account the Steiner theorem, is

$$I_{z2} = \frac{m_2 R^2}{2} + m_1 a_2^2, \qquad \text{(3.1.6)}$$

where a_2 is the distance between the axis Oz and the parallel axis passing through the center of mass of the disk; $R = l/4$ is the radius of the disk.

We write the relation for the quantity a_2

$$a_2 = \frac{2l}{3} + \frac{l}{4} = \frac{11}{12} l. \qquad \text{(P.3.1.7)}$$

The moment of inertia of the disk, taking into account formula (P.3.1.7), is

$$I_{z2} = \frac{m_2}{2}\left(\frac{l}{4}\right)^2 + m_2\left(\frac{11}{12} l\right)^2 \approx 0.871 m_2 l^2. \qquad \text{(P.3.1.8)}$$

We substitute formulas (P.3.1.5) and (P.3.1.8) into the formula (P.3.1.1)

$$I_z \approx 0.111\, m_1 l^2 + 0.871\, m_2 l^2. \qquad \text{(P.3.1.9)}$$

According to the condition of the problem $m_2 = 0.5 m_1$. We rewrite formula (P.3.1.9) with this relation

$$I_z \approx 0.547 m_1 l^2. \qquad \text{(P.3.1.10)}$$

We substitute the numerical relations in the formula (P.3.1.10)

$$I_z = 0.547 \times 1 \times 0.5^2 = 0.137 \text{ kg}\cdot\text{m}^2.$$

Answer. The moment of inertia of the physical pendulum is $I_z \approx 0.137$ kg·m^2.

Problem 3.2

<u>*Problem description*</u>. Flat disk has mass $m = 80$ g. A thin flexible thread is thrown through the block, to the ends of which weights of masses $m_1 = 100$ g and $m_2 = 200$ g are suspended. Determine the acceleration with which the weights will move. The friction between the thread and the disc is neglected.

<u>*Known quantities*</u>: $m = 80$ g, $m_1 = 100$ g, $m_2 = 200$ g.

<u>*Quantities to be calculated*</u>: a.

<u>*Problem solution*</u>. We use the basic laws of translational and rotational motion to solve the problem. Two forces act on each of the weights: gravity mg, directed downward, and the force of tension of the thread, directed upward.

Suppose that the acceleration vector $\vec{a}$ of the weight m_1 is directed upwards. In this case, we can write

$$T_1 > m_1 g\,, \tag{P.3.2.1}$$

where T_1 is the thread tension force acting on a weight with mass m_1; g is the acceleration of gravity.

The resultant force causes uniformly accelerated movement of the weight. According to Newton's second law

$$T_1 - m_1 g = m_1 a \tag{P.3.2.2}$$

or

$$T_1 = m_1 g + m_1 a\,. \tag{P.3.2.3}$$

The vector of acceleration $\vec{a}$ of the weight m_2 is directed downward, therefore $T_1 < m_2 g$. We write Newton's second law for this weight

$$m_2 g - T_2 = m_2 a \tag{P.3.2.4}$$

or

$$T_2 = m_2 g - m_2 a\,. \tag{P.3.2.5}$$

According to the basic law of the dynamics of rotational motion, the torque M applied to the disk is equal to the product of the moment of inertia J of the disk and its angular acceleration ε

$$M = J\varepsilon\,. \tag{P.3.2.6}$$

Define the torque. The forces of the tension of the threads act not only on the weights, but also on the disk. According to Newton's third law, the forces T_1' and T_2' applied to the rim of the disk are equal, respectively, to forces T_1 and T_2, but opposite in direction. The disk rotates with acceleration when moving weights. According to the earlier assumption, we can conclude that $T_2' > T_1'$.

The torque applied to the disk is equal to the product of the difference of these forces by the force arm equal to the radius r of the disk, i.e.

$$M = (T_2' - T_1')r. \tag{P.3.2.7}$$

The moment of inertia of the disk is

$$J = \frac{mr^2}{2}. \tag{P.3.2.8}$$

The angular acceleration of the disc is related to linear acceleration by the ratio

$$\varepsilon = \frac{a}{r}. \tag{P.3.2.9}$$

We substitute formulas (P.3.2.7), (P.3.2.8), (P.3.2.9) into formula (P.3.2.6)

$$(T_2' - T_1')r = \frac{mr^2}{2}\frac{a}{r}. \tag{P.3.2.10}$$

We rewrite equation (P.3.2.10)

$$T_2' - T' = \frac{ma}{2}. \tag{P.3.2.11}$$

Due to the following relations $T_1' = T_1$ and $T_2' = T_2$, we can replace the forces T_1', T_2' with formulas (P.3.2.3), (P.3.2.5), then

$$m_2 g - m_2 a - m_1 g - m_1 a = \frac{ma}{2} \tag{P.3.2.12}$$

or

$$(m_2 - m_1)g = \left(m_2 + m_1 + \frac{m}{2}\right)a. \tag{P.3.2.13}$$

We express the acceleration from the equation (P.3.2.13)

$$a = \frac{m_2 - m_1}{m_2 + m_1 + \frac{m}{2}} g. \tag{P.3.2.14}$$

We substitute the numerical values in the formula (P.3.2.14)

$$a = \frac{200\times10^{-3} - 100\times10^{-3}}{200\times10^{-3} + 100\times10^{-3} + \frac{80\times10^{-3}}{2}} 9.8 = 2.88 \text{ m/s}^2.$$

Answer. The acceleration with which the loads will move is equal to a = 2.88 m/s^2.

Problem 3.3

Problem description. A disk of mass m = 50 kg and radius r = 20 cm had an initial rotation frequency n = 480 min^{-1}. After some time, under the action of friction, the disk stopped. Find the moment M of friction. The moment of friction forces shall be considered constant in magnitude. Consider two cases: 1) disk stopped after a time t = 50 s; 2) the disk made N = 200 turns to a complete stop.

Known quantities: m = 50 kg, r = 20 cm, n = 480 min^{-1}, t = 50 s, N = 200.

Quantities to be calculated: M.

Problem solution. 1. According to the law of the dynamics of rotational motion, the change in the angular momentum of a rotating body is equal to the product of the angular momentum M acting on the body and the duration of this moment

$$M\Delta t = J\omega_2 - J\omega_1, \qquad \text{(P.3.3.1)}$$

where J is the moment of inertia of the disk; ω_1, ω_2 are the initial and final angular velocities of the disk.

As ω_2 = 0 and Δt = t, then $Mt = -J\omega_1$, whence follows

$$M = -\frac{J\omega_1}{t}. \qquad \text{(P.3.3.2)}$$

The moment of inertia of the disk relative to its geometric axis is

$$J = mr^2/2, \qquad \text{(P.3.3.3)}$$

were m is the mass of the disk; r is the radius of the disk.

We substitute the formula (P.3.3.3) into the formula (P.3.3.2)

$$M = -\frac{mr^2\omega_1}{2t}. \qquad \text{(P.3.3.4)}$$

We rewrite formula (P.3.3.4) taking into account the relation $\omega_1 = 2\pi n$

$$M = -\frac{mr^2 2\pi n}{2t}. \qquad \text{(P.3.3.5)}$$

Substitute the numerical values in the formula (P.3.3.5)

$$M = -\frac{50 \times \left(20 \times 10^{-2}\right)^2 \times 2 \times 3.14 \times \left(\frac{480}{60}\right)}{2 \times 50} = -1\ \text{N} \cdot \text{m}.$$

2. We write a formula that expresses the relationship of work with a change in kinetic energy

$$A = \frac{J\omega_2^2}{2} - \frac{J\omega_1^2}{2}. \qquad \text{(P.3.3.6)}$$

We take into account that the final angular velocity is zero $\omega_2 = 0$, then

$$A = -\frac{J\omega_1^2}{2}. \qquad \text{(P.3.3.7)}$$

Work during rotational motion is determined by the formula $A = M\varphi$. Substituting the relations for the work and the moment of inertia of the disk into the formula (P.3.3.7), we get

$$M\varphi = -\frac{mr^2\omega_1^2}{4}. \qquad \text{(P.3.3.8)}$$

We rewrite the formula in order to obtain an explicit expression for the moment of friction

$$M = -\frac{mr^2\omega_1^2}{4\varphi} = -\frac{mr^2\omega_1^2}{4 \cdot 2\pi N}. \qquad \text{(P.3.3.9)}$$

We substitute the numerical values in the formula (P.3.3.9)

$$M = -\frac{50 \times \left(20 \times 10^{-2}\right)^2 \times 2^2 \times 3.14^2 \times \left(\frac{480}{60}\right)^2}{4 \times 2 \times 3.14 \times 200} = -1\ \text{N} \cdot \text{m}.$$

<u>*Answer*</u>. The moment of friction forces is M = – 1 N·m.

Problems

Problem A

<u>*Problem description*</u>. Determine the moment of inertia J of a thin uniform rod with a length of l = 30 cm and a mass of m = 100 g relative to the axis perpendicular to the rod and passing through its middle.

<u>*Answer*</u>. J = 7.5 kg·m^2.

Problem B

<u>*Problem description*</u>. Determine the moment of inertia J of a flat homogeneous rectangular plate with a mass m = 800 g relative to the axis coinciding with one of its sides if the length of the other side is 40 cm.

<u>*Answer*</u>. J = 4.27 kg·m^2.

Problem C

<u>*Problem description*</u>. A thin uniform rod with a length of l = 50 cm and a mass of m = 400 g rotates with an angular acceleration ε = 3 rad/s^2 about an axis passing perpendicular to the rod through its middle. Determine the torque M.

<u>*Answer*</u>. M= 0.025 N·m.

Problem D

<u>*Problem description*</u>. A platform in the form of a disk of radius R = 1 m rotates by inertia with a frequency of n_1 = 6 min^{-1}. At the edge of the platform is a man whose mass is 80 kg. Calculate the rotation frequency of the platform if a man moves to its centre. The moment of inertia of the platform is J = 120 kg·m^2. The moment of inertia of a man is considered as the moment of inertia of a material point.

<u>*Answer*</u>. n_2 = 0.1 min^{-1}.

Problem E

<u>*Problem description*</u>. A thin straight rod of length l = 1 m is attached to a horizontal axis passing through its end. The rod was rejected at an angle φ = 60° and released. Determine the linear velocity v of the lower end of the rod at the time of passage through the equilibrium position.

<u>*Answer*</u>. v = 3.84 m/s.

CHAPTER 4. MECHANICAL OSCILLATIONS

4.1. Harmonic Oscillations

Many physical processes are reduced to the study of the behaviour of the system with small deviations from the equilibrium position. In most practically important cases, researchers are interested in the behaviour of the system not with possible deviations from the equilibrium position, but only with small deviations. No matter how complex the dependence of the force $f(x)$ acting on the system on the coordinate is, this dependence can be represented in the form of a Taylor series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3} f'''(0) + \ldots \qquad (4.1.1)$$

The laws of action of forces $f(x)$ encountered in physics usually satisfy the expansion of this function in a ***Taylor series***. This subject was formulated by James Gregory (11.1638 – 10.1675) and Brook Taylor (18.08.1685 – 29.12.1731) in 1715.

Obviously $f(x) = 0$, since point $x = 0$ is an equilibrium point and, therefore, the force at this point is zero. There are two possible cases: 1) $f'(0) \neq 0$, and 2) $f'(0) = 0$.

In the first case, the term $xf'(0)$ is the main member of the Taylor series (4.1.1). All subsequent terms of the series are proportional to x^2, x^3, …, and for sufficiently small values of x they are arbitrarily small compared to the first term. Therefore, with a sufficiently small deviation x from the equilibrium position, we can assume that the force acting on the system is $xf'(0)$. Point $x = 0$ is an equilibrium point, so the force $xf'(0)$ should always be directed to this point. It means that $f'(0) < 0$.

For the second case, when $f'(0) = 0$, it is necessary to consider the third term, proportional to x^2. This term $(x^2/2)f''(0)$ must be zero, because $x = 0$ is an equilibrium point. This statement follows from the fact that this term has the same sign, both for positive and negative values x. Finally we get that $f''(0) = 0$. Thus, the next non-zero term can be a term proportional to x^3. When analyzing small deviations in case $f'(0) = 0$, term $(x^3/3)f'''(0)$ must be used as an expression for force.

Usually, in real physical systems, the term $xf'(0)$ is nonzero, and the equation of motion for small deviations x from the equilibrium position has the form

$$m\left(\frac{d^2x}{dt^2}\right) = xf'(0) = -kx, \qquad (4.1.2)$$

where $f'(0) < 0$; $k = -f'(0) > 0$.

Equation (4.1.2) is called the ***equation of harmonic oscillations***. A system that implements such small oscillations is called a ***harmonic oscillator***. Examples of a harmonic oscillator are mathematical and physical pendulums with sufficiently small angles of deviation from the equilibrium position.

Consider the system on which the force $f(x)$ acts and in the expansion for this force, along with the term proportional to x, it is also necessary to consider the terms proportional to x^2, x^3, … In this case, nonlinear oscillations occur. A system that performs such oscillations is called an ***anharmonic oscillator***.

Equation (4.1.2) of the motion of a harmonic oscillator can be represented in the form of

$$\ddot{x} + \omega^2 x = 0, \tag{4.1.3}$$

where $\omega^2 = \dfrac{k}{m} > 0$.

A direct check can verify that the quantities $\sin(\omega t)$ and $\cos(\omega t)$ are solutions of (4.1.3). The differential equation (4.1.3) is linear. The sum of the solutions of the linear equation and the multiplication of any solution by an arbitrary constant value also makes up the solution. Therefore, the general solution of equation (4.1.3) has the form

$$x(t) = A_1 \sin(\omega t) + A_2 \cos(\omega t), \tag{4.1.4}$$

where A_1 and A_2 are constant values. A function of type $x(t)$ is called a ***harmonic function***.

Expression (4.1.4) is often converted to another form

$$A_1 \sin(\omega t) + A_2 \cos(\omega t) =$$

$$= \sqrt{A_1^2 + A_2^2}\left[\frac{A_1}{\sqrt{A_1^2 + A_2^2}}\sin(\omega t) + \frac{A_2}{\sqrt{A_1^2 + A_2^2}}\cos(\omega t)\right] =$$

$$A[\cos\varphi \sin(\omega t) + \sin\varphi \cos(\omega t)] = A\sin(\omega t + \varphi), \tag{4.1.5}$$

where $A = \sqrt{A_1^2 + A_2^2}$; $\cos\varphi = \dfrac{A_1}{\sqrt{A_1^2 + A_2^2}}$; $\sin\varphi = \dfrac{A_2}{\sqrt{A_1^2 + A_2^2}}$.

Therefore, equation (4.1.4) can be represented as

$$x = A\sin(\omega t + \varphi) \quad \text{or} \quad x = A\cos(\omega t + \varphi'). \tag{4.1.6}$$

The value A is called the ***amplitude***. The value ω is called the ***harmonic frequency***. The value $(\omega t + \varphi)$ in the argument of sine or cosine in expression (4.1.6) is called the ***oscillation phase***. The phase value at the initial time $t = 0$, i.e. $(\omega t + \varphi)\big|_{t=0} = \varphi$ is called the ***initial phase***. The value $T = \dfrac{2\pi}{\omega}$ is called the ***period of harmonic oscillation***.

The description of the processes associated with the addition of harmonic oscillations is simplified if the oscillations are depicted graphically as vectors in the plane, or the vibrations are presented in complex form. The graphic oscillation diagram is called a ***vector diagram***.

Consider x axis, which contains point O. We will build a vector from a point O. The vector has a length A and makes an angle φ with the x axis. We bring the vector into rotation with an angular velocity ω. The projection of the end of the vector will move along the x axis from $-A$ to $+A$. The coordinate of this projection will change over time according to the law $x = A\cos(\omega t + \varphi)$. Consequently, the projection of the end of the vector onto x axis will perform harmonic oscillation with amplitude, equal to the length of the vector, with a circular frequency equal to the angular velocity of rotation of the vector, and with an initial phase equal to the angle formed by the vector with the axis at the initial time.

Consider the addition of two harmonic oscillations of the same direction and the same frequency

$$x_1 = A_1\cos(\omega t + \varphi_1)$$

$$x_2 = A_2\cos(\omega t + \varphi_2). \qquad (4.1.7)$$

We represent both vibrations using vectors $\vec{A}_1$ and $\vec{A}_2$. We construct the resulting vector $\vec{A} = \vec{A}_1 + \vec{A}_2$ using the rules of addition of vectors. The resulting vector will rotate at the same angular velocity ω as the vector $\vec{A}_1$ and $\vec{A}_2$. Using the cosine theorem, for the modules of vectors $|\vec{A}| = A$, $|\vec{A}_1| = A_1$, $|\vec{A}_2| = A_2$, as well as the initial phase φ of the resulting vector, we can write

$$|\vec{A}| = A = \left[\,|\vec{A}_1|^2 + |\vec{A}_2|^2 + 2|\vec{A}_1|\cdot|\vec{A}_2|\cos(\varphi_2 - \varphi_1)\right]^{1/2}$$

$$\tan\varphi = \frac{A_1\sin\varphi_1 + A_2\sin\varphi_2}{A_1\cos\varphi_1 + A_2\cos\varphi_2}. \qquad (4.1.8)$$

So, the resulting oscillation is also harmonic with amplitude A, frequency ω and initial phase φ.

Consider the addition of two equally directed oscillations, which vary slightly in frequency: ω and $\omega + \Delta\omega$ $(\Delta\omega << \omega)$.This difference allows you to choose the point in time when the initial phases of the oscillations are equal to zero. Consider case $A_1 = A_2 = A$ for simplicity. The equation of these oscillations will have the form

$$x_1 = A\cos(\omega t)$$

$$x_2 = A\cos[(\omega + \Delta\omega)t]. \qquad (4.1.9)$$

We use the formula for the sum of cosines

$$x = x_1 + x_2 = \left[2A\cos\left(\frac{\Delta\omega}{2}t\right)\right]\cos(\omega t). \qquad (4.1.10)$$

The first factor in square brackets changes much more slowly than the second factor. Consequently, the first factor will hardly change in the time it takes the second factor to complete several complete oscillations. This gives us reason to consider the oscillation (4.1.10) as a harmonic oscillation of the frequency ω, the amplitude of which varies according to some periodic law. Oscillations of this type are called ***beats***.

Consider a system with two degrees of freedom. Two quantities are needed to uniquely specify the position of such a system in space. Such a system can be considered as a system that performs two mutually perpendicular vibrations.

Consider the addition of two mutually perpendicular harmonic oscillations of the same frequency ω, which occur along the coordinate axes x and y. We choose the time reference so that the initial phase of the first oscillation is equal to zero, and the initial phase of the second oscillation is equal to φ. Then the equations of mutually perpendicular vibrations will have the form

$$\begin{aligned} x &= A\cos(\omega t) \\ y &= B\cos(\omega t + \varphi). \end{aligned} \qquad (4.1.11)$$

Equations (4.1.11) is the equation of the trajectory specified in the parametric form along which the body participating in both vibrations moves. We exclude time from these equations.

Let's make preliminary transformations: $\cos(\omega t) = x/A$; $\sin(\omega t) = \sqrt{1-(x/A)^2}$. Using these transformations, we rewrite (4.1.11):

$$\frac{y}{b} = \frac{x}{A}\cos\varphi - \sin\varphi\sqrt{1-\left(\frac{x}{A}\right)^2}$$

or

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}\cos\varphi = \sin^2\varphi. \qquad (4.1.12)$$

Equation (4.1.12) is the equation of an ellipse whose axes are arbitrarily oriented relative to the coordinate axes x and y.

We study the shape of the trajectory in some special cases.

1. Phase difference φ is zero. In this case (4.1.12) takes the form

$$\left(\frac{x}{A}-\frac{y}{B}\right)=0. \qquad (4.1.13)$$

We get a straight line equation

$$y=\frac{B}{A}x. \qquad (4.1.14)$$

The point during vibrations moves along this straight line, and its distance from the origin is $r=\sqrt{x^2+y^2}$. We take into account that φ = 0, then for dependence $\varphi = \varphi(t)$ we get

$$r=\sqrt{A^2+B^2}\cos(\omega t). \qquad (4.1.15)$$

Therefore, the resulting movement is harmonic oscillation along a straight line.

2. The phase difference φ is ± π. In this case (4.1.12) takes the form

$$\left(\frac{x}{A}+\frac{y}{B}\right)^2=0. \qquad (4.1.16)$$

Analysis of the equation leads to the conclusion that the resulting movement is a harmonic oscillation along a straight line

$$y=-\frac{B}{A}x. \qquad (4.1.17)$$

3. The phase difference φ is ± π/2. In this case (4.1.12) takes the form

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1. \qquad (4.1.18)$$

Equation (4.1.18) is the equation of an ellipse reduced to coordinate axes x and y. The semiaxes of the ellipse are equal to the corresponding amplitudes of the oscillations. The ellipse turns into a circle with equal amplitudes $A = B$.The cases $\varphi = +\pi/2$ and $\varphi = -\pi/2$ differ in the direction of movement along the ellipse (or around the circumference). Consider mutually perpendicular vibrations whose frequencies differ by a small amount. The resulting movement in this case occurs along a slowly changing curve, which will sequentially take the form corresponding to all the values of the phase difference from $-\pi$ to $+\pi$.

Consider the case when the frequencies of mutually perpendicular vibrations differ by a significant amount. In this case, the trajectory of the resulting movement has the form of rather complex curves, which are called ***Lissajous figures*** or ***Bowditch curve***. This family of curves was investigated by Nathaniel Bowditch

(26.03.1773 – 16.03.1838) and Jules Antoine Lissajous (4.03.1822 – 24.06.1880). For example, with a frequency ratio of 1 : 2 and phase differences $\varphi = \pi/2$ and $\varphi = 0$, the motion paths are a closed and an open curve, respectively.

4.2. Natural Oscillations

Oscillations that occur under the action of only internal forces without external influences are called ***natural oscillations***. Natural oscillations may not be harmonic. But with sufficiently small deviations from the equilibrium position, these oscillations can be considered as harmonic oscillations.

Harmonic oscillation is completely characterized by frequency, amplitude and initial phase. The oscillation frequency depends on the physical properties of the system. For example, in the case of a harmonic oscillator in the form of a material point oscillating under the action of the spring's elastic forces, the elastic properties of the spring are taken into account by the elastic coefficient k, and the properties of the point are taken into account by its mass m. In this case, for the frequency we get the expression $\omega = k/m$.

The calculation of the amplitude and initial phase of the oscillations requires knowledge of the position and velocity of the material point at some point in time. If the oscillation of the material point is expressed by equation $x = A\cos(\omega t + \varphi)$, and the coordinate and velocity at the time $t = 0$ are equal, respectively x_0 and v_0, then we can write the corresponding relationship

$$x_0 = A\cos\varphi,\ \dot{x}_0 = v_0 = \left.\frac{dx}{dt}\right|_{t=0} = -A\omega\sin\varphi. \qquad (4.2.1)$$

Formulas (4.2.1) allow us to calculate unknown values of the amplitude and initial phase

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \quad \tan\varphi = -\frac{v_0}{(x_0\omega)}. \qquad (4.2.2)$$

Thus, the known initial conditions make it possible to completely describe the harmonic oscillation.

The idea of potential oscillation energy makes sense only when the acting forces are potential. Consider a harmonic oscillator. In this case, it is convenient to assume that the potential energy of the point is zero in the equilibrium position. We take into account that the restoring force can be represented in the form $F = -kx$. The relationship between potential energy and potential force has the form $F = -\partial W_p / \partial x$. Consequently, the potential energy of a harmonic oscillator can be represented as

$$W_p(x) = \frac{kx^2}{2} = \frac{m\omega^2 x^2}{2}. \qquad (4.2.3)$$

The law of conservation of mechanical energy for an oscillating system in which only potential forces act has the form

$$\frac{m\dot{x}^2}{2}+\frac{m\omega^2 x^2}{2}=\text{const}. \qquad (4.2.4)$$

The law of conservation of mechanical energy of the considered oscillating system allows us to draw two important conclusions.

1. The maximum kinetic energy of a harmonic oscillator is equal to its maximum potential energy. The harmonic oscillator has the maximum kinetic energy at the moment of its passage, the equilibrium position point $x = 0$. The potential energy at this moment is zero. Therefore, denoting the maximum speed as v_m, we can write

$$\frac{mv_m^2}{2}=\frac{m\omega^2 A^2}{2}. \qquad (4.2.5)$$

2. The average kinetic energy of a harmonic oscillator is equal to its average potential energy. The law of motion for a harmonic oscillator corresponds to formula $x(t) = A\cos(\omega t + \varphi)$, and the velocity in harmonic oscillations is $\dot{x}=-A\omega\sin(\omega t+\varphi)$. Therefore, the time dependences for the kinetic and potential energy have the form

$$W_k(t)=\frac{m\dot{x}^2}{2}=\frac{m\omega^2 A^2}{2}\sin^2(\omega t+\varphi)$$

$$W_p(t)=\frac{m\omega^2 A^2}{2}\cos^2(\omega t+\varphi). \qquad (4.2.6)$$

Consider the period of one oscillation as a period of time on which the average value is determined. The calculation of the average values of $\langle W_k\rangle$ and $\langle W_p\rangle$ is reduced to finding the average values $\langle\cos^2(\omega t+\varphi)\rangle$ and $\langle\sin^2(\omega t+\varphi)\rangle$. We can write the following relations

$$\langle\cos^2(\omega t+\varphi)\rangle=\frac{1}{T}\int_0^T\cos^2(\omega t+\varphi)dt=\frac{1}{2}$$

$$\langle\sin^2(\omega t+\varphi)\rangle=\frac{1}{T}\int_0^T\sin^2(\omega t+\varphi)dt=\frac{1}{2}. \qquad (4.2.7)$$

An analysis of the formulas (4.2.6) and (4.2.7) allows us to write the following relation

$$\langle W_k(t)\rangle=\langle W_p(t)\rangle. \qquad (4.2.8)$$

The dependence of deviation, velocity and acceleration during harmonic oscillations on time are represented by completely identical curves, but shifted relative to each other in the direction of the axis ωt. The phase of velocity in harmonic oscillations is greater than the displacement phase by an amount of $\pi/2$. The phase of acceleration in harmonic oscillations is greater than the phase of velocity by a value of $\pi/2$. Thus, the acceleration phase is greater than the displacement phase by an amount of π.

Consider the case when, in the expansion for the force, along with the linear term $xf'(0)$, the next term is also significant, for example $x^2 f''(0)/2!$ In this case, the Taylor series for the force will have the form

$$m\frac{d^2x}{dt^2} = xf'(0) + \frac{x^2}{2!} f''(0). \tag{4.2.9}$$

It was previously noted that if the system oscillates around the position of stable equilibrium $x = 0$, then when condition $f'(0) = 0$ is fulfilled, condition $f''(0) = 0$ must also be fulfilled. Otherwise, point $x = 0$ cannot be a point of stable equilibrium. Obviously, if condition $f'(0) \neq 0$ is satisfied, then condition $f'(0) < 0$ must also be fulfilled, and in addition, derivative $f''(0)$ need not be equal to zero and can have any sign. In addition, it is assumed that the value of $f''(0)$ is a sufficiently small value.

We divide both sides of equation (4.2.9) by mass m

$$\ddot{x} + \omega_0^2 x = \varepsilon \omega_0^2 x^2, \tag{4.2.10}$$

where $\omega_0^2 = -\dfrac{f'(0)}{m}$; $\varepsilon = \dfrac{f''(0)}{2m\omega_0^2} = -\dfrac{f''(0)}{2f'(0)}$.

The value of ε is a parameter of the smallness of the term proportional to the square of the displacement. This value has a dimension inversely proportional to the length, and therefore can be represented in the form $\varepsilon = 1/L$, where L is a large quantity having a length dimension.

If the displacement x is sufficiently small $x << L = \varepsilon^{-1}$, then the term on the right-hand side of (4.2.10) can be considered small. In this case, this term is called a ***perturbation.*** The method by which an approximate solution of equation (4.2.10) is found is called the perturbation theory method.

The oscillation that occurs when condition $\varepsilon = 0$ is satisfied is called unperturbed motion

$$x_0(t) = A_0 \sin \omega_0 t. \tag{4.2.11}$$

The condition under which the right-hand side of equation (4.2.10) can be considered as a perturbation has the form $\varepsilon A_0 << 1$ (the amplitude A_0 is not very large). The solution of the equation of oscillation motion in the presence of a disturbance ($\varepsilon \neq 0$) can be represented as

$$x = A_0 \sin \omega_0 t + x_1(t), \tag{4.2.12}$$

where $x_1(t)$ is a correction to the unperturbed motion.

If $\varepsilon \to 0$, then the value $x_1(t)$ should also tend to zero. Therefore, $x_1(t)$ should also be a small value $|x_1| << A_0$. We substitute equation (4.2.12) into equation (4.2.10)

$$\ddot{x}_1 + \omega_0^2 x_1 = \varepsilon\omega_0^2\left(A_0^2 \sin^2 \omega_0 t + 2A_0 x_1 \sin \omega_0 t + x_1^2\right). \tag{4.2.13}$$

The second and third terms in brackets on the right side of equation (4.2.13) are much smaller than the first term, therefore, these terms can be neglected. Using these considerations, we rewrite equation (4.2.13) in the form

$$\ddot{x}_1 + \omega_0^2 x_1 = \frac{\varepsilon\omega_0^2}{2} A_0^2 (1 - \cos 2\omega_0 t). \tag{4.2.14}$$

The solution to equation (4.2.14) should be sought in the form

$$x_1 = a_1 + b_1 \cos 2\omega_0 t, \tag{4.2.15}$$

where a_1 and b_1 are constant values.

The simultaneous solution of equations (4.2.14) and (4.2.15) with subsequent analysis leads to the following relations for a_1 and b_1

$$a_1 = \frac{\varepsilon A_0^2}{2}, \quad b_1 = \frac{\varepsilon A_0^2}{6}. \tag{4.2.16}$$

Therefore, the solution of equation (4.2.12), taking into account the first correction, can be written in the form

$$x = A_0 \sin \omega_0 t + \frac{1}{2}\varepsilon A_0^2 + \frac{1}{6}\varepsilon A_0^2 \cos 2\omega_0 t. \tag{4.2.17}$$

The most significant feature of this decision is the presence of a member proportional to $\cos 2\omega_0 t$. Therefore, due to the presence in the expansion for the force of a nonlinear term proportional to x^2, a term with a doubled frequency $2\omega_0$ appears in the vibrations. This term is called the ***second harmonic*** of oscillations. If we continue the solution of equation (4.2.10) and find the following smaller corrections, we can make sure that they contain frequencies $n\omega_0$, which are multiples of the fundamental frequency ω_0. In this case, it can be argued that the oscillation has higher harmonics. We can say that the most characteristic consequence of the presence of nonlinearity in the Taylor series for a force is the appearance of higher harmonics in the oscillations.

An analysis of equation (4.2.17) allows us to state that both components of the oscillations with frequencies ω_0 and $2\omega_0$ do not occur near point $x = 0$, but in the vicinity of point $x = \varepsilon A_0/2$. Consequently, the presence of a nonlinear term proportional to x^2 shifts the equilibrium point, near which oscillations occur. This result is understandable if we take into account that a force proportional to x^2 is directed all the time in the same direction and, therefore, must inevitably shift the point near which the oscillations occur.

4.3. Damped Oscillations

The presence of friction, which is an external force, leads to a decrease in the energy of the harmonic oscillator. A decrease in energy leads to a decrease in the amplitude of the oscillations. Fluctuations in the presence of friction become ***damped oscillations***. Friction acts against speed. Therefore, for a harmonic oscillator, the action of the friction force is equivalent to a decrease in the returning force. For a spring pendulum, this is equivalent to a decrease in spring elasticity, i.e. quantities k. The oscillation frequency is $\omega = k/m$. This means that the oscillation frequency should decrease, and the oscillation period should increase.

Consider the force of liquid friction. It is necessary to add the force of liquid friction to the right side of the equation of motion

$$m\ddot{x} = -kx - b\dot{x}, \tag{4.3.1}$$

where b is the coefficient of friction.

It is convenient to rewrite equation (4.3.1) as follows

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \tag{4.3.2}$$

where $\gamma = \dfrac{b}{2m}$; $\omega_0^2 = \dfrac{k}{m}$.

The solution to equation (4.3.2) is conveniently sought in the form

$$x = A_0 \exp(i\beta t). \tag{4.3.3}$$

Considering that

$$\frac{d}{dt}\exp(i\beta t) = i\beta \exp(i\beta t), \qquad \frac{d^2}{dt^2}\exp(i\beta t) = -\beta^2 \exp(i\beta t) \tag{4.3.4}$$

and substituting (4.3.3) in (4.3.2), we find

$$A_0 \exp(i\beta t)\cdot\left(-\beta^2 + 2i\gamma\beta + \omega_0^2\right) = 0. \tag{4.3.5}$$

The factor $A_0\exp(i\beta t)$ is not equal to zero. Therefore, another factor must be equal to zero

$$-\beta^2 + 2i\gamma\beta + \omega_0^2 = 0. \tag{4.3.6}$$

Equation (4.3.6) is a quadratic equation with respect to β. The solutions of this equation have the form

$$\beta = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2} = i\gamma \pm \Omega, \quad \Omega = \sqrt{\omega_0^2 - \gamma^2}. \tag{4.3.7}$$

Substituting these values for quantity β into equation (4.3.3), we obtain

$$x = A_0 \exp(-\gamma t) \cdot \exp(\pm i\Omega t). \tag{4.3.8}$$

The presence of the signs "±" in equation (4.3.8) reflects the fact that equation (4.3.2) is a second-order equation and, therefore, must have two independent solutions that are obtained with different signs.

Considering the case of not very large coefficients of friction, we can write the following relation

$$\gamma = \frac{b}{2m} < \omega_0. \tag{4.3.9}$$

In this case, $\omega_0^2 - \gamma^2 > 0$ and, therefore, Ω is a real quantity. Therefore, $\exp(i\Omega t)$ is a harmonic function. The oscillation described by equality (4.3.8) is represented in real form by formula

$$x = A_0 \exp(-\gamma t) \cos \Omega t. \tag{4.3.10}$$

This oscillation is not harmonic and is not periodic. Nevertheless, it is convenient to talk about the period T of such oscillations, meaning by the period the time intervals through which the displacement vanishes. In the same sense, we can speak of the oscillation frequency $\Omega = \frac{2\pi}{T}$. The amplitude of the oscillations in this case is $A = A_0\exp(-\gamma t)$. The amplitude of damped oscillations depends on time and can be represented as maximum deviations during successive oscillations. The time during which the amplitude of the damped oscillations decrease by e times

$$\tau_d = \frac{1}{\gamma} \tag{4.3.11}$$

is called the ***decay time***. The value γ is called the ***damping decrement***.

The attenuation value must be attributed to the natural decay time scale, i.e. to the period of oscillation. The attenuation intensity is characterized by the attenuation of the amplitude in one oscillation period, and therefore, instead of the attenuation decrement γ, it is convenient to use the so-called logarithmic attenuation decrement.

We calculate the oscillation amplitudes in two consecutive time intervals separated by the oscillation period

$$A_1 = A_0 \exp(-\gamma t_1),\ \ A_2 = A_0 \exp(-\gamma t_2). \tag{4.3.12}$$

It follows that

$$\frac{A_1}{A_2} = -\exp(\gamma t). \tag{4.3.13}$$

Therefore, the change in the amplitude of oscillations for period T is characterized by a value

$$\theta = \gamma T, \tag{4.3.14}$$

The value θ is called the ***logarithmic damping decrement***.

Formula (4.3.13) can be transformed as follows

$$\theta = \ln\left(\frac{A_1}{A_2}\right). \tag{4.3.15}$$

The logarithmic decrement of attenuation is the logarithm of the ratio of the amplitudes of the oscillations in one period.

A different interpretation can be given to the logarithmic decrement of attenuation. Consider a decrease in the amplitude of oscillations over N periods, i.e. in time NT. Instead of formulas (4.3.12), we can write the following formulas

$$A_1 = A_0 \exp(-\gamma t_1),\ \ A_{N+1} = A_0 \exp[-\gamma(t_1 + NT)]. \tag{4.3.16}$$

Therefore, the ratio of amplitudes separated by a time interval containing N periods is

$$\frac{A_{N+1}}{A_1} = \exp(\gamma NT) = \exp(N\theta). \tag{4.3.17}$$

The fulfillment of condition $N\theta = 1$ corresponds to a decrease in amplitude by e times. Consequently, we can say that the logarithmic damping decrement

$$\theta = \frac{1}{N}. \tag{4.3.18}$$

is the inverse of the number of periods during which the amplitude decays e times.

Let's look at two examples.

1. Assume that $\theta = 10^{-2}$. The oscillations decay only after about 10^2 oscillations. During $N_1 = 10$ oscillations, the amplitude changes insignificantly, approximately by one tenth of its initial value. Consequently, during this small number of periods the oscillations can be considered as undamped oscillations.
2. Assume that $\theta = 0.1$. In this case, after $N_2 = 10$ oscillations, complete attenuation will occur. The attenuation is significant even after several fluctuations. Therefore,

in this case, when considering processes that occur even over several periods, it is impossible to consider fluctuations as undamped as an approximation.

4.4. Forced Oscillations

Along with friction, some other external force can act on the harmonic oscillator. The most important from an applied point of view is the case of harmonic external force. We assume that an external force acts on a harmonic oscillator according to the following law

$$F = F_0 \cos \omega t, \tag{4.4.1}$$

where F_0 is the amplitude of the force; ω is the frequency with which an external force acts on a harmonic oscillator.

Instead of equation (4.3.2), motion is described by the following equation

$$m\ddot{x} = -kx - b\dot{x} + F_0 \cos \omega t. \tag{4.4.2}$$

Dividing both sides of this equation by mass m, we obtain the canonical form of the equation of forced oscillations

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos \omega t. \tag{4.4.3}$$

The influence of the initial conditions weakens over time, and the oscillator moves into steady harmonic oscillations. The process of establishing oscillations is called ***transient regime***.

When considering a transitional regime, the most important is the question of its duration. This duration is determined by the decay time of the oscillations $\tau_d = 1/\gamma$ that existed at the time the external force started. Consequently, the value of τ_d is the period of time after which one can forget about the initially existing oscillations and consider only the oscillations that are established under the action of an external force. It can be shown that the time to establish the stationary regime of forced oscillations after the onset of an external force is also τ_d.

Consider the steady forced oscillations. In this case, it is necessary to consider that the oscillations were established at the infinitely distant past moment of time. Therefore, equation (4.4.3) can be considered for all instants of time. To solve this equation, it is convenient to use the complex form of harmonic oscillations

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\exp(i\omega t). \tag{4.4.4}$$

The solution to the equation will be sought in the form

$$x = A\exp(i\beta t). \tag{4.4.5}$$

We substitute (4.4.5) into (4.4.4)

$$A\exp(i\beta t)\cdot\left(-\beta^2+2i\gamma\beta+\omega_0^2\right)=\frac{F_0}{m}\exp(i\omega t). \tag{4.4.6}$$

This equality should be valid for all moments of time, i.e. time t should not ultimately enter it. From this condition it follows that $\beta = \omega_0$. We will find amplitude A from equation (4.4.6)

$$A=\frac{F_0}{m}\frac{1}{\left(\omega_0^2-\omega^2+2i\gamma\omega\right)}=\frac{F_0}{m}\frac{\left(\omega_0^2-\omega^2-2i\gamma\omega\right)}{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}. \tag{4.4.7}$$

The complex number (4.4.7) is more convenient to represent in exponential form

$$A=A_0\exp(i\varphi)$$

$$A_0=\frac{F_0}{m}\frac{1}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}$$

$$\tan\varphi=-\frac{2\gamma\omega}{\omega_0^2-\omega^2}=\frac{2\gamma\omega}{\omega^2-\omega_0^2}. \tag{4.4.8}$$

Therefore, the solution (4.4.5) can be represented in complex form

$$x=A_0\exp\left[i(\omega t+\varphi)\right]. \tag{4.4.9}$$

The real part of the expression (4.4.9) is equal to

$$x=A_0\cos(\omega t+\varphi), \tag{4.4.10}$$

where quantities A_0 and φ are determined by equations (4.4.8).

Thus, under the influence of an external harmonic force, the oscillator performs forced harmonic oscillations with the frequency of this force. The phase and amplitude of these oscillations are determined by both the properties of the force and the characteristics of the oscillator.

Consider the change in the phase and amplitude of the forced oscillations.

A curve describing the dependence of the amplitude of steady-state forced oscillations on the frequency of an external force is called the ***amplitude phase curve***. An analytical expression of such a curve is given by dependence (4.4.8). The amplitude reaches its maximum value at a frequency of external force close to the frequency of natural oscillations of the harmonic oscillator ($\omega \approx \omega_0$).Oscillations with a resonant frequency are called ***resonant oscillations***. The phenomenon of increasing the amplitude of oscillations to a maximum value at a frequency of $\omega \approx \omega_0$

is called ***resonance***. The frequency ω_0 in this case is called the ***resonant frequency***. When the frequency deviates from ω_0, the amplitude decreases sharply.

Forced vibrations with low friction are of the greatest interest. Therefore, we assume that $\gamma << \omega_0$.

Consider the case of low frequencies $\omega << \omega_0$ (static case). We write the expression for the amplitude from equation (4.4.8)

$$A_{0,s} \approx \frac{F_0}{m\omega_0^2}. \tag{4.4.11}$$

An external force at a low frequency acts on the system as a constant static force. Therefore, the maximum displacement (amplitude) is equal to the displacement under the influence of static force F_0

$$x_m = \frac{F_0}{k} = \frac{F_0}{m\omega_0^2}, \tag{4.4.12}$$

where $k = m\omega_0^2$ is the coefficient of elasticity of the restoring force.

From condition $\omega << \omega_0$ it follows that in equation (4.4.3), the terms $\ddot{x}$ (acceleration) and $2\gamma\dot{x}$ (speed) are much smaller than the term $\omega_0^2 x$ (in proportion to the elastic force). In this case, equation (4.4.3) can be represented as

$$\omega_0^2 x = \frac{F_0}{m}\cos\omega t. \tag{4.4.13}$$

The solution to this equation has the form

$$x = \frac{F_0}{m\omega_0^2}\cos\omega t. \tag{4.4.14}$$

This means that at each moment of time the displacement is what it should be if the force did not change with time. Friction forces do not play a role in these processes.

Consider the case of high frequencies $\omega >> \omega_0$. We write the expression for the amplitude from equation (4.4.8)

$$A \approx \frac{F_0}{m\omega^2}. \tag{4.4.15}$$

The term $\ddot{x}$ in equation (4.4.8), which is proportional to the external force, is much larger than each of the terms associated with speed and elastic force: $|\ddot{x}| \approx |\omega^2 x| >> |\omega_0^2 x|$; $|\ddot{x}| \approx |\omega^2 x| >> |2\gamma\dot{x}| \approx |2\gamma\omega x|$. Therefore, equation (4.4.3) can be written as

$$\ddot{x} \approx -\frac{F_0}{m}\cos\omega t. \tag{4.4.16}$$

The solution to this equation has the form

$$x \approx \frac{F_0}{m\omega^2} \cos \omega t . \qquad (4.4.17)$$

An analysis of equation (4.4.17) shows that elastic forces and friction forces in comparison with an external force do not play any role in forced oscillations. An external force acts on the harmonic oscillator as if there were no elastic and friction forces.

Let us consider forced oscillations at a frequency close to the natural frequency of a harmonic oscillator $\omega \approx \omega_0$. Such an oscillation can be considered as a resonant oscillation. The amplitude has a maximum value at resonance

$$A_{0,r} = \frac{F_0}{m} \cdot \frac{1}{2\gamma\omega_0} . \qquad (4.4.18)$$

The term of equation (4.4.3) associated with acceleration is equal to the term due to elastic force, hence $\ddot{x} = -\omega^2 x = -\omega_0^2 x$. This means that the acceleration of the harmonic oscillator is due to the action of the elastic force, and the external force and the elastic force are mutually compensated. Equation (4.4.3) in this case can be rewritten in the form

$$2\gamma \dot{x} = \frac{F_0}{m} \cos \omega_0 t . \qquad (4.4.19)$$

The solution of equation (4.4.19) has the form

$$x = \frac{F_0}{2\gamma m \omega_0} \sin \omega_0 t . \qquad (4.4.20)$$

The maximum amplitude is reached not exactly at $\omega = \omega_0$, but near this value $\omega = \omega_1$. However, with not very large friction, when $\gamma << \omega_0$, the displacement of the position of the maximum at $\omega = \omega_1$ from the position of the maximum at $\omega = \omega_0$ is insignificant and can not be taken into account.

An important characteristic of the properties of a harmonic oscillator is the increase in the amplitude of its oscillations at resonance in comparison with the amplitude value for the static case. A consequence of formulas (4.4.11) and (4.4.18) is the equation

$$Q = \frac{A_{0,r}}{A_{0,s}} = \frac{\omega_0}{2\gamma} = \frac{2\pi}{2\gamma T} = \frac{\pi}{\theta}, \qquad (4.4.21)$$

where θ is the logarithmic damping decrement.

The value Q is called the quality factor (***Q-factor***) of the oscillating system.

It follows from equation (4.4.21) that the smaller the oscillator attenuation, the more vigorously the oscillator increases its amplitude in resonance: $A_{0,r} = A_{0,s}Q = A_{0,s}(\pi / \theta)$.

4.5. Self-Oscillations and Parametric Oscillations

Maintaining undamped natural oscillations in the system requires an energy source. This energy source replenishes the energy loss associated with attenuation. The oscillations will be stationary provided that in one period exactly as much energy enters the oscillation system as it is spent on the damping process during the same time.

Oscillating systems, which provide special circuits for receiving from a special energy source to compensate for energy losses due to attenuation, are called ***self-oscillating systems***. In contrast to forced oscillations, the frequency of self-oscillations coincides with the natural frequency of the system. In phase space, a periodic self-oscillation corresponds to a closed trajectory, to which all middle trajectories tend.

If the friction in the system is small, then in one period a small part of the total energy of the oscillator enters the system. In this case, self-oscillations are harmonic with great accuracy and their frequency is close to the frequency of natural vibrations. In the case of large friction forces in a single period, a significant part of the oscillator energy is supplied to the system and therefore the oscillations are very different from harmonic oscillations.

Consider the oscillations of a pendulum suspended on an axis in a rotating sleeve. Let the pendulum initially rest. Then the rotating sleeve as a result of sliding about the axis does the work to overcome the friction force. This work is completely converted into internal energy, and as a result, the axis and the sleeve are heated.

Now let the pendulum oscillate. In that half-period of oscillations of the pendulum, when the directions of rotation of the axis of the pendulum and the sleeve coincide, the friction forces coincide in the direction with the motion of the points of the axis surface. Therefore, these forces cause an increase in the amplitude of oscillations of the pendulum. On the other hand, the energy converted into internal energy decreases during the half-cycle of oscillations in comparison with the case of a resting pendulum. Therefore, only part of the energy from the machine, which rotates the sleeve, is converted into internal energy. Another part of the energy goes to increase the energy of the pendulum.

If the friction forces are independent of speed, then the energy acquired by the pendulum in the half-cycle of oscillations, when the directions of rotation of its axis and shaft coincide, is equal to the energy lost by this pendulum to work against the forces of friction in another half-period.

Self-oscillations are widely used in technology. A well-known example is a pendulum clock. In this watch, energy is transferred to the pendulum by the jerks as a result of the force exerted on the pendulum by the spring. Another example is an electric bell. The oscillations of the electric bell hammer turn on and off the electric current, which, in turn, transfers energy to the bell system, due to which the oscillations of the hammer are supported.

A special case of self-oscillations are resonant vibrations. A system that makes resonant oscillations accumulates energy for a fairly long time. At a certain point in time, sharp changes occur in the system, and it returns to its original state.

Undamped oscillations can be excited with a periodic change in the parameters of the oscillating system. Such excitation of oscillations is called parametric resonance. An example of a system that performs such oscillations is a pendulum whose length varies periodically. Suppose that at any time when the pendulum thread passes through a vertical (equilibrium) position, its length L decreases by ΔL. During each period, a decrease in the length L of the pendulum by ΔL will occur twice.

Consider the simplest case when the decrease in length L will occur according to the law

$$L = L_0 - \Delta L \cos \alpha_0, \tag{4.5.1}$$

where α_0 is the angular amplitude of the oscillations of the pendulum.

One change in the length of the pendulum corresponds to the performance of work against gravity

$$A_1 = mg\Delta L(1 - \cos \alpha_0) \approx \frac{mg\Delta L \alpha_0^2}{2} \tag{4.5.2}$$

and works against centrifugal force

$$A_2 = \frac{mv_0^2}{L}, \tag{4.5.3}$$

where $v_0 = L\omega\alpha_0$ is the maximum speed of the pendulum.

The total work performed by an external force in one period with a decrease in the length of the pendulum is

$$A = A_1 + A_2 = 3\frac{\Delta L}{L} mv_0^2. \tag{4.5.4}$$

This work is proportional to the oscillation energy of the pendulum. Thus, the energy of the pendulum with a parametric resonance systematically increases, and the amplitude of the oscillations in the case of low friction in the system increases exponentially.

An example of parametric resonance is a person swinging a swing. A person periodically crouches at a time when the swing is deflected to the maximum angle, and straightens when they pass the equilibrium position. Parametric resonance also occurs in an electric oscillatory circuit with variable reactive parameters.

4.6. Waves in Elastic Medium

An ***elastic medium*** is a large number of interconnected particles. The action of external forces in an elastic medium can lead to oscillatory processes or ***mechanical waves***. The kinematic sign of wave motion is the propagation of the oscillation phase, and the dynamic sign is energy transfer.

The wave process is characterized by a phase velocity (wave propagation velocity v), a wavelength of λ, a frequency of f, or a period $T = 1/f$ of oscillations. There is a simple relation between these quantities

$$v = \lambda f = \frac{\lambda}{T}. \tag{4.6.1}$$

The wavelength λ is the distance between particles that oscillate with the same phase. The wavelength is independent of coordinates and time. The propagation velocity of an elastic shear wave in solids is

$$v = \sqrt{\frac{F}{\rho S}}, \tag{4.6.2}$$

where F is the tension force of a solid (strings, wires, etc.); ρ is the density of the material from which the oscillating body is made; S is the cross-sectional area of a solid.

The phase velocity of an elastic longitudinal wave in a solid is

$$v = \sqrt{\frac{E}{\rho}}, \tag{4.6.3}$$

where E is Young's modulus.

The propagation velocity of a longitudinal wave in liquids is

$$v = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{1}{\kappa\rho}}, \tag{4.6.4}$$

where $K = 1/\kappa$ is a compression module; ρ is fluid density.

The propagation velocity of a longitudinal wave in a gas is

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\gamma RT}, \tag{4.6.5}$$

where ρ is gas density; P is pressure; γ is an adiabatic index; T is the thermodynamic temperature; R is a gas constant.

Transverse waves are called waves in which the direction of particle velocity is perpendicular to the direction of phase velocity. Longitudinal waves are waves in which the directions of particle velocity and phase velocity coincide.

Depending on the nature of propagation, one can consider linear (in one dimension), surface (in two dimensions), and spatial (in three dimensions) waves. The direction of wave propagation is called the ***beam***. The ***wave front*** is the geometrical location of all particles oscillating with the same phase. The wave front is perpendicular to the beam. The distance between adjacent wave fronts is equal to the ***wavelength***. Important examples of waves are spherical and plane waves. Spherical waves arise from a point source in space. The rays of spherical waves are directed along the radius, and the wave fronts are spheres. Spherical waves in the plane will be circular, and their wave fronts will be circles. Plane waves arise from a plane or remote source. The rays of plane waves are parallel to each other, and the wave fronts are planes.

The motion of particles of a medium oscillates with the transfer of energy due to the transfer of this energy from one particle to another. ***Energy density*** is the amount of energy per unit volume of the medium. The energy density of the medium is

$$u = \frac{dE}{dV} = \frac{\rho A^2 \omega^2}{2}, \qquad (4.6.6.)$$

where dE is energy in volume dV; A is the amplitude of the particle's oscillations; ω is the cyclic oscillation frequency; $\rho = dm/dV$ is the density of the medium; dm is mass in volume dV.

The ***flux of energy*** W is the energy passing through a surface of S in time t. The flux of energy equals

$$W = uSvt = \frac{\rho \omega^2 A^2 Svt}{2}. \qquad (4.6.7)$$

Power can be expressed as

$$P = \frac{dW}{dt} = uSv = \frac{\rho \omega^2 A^2 Sv}{2}. \qquad (4.6.8)$$

A value of $I = P/S$ is called ***intensity***. The intensity of the wave processes is equal to

$$I = uv = \frac{1}{2} \rho \omega^2 A^2 v. \qquad (4.6.9)$$

The dependence of the energy density on the distance r to the source has the form $u \sim 1/r$ for circular waves and $u \sim 1/r^2$ for spherical waves.

Consider a medium that contains a sufficiently large number of particles. The displacement of these particles during wave motion can be described by a continuous function $\psi(x,y,z,t)$. Displacements along the selected z axis are called ***longitudinal displacements***, and along the axes x and y are called ***transverse displacements***. Oscillations are called ***linearly polarized oscillations*** along the x axis or y axis if these oscillations occur only along one of these axes.

Let us consider the case when the quantity $\psi(z,t)$ is the instantaneous transverse displacement of particles in a linearly polarized oscillation with an equilibrium position z. In this case, the velocity and acceleration are described, respectively, by the relations $\partial\psi(z,t)/\partial t$ and $\partial^2\psi(z,t)/\partial t^2$. The equation of motion of such oscillations has the form

$$\frac{\partial^2\psi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0. \qquad (4.6.10)$$

Equation (4.6.10) is called the ***wave equation***. The general solution of the wave equation has the form

$$\psi(z,t) = A_1 f_1\left(t - \frac{z}{v}\right) + A_2 f_2\left(t + \frac{z}{v}\right), \qquad (4.6.11)$$

where f_1 and f_2 are arbitrary twice differentiable functions in z and t, which determine the wave profile.

In particular, the solution to equation (4.6.10) are ***plane monochromatic waves***

$$\psi_1 = A\cos\left(\omega t - \frac{\omega}{v}z\right) \quad \text{and} \quad \psi_2 = A\cos\left(\omega t + \frac{\omega}{v}z\right), \qquad (4.6.12)$$

where $v = dz/dt$ is the speed of wave propagation; $z = \vec{n}\vec{r}$; $\vec{n}$ is the unit normal to the wave front (along the z axis); $\vec{r}$ is the radius vector of any point on the surface of the wave front; $\omega = 2\pi/T$ is the circular frequency; T is a period of oscillation.

The value $k = \omega/v$ is called the ***wave number***, and the vector $\vec{k} = k\vec{n}$ is called the ***wave vector***. Waves ψ_1 and ψ_2 correspond to increasing and decreasing values of z.

In the general case, for waves that arise in a certain small region and propagate in a homogeneous isotropic medium, the ***d'Alembert's formula*** is valid

$$\Delta\vec{\psi} - \frac{1}{v^2}\frac{\partial^2\vec{\psi}}{\partial t^2} = 0, \qquad (4.6.13)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the ***Laplace operator*** (Laplacian).

d'Alembert's formula is named after Jean-Baptiste le Rond d'Alembert (16.11.1717 – 29.10.1783), who derived it in 1747. The Laplace operator is named after Pierre-Simon de Laplace (23.03.1749 – 5.03.1827).

The general solution of equation (4.6.13) for components ψ_i of vector $\vec{\psi}$ in the presence of spherical symmetry has the form

$$\psi_i = A_1\frac{1}{r}f_1(vt - r) + A_2\frac{1}{r}f_2(vt + r), \quad i = x, y, z, \qquad (4.6.14)$$

where f_1 and f_2 are arbitrary doubly differentiable functions in x, y, z, t; r is the distance from the points where the wave source is located to a given material point.

The first term in equation (4.6.14) describes the diverging wave, and the second term describes the converging wave.

Test questions

1. Is it possible to consider a mathematical pendulum, with an angle $\alpha \sim \pi/4$ of deviation from the equilibrium position, as a harmonic oscillator?
2. Write down the solution of the equation of motion of the harmonic oscillator in general form.
3. What restrictions are imposed on the frequency of oscillations that form the beats?
4. Give an example of the conditions necessary for the appearance of closed figures of Lissajous.
5. What parameters of the spring pendulum affect the oscillation frequency?
6. Indicate the conditions under which the harmonic oscillator has maximum kinetic energy.
7. Record the numerical values of the phase difference between displacement, velocity and acceleration during harmonic oscillations.
8. Describe the causes of higher harmonics in the oscillating system.
9. Can the law of conservation of mechanical energy be used for damped oscillations?
10. Write down the equation of damped oscillations.
11. Define the logarithmic attenuation decrement.
12. Write down the equation of forced oscillations in a general form.
13. Give a formula that determines the time of the transition period of forced oscillations.
14. Describe the dependence of the amplitude of the forced oscillations on the frequency.
15. Write down the formula for the initial phase of forced oscillations.
16. Define the amplitude phase curve.
17. Under what conditions does a resonance of forced oscillations occur?
18. Record the relationship between the Q factor and the logarithmic attenuation decrement.
19. Write down the formula for the phase velocity of an elastic longitudinal wave.
20. What properties should a medium have in order for the D'Alembert equation to be applied to it?

Problem-solving examples

Problem 4.1

<u>*Problem description*</u>. A material point of mass $m = 5$ g performs harmonic oscillations with a frequency of $f = 0.5$ Hz. The amplitude of the oscillations is $A = 3$ cm. Calculate the following values: 1) point speed v at the time when the displacement is $x = 1.5$ cm; 2) maximum force F_m, acting on a point; 3) full energy E of the oscillating point.

<u>*Known quantities*</u>: $m = 5$ g, $f = 0.5$ Hz, $A = 3$ cm, $x = 1.5$ cm.

<u>*Quantities to be calculated*</u>: v, F_m, E.

<u>*Problem solution*</u>. The displacement of the material point, which performs harmonic oscillation, at time t is

$$x = A\cos(\omega t + \varphi), \tag{P.4.1.1}$$

where A is the amplitude of the oscillation; ω is the cyclic oscillation frequency; φ is the initial phase.

The speed of the material point is

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t + \varphi). \tag{P.4.1.2}$$

We transform equations (P.4.1.1) and (P.4.1.2). To do this, we square both equations, divide the first by A^2, and divide the second equation by $A^2\omega^2$ and add

$$\frac{x^2}{A^2} + \frac{v^2}{A^2\omega^2} = 1 \tag{P.4.1.3}$$

or

$$\frac{x^2}{A^2} + \frac{v^2}{4\pi^2 f^2 A^2} = 1, \tag{P.4.1.4}$$

where f is the oscillation frequency.

We determine the speed v from equation (P.4.1.4)

$$v = \pm 2\pi f\sqrt{A^2 - x^2}\ . \tag{P.4.1.5}$$

Substitute the numerical values in the formula (P.4.1.5)

$$v = \pm 2 \times 3.14 \times 0.5 \times \sqrt{\left(3 \times 10^{-2}\right)^2 - \left(1.5 \times 10^{-2}\right)^2} = 8.16 \times 10^{-2} \text{ m/s}.$$

The plus sign corresponds to the case when the direction of speed coincides with the positive direction of the x axis. The minus sign corresponds to the case when the direction of speed coincides with the negative direction of the x axis.

We find the force F acting on the point based on Newton's second law

$$F = ma, \qquad \text{(P.4.1.6)}$$

where m is the point mass; a is the point acceleration.

Find the acceleration point

$$a = \frac{dv}{dt} = -A^2\omega^2 \cos(\omega t + \varphi), \qquad \text{(P.4.1.7)}$$

or

$$a = -4\pi^2 f^2 A\cos(\omega t + \varphi). \qquad \text{(P.4.1.8)}$$

We substitute the acceleration from the formula (P.4.1.8) into the formula (P.4.1.6)

$$F = -4\pi^2 f^2 mA\cos(\omega t + \varphi). \qquad \text{(P.4.1.9)}$$

An analysis of formula (P.4.1.9) shows that the maximum value of the force is

$$F_m = 4\pi^2 f^2 mA. \qquad \text{(P.4.1.10)}$$

We substitute the numerical values in the formula (P.4.1.10)

$$F_m = 4 \times 3.14^2 \times 0.5^2 \times 5 \times 10^{-3} \times 3 \times 10^{-2} = 1.49 \times 10^{-3} \text{ N}.$$

The total energy of the oscillating point is equal to the sum of the kinetic and potential energies. We calculate the total energy at the time when the kinetic energy reaches its maximum value. At this point in time, the potential energy is zero. Therefore, the total energy E of the oscillating point is equal to the maximum kinetic energy $W_{k,m}$

$$E = W_{k,m} = \frac{mv_m^2}{2}, \qquad \text{(P.4.1.11)}$$

where m is the mass of the material point; v_m is the maximum speed of the material point.

The maximum velocity of the material point is determined from formula (P.4.1.2), provided that $(\omega t + \varphi) = 1$

$$v_m = 2\pi f A. \qquad \text{(P.4.1.12)}$$

We substitute the quantity v_m determined from the formula (P.4.1.12) into the formula (P.4.1.11), we find the total energy of harmonic vibrations

$$E = 2\pi^2 m f^2 A^2. \qquad \text{(P.4.1.13)}$$

We substitute the numerical values in the formula (P.4.1.13)

$$E = 2\times 3.14^2 \times (5\times 10^{-3})\times 0.5^2 \times \left(3\times 10^{-2}\right)^2 = 2.21\times 10^{-5}\ \text{J}.$$

Answer. Point speed is v = 8.16×10^{-2} m/s. The maximum force acting on a point is F_m = 1.49×10^{-3} N. The total energy harmonic oscillations is E = 2.21×10^{-5} J.

Problem 4.2

Problem description. The system performs two harmonic oscillations $x_1 = A_1\cos\,\omega(t + \tau_1)$ and $x_2 = A_2\cos\,\omega(t + \tau_2)$ simultaneously. The notation used in the formulas: A_1 = 1 cm, A_2 = 2 cm, τ_1 = 1/6 s, τ_2 = 1/2 s, $\omega = \pi$ s^{-1}. Determine the initial phases φ_1 and φ_2 of these oscillations. Find the amplitude A and the initial phase φ of the resulting oscillation.

Known quantities: $x_1 = A_1\cos\,\omega(t + \tau_1)$, $x_2 = A_2\cos\,\omega(t + \tau_2)$, A_1 = 1 cm, A_2 = 2 cm, τ_1 = 1/6 s, τ_2 = 1/2 s.

Quantities to be calculated: φ_1, φ_2, A, φ.

Problem solution. The displacement during harmonic oscillation at time t is

$$x = A\cos(\omega t + \varphi), \qquad \text{(P.4.2.1)}$$

where A is the amplitude of the oscillation; ω is the cyclic frequency of oscillation; φ is the initial phase of oscillation.

We transform the equations given in the condition of the problem to the form (P.4.2.1)

$$x_1 = A_1\cos(\omega t + \omega\tau_1), \qquad \text{(P.4.2.2)}$$

$$x_2 = A_2\cos(\omega t + \omega\tau_2), \qquad \text{(P.4.2.3)}$$

where τ_1 and τ_2 are initial time shifts of harmonic oscillations.

We determine the initial phases of oscillations φ_1 and φ_2 from a comparison of equations (P.4.2.1), (P.4.2.2), and (P.4.2.3)

$$\varphi_1 = \omega\tau_1 = \pi/6 \text{ rad},$$

$$\varphi_2 = \omega\tau_2 = \pi/2 \text{ rad}.$$

The cosine theorem indicates the relationship of oscillations amplitudes A, A_1, and A_2

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1)}. \quad \text{(P.4.2.4)}$$

Substitute the numerical values in the formula (P.4.2.4)

$$A = \sqrt{(0.01)^2 + (0.02)^2 + 2\times 0.01\times 0.02\times \cos(\pi/6 - \pi/2)} = 2.65\times 10^{-2} \text{ m}.$$

The initial phase of resulting oscillations in one direction can be determined from the formula

$$\varphi = \arctan\left(\frac{A_1\sin\varphi_1 + A_2\sin\varphi_2}{A_1\cos\varphi_1 + A_2\cos\varphi_2}\right). \quad \text{(P.4.2.7)}$$

Substitute the numerical values in the formula (P.4.2.7)

$$\varphi = \arctan\left[\frac{0.01\times\sin(\pi/6) + 0.02\times\sin(\pi/2)}{0.01\times\cos(\pi/6) + 0.02\cos(\pi/2)}\right] = 1.24 \text{ rad}.$$

Answer. The initial phases of oscillations are equal $\varphi_1 = \pi/6$ rad, $\varphi_2 = \pi/2$ rad. The amplitude of the resulting oscillation is equal to $A = 2.65\times 10^{-2}$ m. The initial phase of the resulting oscillation is $\varphi = 1.24$ rad.

Problem 4.3

Problem description. At a distance of $l = 4$ m from a plane wave source with a frequency of $f = 440$ Hz, a wall is laid out. The wall plane is perpendicular to the direction of wave propagation. Determine the distance from the source to the points at which the first three nodes of the standing wave arise. A standing wave occurs as a result of the addition of a travelling wave and a wave that is reflected from the wall. The speed of the wave is $v = 440$ m/s.

Known quantities: $l = 4$ m, $f = 440$ Hz, $v = 440$ m/s.

Quantities to be calculated: x_0, x_1, x_2.

Problem solution. The equation of the travelling wave at time t has the form

$$\xi_1 = A\cos(\omega t - kx), \tag{P.4.3.1}$$

where A is the amplitude of the wave; ω is the cyclic frequency of the wave; k is the wave number; x is the coordinate of the wave.

The reflected wave arrives at the point with coordinate x, having twice passed the path $(l - x)$, where l is the distance from the source of plane waves to the wall. In addition, reflection from the wall is accompanied by a change in the phase of the wave by a value of π. Therefore, the equation of the reflected wave can be written as

$$\xi_2 = A\cos\{\omega t - k[x + 2(l - x)] + \pi\}. \tag{P.4.3.2}$$

We transform the formula (4.3.2)

$$\xi_2 = -A\cos[\omega t - k(2l - x)]. \tag{P.4.3.3}$$

The standing wave equation can be determined by adding equations (P.4.3.1) and (P.4.3.2)

$$\xi = \xi_1 + \xi_2 = A\cos(\omega t - kx) - A\cos[\omega t - k(2l - x)]. \tag{P.4.3.4}$$

We apply the relation of cosines to the relation (P.4.3.4)

$$\xi = -2A\sin[k(l - x)] \cdot \sin(\omega t - kl). \tag{P.4.3.5}$$

Expression $A\sin\,[k(l - x)]$ is time independent. Therefore, the modulus of this expression can be considered as the amplitude of a standing wave

$$A_s = |A\sin[k(l - x)]|. \tag{P.4.3.6}$$

Nodes arise at those points where the amplitude of the standing wave is zero

$$|A\sin[k(l - x)]| = 0. \tag{P.4.3.7}$$

Equality (P.4.3.7) holds for those points whose coordinates x_n are included in the relation

$$k(l - x_n)\pi n, \quad n = 0, 1, \ldots \tag{P.4.3.8}$$

The wave number is

$$k = \frac{2\pi f}{v}, \tag{P.4.3.9}$$

where f is the frequency of the wave; v is the speed of the wave.

Solving formulas (P.4.3.8) and (P.4.3.9) together, we obtain

$$2\pi f\left(l-x_n\right)=\pi nv. \qquad \text{(P.4.3.10)}$$

The coordinates of the nodes can be determined from equation (P.4.3.10)

$$x_n=l-\frac{nv}{2f}. \qquad \text{(P.4.3.11)}$$

We substitute the numerical values in the formula (P.4.3.11), given that the value n takes three values: 0, 1, and 2

$$x_0 = 4\text{ m},\quad x_1 = 3.5\text{ m},\quad x_2 = 3\text{ m}.$$

Answer. The distances from the source to the points at which the first three nodes of the standing wave arise are $x_0 = 4$ m, $x_1 = 3.5$ m, $x_2 = 3$ m.

Problems

Problem A

Problem description. A point oscillates according to the following law $x = A\cos(\omega t)$. At some moment of time, the point displacement is $x_1 = 5$ cm. Then the phase of the oscillations doubled, and the displacement of the point became $x_2 = 8$ cm. Calculate the amplitude A of the oscillations.

Answer. $A = 8.33\times10^{-2}$ m.

Problem B

Problem description. The point moves evenly around the circle counter clockwise. The period of movement of the point is $T = 6$ s. The diameter of the circle is $d = 20$ cm. Determine the projection of the acceleration of the point on x axis at time $t = 1$ s.

Answer. $a = 9.5\times10^{-2}$ m/s^2.

Problem C

Problem description. The point movement is given by the equations $x = A_1\sin(\omega t)$ and $y = A_2\sin(\omega t + \tau)$, where $A_1 = 10$ cm, $A_2 = 5$ cm, $\omega = 2$ s^{-1}, $\tau = \pi/4$ s. Determine the speed of a point at time $t = 0.5$ s.

Answer. $v = 13.7$ m/s.

Problem D

Problem description. The wave propagates with speed v = 15 m/s. The wave period is T = 1.2 s. The amplitude of the oscillations is A = 2 cm. Determine the displacement of a point that is at a distance of x = 45 cm from the wave source at time t = 4 s, counted from the start of the oscillations

Answer. $x = -1.73\times10^{-2}$ m.

Problem E

Problem description. Two points of the medium are at a distance of Δx = 10 cm. The phase difference of the oscillations at these points is $\Delta\varphi = \pi/3$. The oscillation frequency is 25 Hz. Determine the speed of wave propagation in an elastic medium.

Answer. v = 15 m/s.

CHAPTER 5. CLASSICAL RELATIVISTIC MECHANICS

5.1. Galilean Transformation

The position of the points relative to the material body, taken as a reference frame, is described using a coordinate system. In each coordinate system, the spatial position of a point is defined by three numbers called coordinates. Formulas that relate these numbers in one coordinate system to the corresponding numbers in another coordinate system are called a ***coordinate transformation*** (or simply a transformation).

The simplest motion of a rigid body is translational uniform rectilinear motion. A similar statement can be applied to the relative motion of reference systems. All mechanical phenomena occur exactly the same in all reference frames moving uniformly and rectilinearly with respect to the system of fixed stars. In this case, it is assumed that the gravitational fields are negligible. Such reference frames are called inertial, since Newton’s law of inertia operates in them. ***Newton’s law of inertia*** can be formulated as follows: a body far enough away from other bodies moves with respect to coordinate systems in a straight-line and uniform manner.

The statement, first made by Galileo, that mechanical phenomena occur in exactly the same way in all inertial reference frames, is called the ***Galilean principle of relativity***. Galileo's principle of relativity is a postulate that goes beyond experimental verification.

Let's consider two coordinate systems: moving and motionless. Suppose, for simplicity, that the possible movement of reference systems can occur only along the abscissa (x axis). A moving coordinate system at each moment in time occupies a certain position relative to a fixed coordinate system. In the future, we will denote all characteristics of fixed coordinate system K by symbols without a prime (for example, x, y, z, t), and we will denote all characteristics of moving coordinate system K' by primed symbols (e.g. x', y', z', t'). Suppose that at time $t = 0$, the beginnings of the moving and fixed coordinate systems coincide. At time t, the origin of the moving coordinate system is at point $x = vt$ of the fixed system. It is assumed that the time flows in both coordinate systems equally $t = t'$. ***Galileo's transformation*** in this case have the form

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t. \tag{5.1.1}$$

Obviously, as a fixed coordinate system one could take a system with primed symbols. The coordinate system K moves relative to K' system at a speed of v in the direction of negative values x', i.e. with a negative speed. In this case, the Galilean transformation will have the form

$$x = x' + vt', \quad y = y', \quad z = z', \quad t = t'. \tag{5.1.2}$$

Various physical and geometric quantities change their values during coordinate transformations. If a value does not change its numerical value during coordinate transformation, then this means that it has an objective value, independent of the choice of a particular coordinate system. Values whose numerical value does not change during coordinate transformation are called ***transformation invariants***. Transformation invariants are of primary importance in physical theory. Galileo Galilei (02.15.1564 – 8.01.1642) in his work Letter to Francesco Ingoli (1624) proclaimed the homogeneity of space (the absence of a center of the world) and the equality of inertial reference systems.

Consider the invariants of the Galileo transformation.

Assume that there is a rod in the hatched coordinate systems. The coordinates of the ends of the rod are (x_1', y_1', z_1') and (x_2', y_2', z_2'), respectively. The length of the rod in the hatched coordinate system is $L=\sqrt{(x_2'-x_1')^2+(y_2'-y_1')^2+(z_2'-z_1')^2}$. In the coordinate system K, the rod has translational motion, and all its points have a speed v. To measure the length of a moving rod, it is necessary to simultaneously measure the position of the ends of the rod. Simultaneity in this case means measuring with the same clock readings of a fixed coordinate system. Suppose that the coordinates of the ends of the rod in a fixed coordinate system at time t_0 are equal to (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively. We use for this case the formula (5.1.1)

$$\begin{aligned} x_1' &= x_1 - vt_0, & x_2' &= x_2 - vt_0, \\ y_1' &= y_1, & y_2' &= y_2, \\ z_1' &= z_1, & z_2' &= z_2, \\ t_1' &= t_0, & t_2' &= t_0. \end{aligned} \quad (5.1.3)$$

This implies

$$x_2'-x_1' = x_2 - x_1, \qquad y_2'-y_1' = y_2 - y_1, \qquad z_2'-z_1' = z_2 - z_1 \quad (5.1.4)$$

The length of the rod is

$$L=\sqrt{(x_2'-x_1')^2+(y_2'-y_1')^2+(z_2'-z_1')^2} =$$

$$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = L'. \quad (5.1.5)$$

From the formula (5.1.5) it follows that the length of the rod in both reference frames is the same. Therefore, length is an invariant of the Galileo transformation.

An analysis of equations (5.1.3) shows that events that are simultaneous in one frame of reference ($t_1' = t_0$) are simultaneous in another frame of reference ($t_2' = t_0$). The statement about the simultaneity of two events from the point of view of the Galileo transformation is absolute in nature, independent of the choice of the coordinate system. Consider the events that occurred at time points t_1' and t_2' in a moving coordinate system. The time interval between these events is

$$\Delta t = t_2 - t_1 = t_2' - t_1' = \Delta t' . \qquad (5.1.6)$$

A consequence of equation (5.1.6) is the assertion that the time interval is an invariant of Galilean transformation.

Consider the motion of a material point in a hatched coordinate system. The dependence of the coordinates of this point on time has the form

$$x' = x'(t'),\ \ y' = y'(t'),\ \ z' = z'(t'). \qquad (5.1.7)$$

The components of the velocity u of a point in a moving coordinate system are

$$u_x' = \frac{dx'}{dt'},\ \ u_y' = \frac{dy'}{dt'},\ \ u_z' = \frac{dz'}{dt'} . \qquad (5.1.8)$$

In a fixed system, the coordinates of the point change according to the law

$$x(t) = x'(t') + vt',\ \ y(t) = y'(t'),\ \ z(t) = z'(t'),\ \ t = t' . \qquad (5.1.9)$$

The components of the velocity of a point in a fixed coordinate system are

$$u_x = \frac{dx}{dt} = \frac{dx'}{dt'} + v\frac{dt'}{dt} = \frac{dx'}{dt'} + v\frac{dt'}{dt'} = u_x' + v ,$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{dt} = \frac{dy'}{dt'} = u_y' ,$$

$$u_z = \frac{dz}{dt} = \frac{dz'}{dt} = \frac{dz'}{dt'} = u_z' . \qquad (5.1.10)$$

Formulas (5.1.10) are the formulas for the addition of velocities in classical non relativistic mechanics.

We differentiate equalities (5.1.10), taking into account the fact that $dt = dt'$

$$\frac{d^2x}{dt^2} = \frac{d^2x'}{dt'^2} ,\ \frac{d^2y}{dt^2} = \frac{d^2y'}{dt'^2} ,\ \ \frac{d^2z}{dt^2} = \frac{d^2z'}{dt'^2} . \qquad (5.1.11)$$

Formulas (5.1.11) show that acceleration is invariant with respect to the Galilean transformation.

5.2. Speed of Light

The validity of Galileo's transformation can be verified by comparing the consequences of them with experiment. The most important consequence of the Galilean transformations is the formula for the addition of velocities. Verification of this formula showed its approximate nature. Deviations from the formula are especially large when the velocities of bodies approach the speed of light. Consider the features of physical processes associated with the speed of light c.

The speed of light was first measured in 1676 by Ole Christensen Rømer (25.09.1644 – 19.09.1710). Observations of the eclipses of the moons of Jupiter showed that the apparent period of their revolution decreases when the Earth in its annual motion approaches Jupiter. The removal of the Earth from Jupiter leads to an increase in the period of revolution of the moons of Jupiter. Roemer realized that this effect is associated with a finite speed of light propagation, and from the results of observations he calculated this speed. The calculation received the value of the speed of light with c = 214 300 000 m/s. It was the first reliable measurement of the speed of light with satisfactory accuracy for those times. It should be noted that it is currently believed that the speed of light in vacuum is a fundamental physical constant, by definition exactly equal to 299 792 458 m/s.

When considering the issue of the speed of light, mention should be made of a phenomenon such as ***aberration of light***. As a result of light aberration, the apparent direction of the star differs from the true direction by the angle $\beta = (\pi / 2) - \alpha$. The angle β is called the ***angle of aberration***. The following relation holds for the aberration angle

$$tg\beta = \frac{v_{\perp}}{c}, \tag{5.2.1}$$

where $v_{\perp}$ is a component of the speed of the Earth, perpendicular to the direction of the star. The speed of light can be calculated from known values α and $v_{\perp}$.The results of calculations lead to approximately the same values that were obtained by Roemer.

After measuring the speed of light, the question arises of what parameters this speed depends on. The answer to this question within the framework of the ideas that existed at that time is due to a look at the nature of light.

Under the assumption that light is a wave-like motion of a homogeneous medium, the speed of light relative to this medium is a certain constant value determined by the properties of this medium. In this case, the speed of light relative to the source and observer is a variable that depends on the speed of the source or observer relative to this medium. In this case, the speed of light is found by the rule of addition of velocities of classical non relativistic mechanics.

Assuming that light is a stream of fast corpuscles flying from a source, it can be argued that the speed of these corpuscles relative to the source has some constant value. The speed of light relative to the observer is added to the speed of the observer according to the rule of addition of velocities of classical non relativistic mechanics.

Augustin-Jean Fresnel (10.05.1788 – 14.07.1827) in 1818 developed the theory of diffraction based on the wave theory. One of the results of this was the replacement of the corpuscular theory of light by wave theory. The point of view on light as a wave process in some medium has become generally accepted. This medium, which fills the entire Universe, is called the ether. The speed of light relative to the ether was considered a constant value, depending on the properties of the ether. The speed of light relative to the world ether was called absolute. The absolute speed of a given material body does not depend on the movement of other bodies.

Since the speed of light relative to the ether is constant, the speed of light relative to material bodies moving in the ether is variable and depends on the speed of these bodies relative to the ether.

Albert Abraham Michelson (19.12.1852 – 9.05.1931) and Edward Williams Morley (29.01.1838 – 24.02.1923) in 1887 attempted to measure the speed of light in this way (***Michelson–Morley experiment***). The idea of the experience is to compare the passage of light in two ways. One of the paths coincides with the direction of movement of the body in the ether, and the second path is perpendicular to this direction. A beam of monochromatic light is divided (transmitted beam and reflected beam) into two coherent rays on a translucent plate. Two coherent rays after passing through two different paths are found in the interferometer. If these paths are traversed by rays in the same time, then at the meeting point between the oscillations of the rays there will be the same phase difference as at the separation point. Observing the interference of the rays, we can draw a conclusion about the phase difference of the coherent waves that came into the interferometer and then calculate the delay time of one wave relative to the other.

Michelson and Morley measured a lag time of $\Delta t^{(1)}$ and $\Delta t^{(2)}$ for the position of the instrument before and after turning through an angle of 90°, respectively. The analytical expressions for the delay times were

$$\Delta t^{(1)} = \frac{2}{c}\left[\frac{v^2}{c^2}\left(L_1 - \frac{L_2}{2}\right) + \left(L_1 - L_2\right)\right],$$

$$\Delta t^{(2)} = \frac{2}{c}\left[\frac{v^2}{c^2}\left(L_2 - \frac{L_1}{2}\right) + \left(L_2 - L_1\right)\right], \qquad (5.2.2)$$

where L_1, L_2 are the lengths of the arms of the device along which the movement of light rays occurred; v is the orbital velocity of the Earth.

The linear velocity of points on the surface when the Earth rotates around its own axis is approximately 60 times less than the orbital velocity and was not taken into account in the calculations.

Thus, a complete change in time, which corresponds to the difference in the path of the rays when the device is rotated through an angle equal to 90°, is

$$\Delta t = \Delta t^{(1)} + \Delta t^{(2)} = \frac{(L_1 + L_2)}{c} \cdot \frac{v^2}{c^2}. \qquad (5.2.3)$$

To increase the effective distance, Michelson and Morley used multiple reflection of light rays from the mirrors and achieved an increase in $(L_1 + L_2)$ to about 11 m. The wavelengths of visible light are within $(0.4 \div 0.75)\cdot 10^{-6}$ m. The delay value calculated by the formula (5.2.3), and expressed as a displacement along the wavelength, is

$$\Delta\lambda = \Delta t \cdot c = \frac{v^2}{c^2}(L_1 + L_2) \approx (L_1 + L_2)\cdot 10^{-8}, \qquad (5.2.4)$$

where it is taken into account that the relation $v^2/c^2 \approx 10^{-8}$ is valid for the orbital velocity of the Earth.

Therefore, for a wavelength of $\lambda = 5\cdot 10^{-7}$ m, the relative displacement of interference fringes is equal $\Delta\lambda/\lambda = (L_1 + L_2)\cdot 2\cdot 10^{-2}$. Considering that $L_1 + L_2 \approx 11$ m, we obtain for the displacement $\Delta\lambda/\lambda \approx 1/5$, which is much larger than those quantities that can be easily observed. In fact, in the experiment of Michelson and Morley it was possible to observe displacements that correspond to the velocities of the device relative to the ether of only 3 km/s. However, no effect was found. At present, it can be considered proven that the speed of the ether wind, in any case, is less than 10 m/s.

As part of the concept of broadcasting, two ways out of this predicament were proposed.

1. It could be assumed that the ether near massive bodies moves with these bodies. Then, naturally, no etheric wind should be observed near these bodies.
2. It could be assumed that the sizes of material bodies moving in the ether do not remain constant, but change in such a way that the expected difference in the path of light rays according to formula (5.2.3) does not work.

The first assumption of the movement of ether near massive bodies cannot be reconciled with the phenomenon of light aberration. The second assumption, supplemented by the statement of the constancy of the speed of light relative to bodies moving in the ether, is logically truthful. However, strictly speaking, the results of the experiments of Michelson and Morley indicate only the constancy of the average speed of light in various directions in an inertial reference frame.

To explain the results of the experiments of Michelson and Morley, a ballistic hypothesis was put forward, according to which light is a stream of material corpuscles. The speed of these corpuscles relative to the source is constant and is added to the speed of the source according to the parallelogram rule. However, astronomical observations of the motion of binary stars disproved the ballistic hypothesis.

The failure of the ballistic hypothesis forces us to admit that the speed of light does not depend on the speed of the light source. The results of the Michelson and Morley experiment show that the speed of light does not depend on the speed of the observer. Therefore, it is concluded that the speed of light is a constant value that does not depend on either the speed of the source or the speed of the observer.

Armand Hippolyte Louis Fizeau (23.09.1819 – 18.09.1896) in 1860 made an experiment (***Fizeau experiment***) on measuring the speed of light in a moving medium (in water). Fizeau discovered that the speed of light in a liquid and the speed of the liquid itself do not add up according to the formula for adding the velocities of classical non relativistic mechanics.

The constancy of the speed of light is in deep contradiction with the usual notions of everyday experience and with the formula for adding the velocities of classical non relativistic mechanics, which are the result of Galileo's transformations. Thus, we can say that the Galilean transformations contradict the experimental fact of the constancy of the speed of light. However, this contradiction becomes noticeable only for sufficiently high speeds. At speeds much lower than the speed of light in vacuum, deviations from the formula for the addition of speeds in classical non relativistic mechanics are extremely small.

The statement about the constancy of the speed of light in a vacuum, i.e. the independence of the speed of light from the speed of the source and the speed of the observer is a natural conclusion from many experimental facts. The main confirmation of the hypothesis of the constancy of the speed of light is the agreement with experiment of all those conclusions that follow from this hypothesis. All modern physics of high speeds and high energies is based on the postulate of the constancy of the speed of light.

Nevertheless, in its absolute form, the statement about the constancy of the speed of light is a postulate, i.e. an assumption that goes beyond direct experimental verification. This is due to the finite accuracy of all experimental tests, as was explained earlier in connection with the postulate nature of the principle of relativity.

5.3. Lorentz Transformation

Galilean transformation for sufficiently high speeds lead to conclusions that contradict experiments. Therefore, it is necessary to find other transformations that correctly describe the experimental facts and, in particular, lead to a constant speed of light. These transformations are called Lorentz transformation. Lorentz transformations can be derived on the basis of two principles, the justification of which will be given in the future:

1) the principle of relativity;
2) the principle of constancy of the speed of light.

Both of these principles, although confirmed by numerous experiments, have the character of postulates and are therefore sometimes called the postulate of relativity and the postulate of the constancy of the speed of light.

Consider the features of coordinate transformation at high speeds. Since the speeds do not add up according to the classical formula, it can be expected that the time of one coordinate system is not expressed only through the time of another coordinate system, but also depends on the coordinates. Therefore, in the general case, the transformations have the following form

$$x' = \theta_1(x, y, z, t), \qquad y' = \theta_2(x, y, z, t)$$
$$z' = \theta_3(x, y, z, t), \qquad t' = \theta_4(x, y, z, t), \tag{5.3.1}$$

where θ_1, θ_2, θ_3, θ_4 are some functions whose explicit form is to be clarified.

The general form of functions θ_1, θ_2, θ_3, θ_4 is determined by the properties of space and time. Consideration of geometric relationships in the selected reference frame was carried out under the assumption that each point is no different from any other point. The homogeneity and isotropy of space is its main properties in inertial coordinate systems.

Time also has the property of homogeneity. Consider this property. Let some physical process begin at some point in time. At subsequent times, this process will evolve in some way. Suppose that the same situation arises at any other moment in time. If the second situation in subsequent moments of time develops in the same way as the first situation, then they say that time is uniform. ***Homogeneity of time*** is the same development of a given physical situation, regardless of at what point in time this situation began to develop.

It follows from the homogeneity of space and time that the transformations (5.3.1) must be linear. We calculate the total infinitesimal change of coordinates dx' from (5.3.1) by the formula of the full differential

$$dx' = \frac{\partial \theta_1}{\partial x} dx + \frac{\partial \theta_1}{\partial y} dy + \frac{\partial \theta_1}{\partial z} dz + \frac{\partial \theta_1}{\partial t} dt. \tag{5.3.2}$$

Due to the homogeneity of space and time, relations of the type (5.3.2) should be the same for all points of space and for any time moments. This means that the values of $\frac{\partial \theta_1}{\partial x}$, $\frac{\partial \theta_1}{\partial y}$, $\frac{\partial \theta_1}{\partial z}$, $\frac{\partial \theta_1}{\partial t}$ should not depend on coordinates and time, i.e. are constant values. Therefore, function θ_1 has the following form

$$\theta_1(x, y, z, t) = A_1 x + A_2 y + A_3 z + A_4 t + A_5, \tag{5.3.3}$$

where A_1, A_2, A_3, A_4 are constant values.

Thus, function θ_1 is a linear function of its arguments. It is proved in a similar way that, due to the homogeneity of space and time, functions θ_2, θ_3, θ_4 in transformations (5.3.1) will also be linear functions of arguments x, y, z, t

$$\theta_2(x, y, z, t) = B_1 x + B_2 y + B_3 z + B_4 t + B_5$$
$$\theta_3(x, y, z, t) = C_1 x + C_2 y + C_3 z + C_4 t + C_5$$
$$\theta_4(x, y, z, t) = D_1 x + D_2 y + D_3 z + D_4 t + D_5. \tag{5.3.4}$$

Suppose that the direction of the abscissa and the axis of the moving (system K') and fixed (system K) reference systems coincide. In addition, suppose for simplicity that these frames can only move along the x-axis and x'-axis.

Consider in this case, the transformation for the coordinates y and z. The origin point in each coordinate system is given by the equalities $x = y = z = 0$, $x' = y' = z' = 0$. We assume that at time $t = 0$, the origin of the coordinate systems coincide. Then the terms B_5 and C_5 in the linear transformation (5.3.4) must be equal to zero. The transformations for y and z will be as follows

$$y' = B_1 x + B_2 y + B_3 z + B_4 t,$$

$$z' = C_1 x + C_2 y + C_3 z + C_4 t. \tag{5.3.5}$$

It is assumed that the directions of the axes y and y' coincide. The same can be said for axes z and z', as well as x and x'. Therefore, from condition $y = 0$ equality $y' = 0$ always follows, and from condition $z = 0$ equality $z' = 0$ follows.

These relations lead to the equalities

$$0 = B_1 x + B_3 z + B_4 t,$$

$$0 = C_1 x + C_2 y + C_4 t. \tag{5.3.6}$$

Equalities (5.3.6) are possible only under condition $B_1 = B_3 = B_4 = 0$ and $C_1 = C_2 = C_4 = 0$. In this case, we can write

$$y' = B_2 y, \qquad z' = C_3 z. \tag{5.3.7}$$

We take into account that the axes y and z are equal in rights when the reference systems move along the axes x and x'. In addition, the moving coordinate system (system K') and the fixed coordinate system (system K) are also equally valid. As a consequence, the relations $B_2 = C_3 = 1$ are valid. Consequently, the transformations for coordinates y and z have the form

$$y' = y, \quad z' = z. \tag{5.3.8}$$

Now consider the transformations for coordinate x and time t. Since variables y and z are converted separately, variables x and t can only be connected by linear transformations to each other. The origin of a moving coordinate system in a fixed system has a coordinate $x = vt$ (i.e. $x - vt = 0$), and in a moving system, this same point has a coordinate $x' = 0$. Therefore, in view of the linearity of the transformation, there should be

$$x' = \alpha(x - vt), \tag{5.3.9}$$

where α is a coefficient of proportionality.

Similar considerations can be made if the moving and fixed systems are interchanged. Then the origin of the coordinates of the unprimed system has a coordinate $x = -vt$ (i.e. $x + vt = 0$), since the system K moves in the direction of negative axis values relative to the system K'. The origin of the coordinates of the system K relative to the system K' is characterized by equality $x = 0$. As a result, we can write the following transformation

$$x = \alpha'(x' + vt'). \tag{5.3.10}$$

According to the principle of relativity, system K and system K' are equivalent, hence $\alpha = \alpha'$.

We use the postulate of the constancy of the speed of light. Suppose that at the point in time when the origin coincides and when the clock at the origin shows time $t = t' = 0$, a light signal is emitted from them. The propagation of light in the hatched and unshaded frames of reference is described by equalities

$$x' = ct', \qquad x = ct, \tag{5.3.11}$$

in which it is taken into account that in both reference frames the speed of light has the same value. In this case, we rewrite the transformations (5.3.9) and (5.3.10)

$$ct' = \alpha t(c - v), \qquad ct = \alpha t'(c + v). \tag{5.3.12}$$

Multiplying the right and left sides of equations (5.3.2) allows us to find an explicit expression for the quantity α

$$\alpha = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{5.3.13}$$

The transformation of equations (5.3.9) and (5.3.10) allows us to obtain the relationship between the quantities t and t'

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}. \tag{5.3.14}$$

Transformations (5.3.8), (5.3.9), (5.3.13) and (5.3.14) relate the coordinates and time of reference systems moving relative to each other with speed v

$$x' = \frac{x - vt}{\sqrt{1 - \upsilon^2/c^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}. \tag{5.3.15}$$

Inverse transformations have the same form, but only the sign of speed changes

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}, \quad y = y', \quad z = z', \quad t = \frac{t' + \left(v/c^2\right)x'}{\sqrt{1 - v^2/c^2}}. \qquad (5.3.16)$$

Formulas (5.3.15) and (5.3.16) are called ***Lorentz transformation***. The transformation is named after Hendrik Antoon Lorentz (18.07.1853 – 4.12.1928).

We consider the limiting case of velocities much lower than the speed of light in vacuum. In this case, values of the order of v/c can be neglected in the Lorentz transformations, i.e. quantities v^2/c^2 and v/c^2 can be equated to zero. Then the Lorentz transformation are reduced to the Galilean transformation. At low speeds, the difference between the Galileo and Lorentz transformations is insignificant, and therefore the inaccuracy of the Galileo transformation has long gone unnoticed.

5.4. Consequences of Lorentz Transformation

The difference in the Galileo and Lorentz transformations requires careful consideration of the consequences of the Lorentz transformation. First of all, we consider the concept of simultaneity of events. Two events that occurred at different points x_1 and x_2 of the reference system are called simultaneous events if they occur at the same time t_0 by the clock of this coordinate system.

These events would occur at points x_1' and x_2' of the moving coordinate system at points in time t_1' and t_2'. The relationship between the kinematic quantities in the system K and the system K' is given by the Lorentz transformation

$$x_1' = \frac{x_1 - vt_0}{\sqrt{1 - v^2/c^2}}, \qquad x_2' = \frac{x_2 - vt_0}{\sqrt{1 - v^2/c^2}},$$

$$t_1' = \frac{t_0 - \left(v/c^2\right)x_1}{\sqrt{1 - v^2/c^2}}, \qquad t_2' = \frac{t_0 - \left(v/c^2\right)x_2}{\sqrt{1 - v^2/c^2}}. \qquad (5.4.1)$$

Since events occur at points on the x axis , the coordinates y and z in both systems are zero. An analysis of equations (5.4.1) shows that in a moving coordinate system the events in question do not occur simultaneously ($t_1' \neq t_2'$).Events are separated by a time interval

$$\Delta t' = t_2' - t_1' = \frac{\left(v/c^2\right)\cdot\left(x_1 - x_2\right)}{\sqrt{1 - v^2/c^2}}. \qquad (5.4.2)$$

Thus, events that are simultaneous in one coordinate system are not simultaneous in another system. Consequently, the concept of simultaneity does not have an absolute meaning.

The length of the bodies and the time interval are invariants in the Galilean transformations and are not invariants in the Lorentz transformations. Consider the invariant in the Lorentz transforms, which is important. Suppose that two events occurred at time points t_1 and t_2 at points with coordinates x_1, y_1, z_1 and x_2, y_2, z_2, respectively. The ***interval*** between these events is called the quantity s

$$s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2}\,. \tag{5.4.3}$$

This value has the same value in all reference frames, i.e. is an invariant of Lorentz transformation. The invariance of the square of the interval for infinitely close points can be expressed as follows

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 = \text{inv}\,. \tag{5.4.4}$$

We denote the distance between the points at which the events occurred by the symbol l, and the time interval between them by the symbol t. Suppose that in some reference system events cannot have a causal relationship. In this case, the following relations are valid $l > ct$ and $s^2 > 0$. The invariance of the interval means that in all other reference systems these events also do not have a causal relationship.

The interval is called ***space-like interval*** if the relation $s^2 > 0$ is true, and ***time-like interval*** in the case when $s^2 < 0$. If the interval is space-like, then we can choose a frame of reference in which two events occur simultaneously at different spatial points ($s^2 = l^2 > 0$, $t = 0$). There is no such frame of reference in which these events take place at the same point.

The property of an interval to be space-like or time-like does not depend on the coordinate system, but is an invariant property of the events themselves.

Consider the concept of the length of a moving body. We choose the rod as such a body. The length of a moving rod is the distance between the points of the resting reference system, with which the beginning and end of the moving rod at some point in time according to the clock of the resting reference system coincide.

Suppose that a rod with a length of L is resting in reference frame K'. The rod is located along the axis x'. The coordinates of the ends of such a rod are denoted by x_1' and x_2'. Consequently, the ratio $L = x_2' - x_1'$ is valid for the reference frame K'. Since the rod rests in reference frame K', its length is indicated by a symbol without a stroke. Using formulas (5.4.1), we can write the following expression for the length of the rod

$$L = x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} = \frac{L'}{\sqrt{1 - v^2/c^2}}\,, \tag{5.4.5}$$

where $L' = x_2 - x_1$ is the length of the moving rod.

We rewrite equality (5.4.5)

$$L' = L\sqrt{1 - v^2 / c^2} \,. \tag{5.4.5}$$

An analysis of equation (5.4.5) indicates that the length of the moving rod located in the direction of motion is less than the length of the resting one. If the rod is placed perpendicular to the direction of motion, for example, along y' axis or z' axis, then, as can be seen from formulas (5.3.15), its length does not change

Typically, the speed of bodies is much less than the speed of light in a vacuum, i.e. $v/c < 1$. Therefore, up to a first-order value in v^2/c^2, formula (5.4.5) for the case of low velocities can be represented as

$$L' \approx L\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right). \tag{5.4.6}$$

Therefore, the relative change in the length of the rod is

$$\Delta L / L = \left(L' - L\right) / L = -v^2 / \left(2c^2\right). \tag{5.4.7}$$

The reduction in the diameter of the Earth during orbital motion is only 6 cm. However, at high speeds, the relativistic contraction is significant. For example, at a body speed of 0.85c, its length will be reduced by half.

Consider the time interval in the Lorentz transforms. Suppose that at a point x_0' in a moving frame of reference two events occur consecutively at time t_1' and t_2'. These events occur at time moments t_1 and t_2 at different points of the fixed (K system) frame of reference. The time intervals in the moving reference frame and in the fixed reference frame are equal $\Delta t' = t_2' - t_1'$ and $\Delta t = t_2 - t_1$, respectively. Using formulas (5.3.15) and (5.3.16), we can write

$$\Delta t = \frac{t_2' - t_1'}{\sqrt{1 - v^2 / c^2}} = \frac{\Delta t'}{\sqrt{1 - v^2 / c^2}} \,. \tag{5.4.8}$$

Thus, the time interval $\Delta t'$ between events measured by the moving clock,

$$\Delta t' = \Delta t\sqrt{1 - v^2 / c^2} \,. \tag{5.4.9}$$

is less than the time interval Δt between the same events measured by the resting clock. This means that the pace of the moving clock is slowed down relative to the fixed clock.

The time, which is measured by the clock associated with a moving point, is called the proper time of this point. We rewrite equation (5.4.9) for the differentials

$$dt' = d\tau = dt\sqrt{1 - v^2 / c^2}\,, \tag{5.4.10}$$

where $d\tau$ is the differential of the proper time of a moving point; dt is the time differential of that inertial coordinate system in which the point has a current speed v.

We use the fact that the square of the differential of the distance between two points is equal $dr^2 = dx^2 + dy^2 + dz^2$, and we rewrite the formula (5.4.4) (5.4.4)

$$\frac{ds}{i} = cdt\sqrt{1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2} = cdt\sqrt{1 - \frac{v^2}{c^2}}\,, \tag{5.4.11}$$

where $i = \sqrt{-1}$.

A comparison of formulas (5.4.10) and (5.4.11) shows that the relationship between the quantities $d\tau$ and ds has the form

$$d\tau = \frac{ds}{ic}\,, \tag{5.4.12}$$

therefore, proper time is also an invariant of Lorentz transformation.

Let us consider how the velocities add up taking into account the Lorentz transformation. The position of the material point will be determined by functions $x' = x'(t),\ y' = y'(t),\ z' = z'(t)$ in the moving frame of reference and functions $x = x(t),\ y = y(t),\ z = z(t)$ in the fixed frame of reference. Speed components u are equal

$$u'_x = dx'/dt',\ u'_y = dy'/dt',\ u'_z = dz'/dt',$$

$$u_x = dx/dt,\ u_y = dy/dt,\ u_z = dz/dt. \tag{5.4.13}$$

Using the Lorentz transformations, we obtain

$$dx = \frac{dx' + vdt'}{\sqrt{1 - v^2/c^2}}, \quad dy = dy', \quad dz = dz',$$

$$dt = \frac{dt' + \left(v/c^2\right)dx'}{\sqrt{1 - v^2/c^2}} = dt'\frac{1 + vu'_x/c^2}{\sqrt{1 - v^2/c^2}}. \tag{5.4.14}$$

Performing a joint transformation of formulas (5.4.13) and (5.4.14), we get

$$u_x = \frac{u'_x + v}{1 + \upsilon u'_x/c^2}, \quad u_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + vu'_x/c^2}, \ u_z = \frac{u'_z\sqrt{1 - v^2/c^2}}{1 + vu'_x/c^2}. \tag{5.4.15}$$

Equations (5.4.15) are relativistic formulas for velocity addition.

An analysis of formula (5.4.15) shows that the addition of speeds never leads to speeds greater than the speed of light. Consider, for example, the case when $u'_y = u'_z = 0$, $u'_x = c$. Then from formula (5.4.15) we find

$$u_x = \frac{c+v}{1+cv/c^2} = c, \qquad u_y = 0, \qquad u_z = 0. \qquad (5.4.16)$$

5.5. Relativistic Dynamics

The momentum in relativistic dynamics is the quantity $\vec{p}$, which is transformed in accordance with the Lorentz transformation

$$\vec{p} = m\vec{v}\,, \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}. \qquad (5.5.1)$$

Value

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}} \qquad (5.5.2)$$

can be considered as the relativistic mass of a material point (particle) moving with speed relative to a motionless reference frame. Relativistic mass is a function of speed $m = m(v)$. Therefore, the quantity $m_0 = m(0)$ is the mass of the particle at rest, called the ***rest mass***. Consider examples of changes in mass $\Delta m = m - m_0$ for different systems that move at different speeds. The spacecraft has a mass of $m_0 = 5 \cdot 10^3$ kg and a speed of $v = 8 \cdot 10^3$ m/s. In this case, the change in mass of the spacecraft is $\Delta m = 3.5 \cdot 10^{-6}$ kg, those is 0.00000007 % of the rest mass of the spacecraft. The electron has mass $m_0 = 9.1 \cdot 10^{-31}$ kg and speed $v = 0.5c = 1.5 \cdot 10^8$ m/s. In this case, the change in electron mass is $\Delta m = 1.4 \cdot 10^{-31}$ kg, i.e. makes up 15.4% of the rest mass of the electron.

In relativistic mechanics, the law of conservation of momentum of a closed system of particles in all reference frames is fulfilled in accordance with the idea of the homogeneity of space.

The mathematical expression of the momentum conservation law of an isolated system consisting, for example, of two interacting relativistic particles with rest masses m_{01} and m_{02} only formally coincides with the mathematical form of the momentum conservation law in classical mechanics

$$\frac{m_{01}\vec{v}_1}{\sqrt{1-v_1^2/c^2}} + \frac{m_{02}\vec{v}_2}{\sqrt{1-v_2^2/c^2}} = \mathrm{const}\,, \qquad (5.5.3)$$

where $\vec{v}_1$ and $\vec{v}_2$ these are the speeds of the first and second particles.

According to the laws of relativistic mechanics, the work A_{12} of the force $\vec{F}$ acting on the particle when moving it from position 1 to position 2 along a curved path L is determined by the expression

$$A_{12} = \int_L dA = \int_{(1)}^{(2)} \vec{F} d\vec{r} = \int \frac{d\vec{p}}{dt} d\vec{r} = \int \vec{\upsilon} d\vec{p} = c^2 \int_{m_1}^{m_2} dm, \qquad (5.5.4)$$

where $\vec{F}$, $\vec{p}$, and $\vec{v}$ are invariant quantities of force, momentum and speed; m_1 and m_2 are masses of particles in the initial and final positions.

An analysis of equation (5.5.4) shows that in relativistic mechanics the work is determined by the increment of the mass of particles

$$dA = c^2 dm,$$

$$A_{12} = c^2 (m_2 - m_1) = c^2 \Delta m. \qquad (5.5.5)$$

The ***total energy*** or ***relativistic energy*** of a free particle is the quantity

$$W = mc^2. \qquad (5.5.6)$$

The quantity

$$W_0 = m_0 c^2, \qquad (5.5.7)$$

that is determined for a free particle is called the ***rest energy*** of the particle.

Kinetic energy W_k is part of the relativistic energy W due to particle motion. By definition, the kinetic energy of a particle is

$$W_k = W - W_0 = (m - m_0)c^2 = m_0 c^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right). \qquad (5.5.8)$$

By analogy with classical mechanics, work can be written as an increment of the kinetic energy of a particle

$$A_{12} = W_{k,2} - W_{k,1}, \qquad (5.5.9)$$

where $W_{k,1} = (m_1 - m_0)\cdot c^2$ and $W_{k,2} = (m_2 - m_0)\cdot c^2$ is the kinetic energy of a particle in positions (1) and (2).

The consequence of formulas (5.5.1) and (5.5.2) is the expression for the total energy

$$W = p^2 c^2 + m_0^2 c^4. \qquad (5.5.10)$$

The expression (5.5.10) is invariant under the Lorentz transformations. Formula (5.5.10) is valid both for elementary (structureless) particles and for systems consisting of a large number of particles. Momentum $\vec{p}$, energy W, and velocity $\vec{v}$ for a free particle are related by

$$\vec{p} = \vec{v}\frac{W}{c^2}. \tag{5.5.11}$$

Consider the case when a particle has potential energy W_p. Then the total energy of the particle is

$$W = mc^2 + W_p = W_k + m_0c^2 + W_p. \tag{5.5.12}$$

The law of conservation of the total energy of a closed system of particles is performed in relativistic mechanics. For the case of a system of two particles, we can write

$$W_1 + W_2 = const, \tag{5.5.13}$$

where W_1 and W_2 are the relativistic energies of the first and second particles.

The total energy W and speed v of the particle with a rest mass equal to zero $m_0 = 0$ is

$$E = pc, \qquad v = c. \tag{5.5.14}$$

Consequently, a particle with a rest mass equal to zero always moves at the speed of light, and its momentum is nonzero and equal

$$p = E/c. \tag{5.5.15}$$

The components of the particle momentum and the particle energy in a fixed (system K) reference frame and a moving (system K') frame of reference are related by the following relations

$$p_x = \frac{p'_x + uW'/c^2}{\sqrt{1-u^2/c^2}}, \qquad p'_x = \frac{p_x - uW/c^2}{\sqrt{1-u^2/c^2}},$$

$$p_y = p'_y, \;\; p'_y = p_y,$$

$$p_z = p'_z, \;\; p'_z = p_z,$$

$$W = \frac{W' + p'_x u}{\sqrt{1-u^2/c^2}}, \qquad W' = \frac{W - p_x u}{\sqrt{1-u^2/c^2}}, \tag{5.5.16}$$

where u is the speed of the moving reference system relative to the fixed reference system. The speed u is directed along the x axis.

The basic equation of relativistic dynamics, generalizing Newton’s second law to the case of the motion of a material point with a speed comparable to the speed of light in a vacuum, has the form formally coinciding with a similar equation for classical mechanics

$$\frac{d\vec{p}}{dt} = \vec{F}, \tag{5.5.17}$$

where $\vec{F}$ is the resultant of all forces applied to the material point; $\vec{p}$ is a relativistic momentum defined by equation (5.5.1).

In explicit form, the basic equation of relativistic dynamics can be written as follows

$$m_0 \frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2/c^2}}\right) = \vec{F},$$

$$m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dv}\frac{dv}{dt} = \vec{F},$$

$$\frac{m_0\vec{a}}{\left(1-v^2/c^2\right)^{3/2}} = \vec{F}, \tag{5.5.18}$$

where $\vec{a} = d\vec{\upsilon}/dt$ is an acceleration of a material point.

Test questions

1. State the Galilean principle of relativity.
2. Write down the Galilean transformation in the Cartesian coordinate system.
3. Is speed an invariant of Galilean transformations?
4. Give the formulas for the addition of velocities in classical non relativistic mechanics.
5. Describe the phenomenon of light aberration.
6. Specify the limits of angle of aberration change.
7. Describe the methodology and results of the experiments of Michelson and Morley.
8. Describe the methodology and results of the Fizeau experiments.
9. List the principles that underlie the Lorentz transformations.
10. Write down the Lorentz transformation for the Cartesian coordinates.
11. Show that in the limiting case, the Lorentz transformation are reduced to Galileo transformation.
12. Define simultaneous events.
13. Write down the formula for the time interval that follows from the Lorentz transformation.

14. Can two events occur simultaneously at different spatial points if the interval between events is time-like?
15. Show, using the formula for the length of the rod, which follows from the Lorentz transformation, that in the limiting case $\upsilon << c$, the length of the rod is the same for all inertial reference systems.
16. List the invariants of the Lorentz transformation.
17. Write down the relativistic velocity addition formula.
18. Calculate the change in mass of an electron if it has changed its speed from 0 to 1.2×10^8 m/s.
19. Give the formula for the rest energy of a free particle.
20. Write down the relations that connect the components of the momentum and the energy of the particle in a fixed and moving coordinate system.

Problem-solving examples

Problem 5.1

Problem description. The kinetic energy of an electron is W_k = 1 MeV. Determine the speed of an electron.

Known quantities: W_k = 1 MeV.

Quantities to be calculated: v.

Problem solution. The relativistic formula for kinetic energy W_k has the form

$$W_k = W_0\left(\frac{1}{\sqrt{1-\beta^2}}-1\right), \tag{P.5.1.1}$$

where W_0 is the rest energy of an electron; $\beta = \frac{v}{c}$ is a relativistic factor; c is the speed of light in a vacuum.

We transform the formula (5.1.1) with respect to the quantity β

$$\beta = \frac{\sqrt{(2W_0 + W_k)W_k}}{W_0 + W_k}. \tag{P.5.1.2}$$

Calculations by the formula (P.5.1.2) can be made in any energy units, since the unit names on the right side of the formula will be reduced, and as a result of the calculation, a dimensionless quantity will be obtained. The rest energy of an electron is W_0 = 0.511 MeV.

Substitute the numerical data in the formula (P.5.1.2)

$$\beta = \frac{\sqrt{(2\times 0.511+1)\times 1}}{0.511+1} = 0.941\,.$$

The electron velocity can be expressed as follows

$$v = \beta c. \qquad \text{(P.5.1.3)}$$

Substitute the numerical data in the formula (P.5.1.3)

$$v = 0.941\times 3\times 10^8 = 2.82\times 10^8 \text{ m/s}.$$

Answer. Electron speed is v = 2.82×10^8 m/s.

Problem 5.2

Problem description. Calculate the relativistic momentum p and kinetic energy W_k of an electron moving at a speed of v = 0.9c (c is the speed of light in a vacuum).

Known quantities: v = 0.9c.

Quantities to be calculated: p, W_k.

Problem solution. The relativistic momentum of an electron is

$$p = m_0 c\frac{\beta}{\sqrt{1-\beta^2}}\,, \qquad \text{(P.5.2.1)}$$

where m_0 is the rest mass of the electron; β is a relativistic factor.

Relativistic factor is determined by the ratio $\beta = v/c$. By the condition of the problem, the electron velocity is v = 0.9c. Hence β = 0.9c/c = 0.9. The rest mass of an electron is m_0 = 9.1×10^{-31} kg. Substituting these values in the formula (P.5.2.1), we obtain

$$p = 9.1\times 10^{-31}\times 3\times 10^8\frac{0.9}{\sqrt{1-0.9^2}} = 5.64\times 10^{-22}\ \text{kg}\cdot\text{m/s}.$$

Kinetic energy W_k in relativistic mechanics is defined as the difference between the total energy W and the rest energy W_0

$$W_k = W - W_0\,. \qquad \text{(P.5.2.2)}$$

The total energy and the rest energy are determined using the following relations

$$W = mc^2, \tag{P.5.2.3}$$

$$W_0 = m_0 c^2. \tag{P.5.2.4}$$

In this case, taking into account the dependence of mass on speed, we can write

$$W_k = \frac{m_0 c^2}{\sqrt{1-\beta^2}} - m_0 c^2 \tag{P.5.2.5}$$

or

$$W_k = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right). \tag{P.5.2.6}$$

We substitute the numerical values in the formula (P.5.2.6)

$$W_k = 9.1 \times 10^{-31} \times \left(3 \times 10^8\right)^2 \times \left(\frac{1}{\sqrt{1-0.9^2}} - 1 \right) = 1.06 \times 10^{-13} \text{ J}.$$

<u>Answer</u>. The momentum of an electron is p = 5.64×10^{-22} kg·m/s . The kinetic energy of an electron is W_k = 1.06×10^{-13} J.

Problem 5.3

<u>Problem description</u>. A relativistic particle with kinetic energy of $W_k = m_0 c^2$ (m_0 is the rest mass of the particle; c is the speed of light in vacuum) experiences an inelastic collision with the same resting particle in the laboratory reference frame. As a result of the collision, a composite particle is formed. Define the following quantities: 1) relativistic mass m of a moving particle; 2) relativistic mass m' and rest mass m_0' of the composite particle; 3) kinetic energy W_k' of a composite particle.

<u>Known quantities</u>: $W_k = m_0 c^2$.

<u>Quantities to be calculated</u>: m, m', m_0', W_k'.

<u>Problem solution</u>. The total relativistic mass of particles remains constant

$$m + m_0 = m', \tag{P.5.3.1}$$

where $m + m_0$ is the total relativistic mass of particles before the collision; m is the mass of a moving particle; m' is the relativistic mass of a composite particle.

Since $W_k = m_0c^2 = W_0$, then relativistic energy of moving particle $W = W_k + W_0 = 2m_0c^2$, therefore $m = 2m_0$, and we can write

$$m' = m + m_0 = 3m_0 . \tag{P.5.3.2}$$

The rest mass m_0' of a composite particle can be found from the relation

$$m' = \frac{m_0'}{\sqrt{1-\left(\frac{v'}{c}\right)^2}} , \tag{P.5.3.3}$$

The velocity v' of a composite particle can be determined from the law of conservation of momentum $p = p'$ (p is the momentum of a relativistic particle before a collision; p' is the momentum of a composite relativistic particle).

We will express momentum p through kinetic energy W_k

$$p = \frac{1}{c}\sqrt{(2W_0 + W_k)W_k} . \tag{P.5.3.4}$$

Since $W_k = W_0 = m_0c^2$, then

$$p = \frac{1}{c}\sqrt{\left(2m_0c^2 + m_0c^2\right)m_0c^2} = m_0c\sqrt{3} . \tag{P.5.3.5}$$

The relativistic momentum of a compound particle is $p' = m'v'$. Using relation $m' = 3m_0$, the law of conservation of momentum can be written as

$$m_0c\sqrt{3} = 3m_0v' . \tag{P.5.3.6}$$

Express the speed v' from the formula (5.3.6)

$$v' = \frac{c}{\sqrt{3}} . \tag{P.5.3.7}$$

We substitute the expressions for the velocity v' and mass m' into the formula (5.3.3). The result of the substitution is the formula for the rest mass of a composite particle

$$m_0' = 3m_0\sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2} \tag{P.5.3.8}$$

or

$$m_0' = m_0\sqrt{6} . \tag{P.5.3.9}$$

The kinetic energy of a composite particle is equal to the difference between the total energy $m'c^2$ and the rest energy $m_0'c^2$ of the composite particle

$$W_k = (m' - m'_0)c^2. \tag{P.5.3.10}$$

Substituting the expressions for the quantities m' and m'_0, we obtain

$$W'_k = (3m_0 - \sqrt{6}m_0)c^2 \approx 0.55 m_0 c^2. \tag{P.5.3.11}$$

Answer. The relativistic mass of a moving particle is $m = 2m_0$. The relativistic mass of a composite particle is $m' = 3m_0$. The rest mass of a composite particle is $m'_0 = m_0\sqrt{6}$. The kinetic energy of a compound particle is $W'_k = 0.55 m_0 c^2$.

Problems

Problem A

Problem description. A photon rocket moves relative to the Earth at a speed of $v = 0.6c$ (c is the speed of light in a vacuum). Determine how many times the course of time in a rocket will slow down from the point of view of an earth observer.

Answer. 1.25 times.

Problem B

Problem description. Two particles are removed from each other in a laboratory reference system. Particle velocities are the same in absolute value. The relative particle velocity in the laboratory reference system is $0.5c$ (c is the speed of light in a vacuum). Determine the particle velocity.

Answer. $v = 8.04 \times 10^7$ m/s.

Problem C

Problem description. Calculate the speed at which a particle moves if its relativistic mass is three times the rest mass.

Answer. $v = 2.83 \times 10^8$ m/s.

Problem D

Problem description. The solar constant is defined as the energy flux density of the electromagnetic radiation of the sun at a distance equal to the average distance from the Earth to the Sun. The value of the solar constant is 1.4 kW/m^2. Determine the mass that the Sun loses in one year.

Answer. $\Delta m = 1.37 \times 10^{17}$ kg.

Problem E

Problem description. The kinetic energy of a relativistic particle is equal to its rest energy. Determine how many times the particle momentum will increase if the kinetic energy of the particle quadruples.

Answer. 2.82 times.

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APPENDICES

Table A1. Greek alphabet

Name	Capital	Lower-case	Name	Capital	Lower-case
Alpha	A	α	Nu	N	ν
Beta	B	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	O	o
Delta	Δ	δ	Pi	Π	π
Epsilon	E	ε	Rho	P	ρ
Zeta	Z	ζ	Sigma	Σ	σ
Eta	H	η	Tau	T	τ
Theta	Θ	θ	Upsilon	ϒ	υ
Iota	I	ι	Phi	Φ	ϕ
Kappa	K	κ	Chi	X	χ
Lambda	Λ	λ	Psi	Ψ	ψ
Mu	M	μ	Omega	Ω	ω

Table A2. SI prefixes

Prefix		Representation	Prefix		Representation
Name	***Symbol***	***Base 10***	***Name***	***Symbol***	***Base 10***
yotta	Y	10^{24}	deci	d	10^{-1}
zeta	Z	10^{21}	centi	c	10^{-2}
exa	E	10^{18}	milli	m	10^{-3}
peta	P	10^{15}	micro	μ or u	10^{-6}
tera	T	10^{12}	nano	n	10^{-9}
giga	G	10^{9}	pico	p	10^{-12}
mega	M	10^{6}	femto	f	10^{-15}
kilo	k	10^{3}	atto	a	10^{-18}
hecto	h	10^{2}	zepto	z	10^{-21}
deca	da	10^{1}	yocto	y	10^{-24}

Table A3. SI base units

Unit name	Unit symbol	Quantity name	Definition
metre	m	length	The distance travelled by light in vacuum in 1/299792458 second.
kilogram	kg	mass	The kilogram is defined by taking the fixed numerical value of the Plank constant h to be 6.62607015×10^{-34} when expressed in the unit J×s, which is equal to $kg\times m^2\times s^{-1}$, where the metre and the second are defined in terms of c and $\Delta\nu_{Cs}$.
second	s	time	The second is define by taking the fixed numerical value of the caesium frequency $\Delta\nu_{Cs}$, the unperturbed ground-state hyperfine transition frequency of the ^{133}C atom, to be 9192631770 when expressed in the unit Hz, which is equal to s^{-1}.
ampere	A	electric current	The ampere is defined by taking the fixed numerical value of the elementary charge e to be $1.602176634\times10^{-19}$ when expressed in unit C, which is equal to A×s, where the second is defined in terms of $\Delta\nu_{Cs}$.
kelvin	K	thermodynamic temperature	The kelvin is defined by taking the fixed numerical value of the Boltzmann constant k to be 1.380649×10^{-23} $J\times K^{-1}$ ($J=kg\times m^2\times s^{-2}$), given the definition of the kilogram, the metre, and the second.
mole	mol	amount of substance	The amount of substance of exactly 6.02214076×10^{23} elementary entities. This number is the fixed numerical value of the Avogadro constant, N_A, when expressed in the unit mol^{-1} and is called the Avogadro number.
candela	cd	luminous intensity	The luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 5.4×10^{14} Hz and that has a radiant intensity in that direction of 1/683 watt per steradian.

Table A4. SI derived units

Unit name	**Unit symbol**	**Unit Equivalents**	**Quantity name**
hertz	Hz	s^{-1}	frequency
radian	rad	One radian is the angle subtended at the center of a circle by an arc that is equal in length to the radius of the circle.	angle
steradian	sr	The solid angle subtended at the center of a unit sphere by a unit area on its surface	solid angle
newton	N	$kg \times m \times s^{-2}$	force, weight
pascal	Pa	$N/m^2 = kg \times m^{-1} \times s^{-2}$	pressure, stress
joule	J	$N \times m = kg \times m^2 \times s^{-2}$	energy, work, heat
watt	W	$J/s = kg \times m^2 \times s^{-3}$	power, radiant flux
coulomb	C	$A \times s$	electric charge
volt	V	$J/C = kg \times m^2 \times s^{-3} \times A^{-1}$	voltage, electromotive force
farad	F	$C/V = A^2 \times s^4 \times kg^{-1} \times m^{-2}$	electrical capacitance
ohm	Ω or Ohm	$V/A = kg \times m^2 \times s^{-3} \times A^{-2}$	electrical resistance, impedance
siemens	S	$1/Ohm = A^2 \times s^3 \times kg^{-1} \times m^{-2}$	electrical conductance
weber	Wb	$V \times s = kg \times m^2 \times s^{-2} \times A^{-1}$	magnetic flux
tesla	T	$Wb/m^2 = kg \times s^{-2} \times A^{-1}$	magnetic field strength
henry	H	$Wb/A = kg \times m^2 \times s^{-2} \times A^{-2}$	electrical inductance
degree Celsius	°C	K	temperature relative to 273.15 K
lumen	lm	$cd \times sr = cd$	luminous flux
lux	lx	$lm/m^2 = cd \times m^{-2}$	illuminance
becquerel	Bq	s^{-1}	radioactivity
gray	Gy	$J/kg = m^2 \times s^{-2}$	absorbed dose
sievert	Sv	$J/kg = m^2 \times s^{-2}$	equivalent dose
katal	kat	$mol/s = mol \times s^{-1}$	catalytic activity

Table A5. Physical constants

Quantity	Symbol	Value
Avogadro constant	N_A	$6.0221415(10)\times10^{23}$ mol^{-1}
Boltzmann constant	k	$1.3806505(24)\times10^{-23}$ J / K
Electric constant	ε_0	$8.854187817\times10^{-12}$ F×m^{-1}
Faraday constant	F	96485.3383(83) C×mol^{-1}
Fine-structure constant	α	$7.297352568(24)\times10^{-3}$
Gravitational constant	G	$6.6742(10)\times10^{-11}$ N×m^2 / kg^2
Magnetic constant	μ_0	$4\pi\times10^{-7}$ T×m /A (exact)
Molar gas constant	R	8.314472(15) J/(mol×K)
Planck constant	h	$6.6260693(11)\times10^{-34}$ J×s
Rydberg constant	R_H	$1.0973731568525(73)\times10^{7}$ m^{-1}
Stefan-Boltzmann constant	σ	$5.670400(40)\times10^{-8}$ W×m^{-2}×K^{-4}
Wien displacement law constant	b	$2.8977685(51)\times10^{-3}$ m×K
Atomic mass unit	u	$1.66053886(28)\times10^{-27}$ kg
Electron mass	m_e	$9.1093826(16)\times10^{-31}$ kg
Neutron mass	m_n	$1.67492728(29)\times10^{-27}$ kg
Proton mass	m_p	$1.67262171(29)\times10^{-27}$ kg
Elementary charge	e	$1.60217653(14)\times10^{-19}$ C
Speed of light in vacuum	c	2.99792458×10^{8} m /s
Bohr magnetron	μ_B	$9.27400949(80)\times10^{-24}$ J/T
Bohr radius	a_0	$5.291772108(18)\times10^{-11}$ m
Compton wavelength	λ_C	$2.426310238(16)\times10^{-12}$ m

Table A6. Astronomical data

Body	Mass, kg	Equatorial radius, m	Perihelion/ Aphelion, m	Sidereal period	Orbital speed, km/s
Sun	1.998×10^{30}	6.955×10^{8}	2.5×10^{20}(*)	2.3×10^{8} y(*)	2.2×10^{2}(*)
Moon	7.342×10^{22}	1.738×10^{6}	$(3.63/4.05)\times10^{8}$	27.321661 d	1.002
Mercury	3.301×10^{23}	2.440×10^{6}	$(4.60/6.98)\times10^{10}$	87.9691 d	47.362
Venus	4.867×10^{24}	6.052×10^{6}	$(1.08/1.09)\times10^{11}$	224.698 d	35.02
Earth	5.973×10^{24}	6.378×10^{6}	$(1.47/1.52)\times10^{11}$	365.25636 d	29.783
Mars	6.417×10^{23}	3.396×10^{6}	$(2.07/2.49)\times10^{11}$	686.971 d	24.007
Jupiter	1.898×10^{27}	7.149×10^{7}	$(7.40/7.78)\times10^{11}$	11.862 y	13.07
Saturn	5.683×10^{26}	6.027×10^{7}	$(1.35/1.51)\times10^{12}$	29.4571 y	9.68
Uranus	8.683×10^{25}	2.556×10^{7}	$(2.75/3.00)\times10^{12}$	84.01 y	6.81
Neptune	1.024×10^{26}	2.476×10^{7}	$(4.45/4.55)\times10^{12}$	164.79 y	5.4349

(*) – Milky Way

Table A7. Periodic table of elements

Name	$_{AN}$Symbol (*AN – atomic number*)	Standard atomic weight	Name	$_{AN}$Symbol (*AN – atomic number*)	Standard atomic weight
1	*2*	*3*	*1*	*2*	*3*
Actinium	$_{89}$Ac	227	Californium	$_{98}$Cf	251
Aluminium	$_{13}$Al	26.9815384	Carbon	$_{6}$C	12.011
Americium	$_{95}$Am	243	Caesium	$_{55}$Cs	132.905452
Antimony	$_{51}$Sb	121.760	Cerium	$_{58}$Ce	140.116
Argon	$_{18}$Ar	39.948	Chlorine	$_{17}$Cl	35.45
Arsenic	$_{33}$As	74.921595	Chromium	$_{24}$Cr	51.9961
Astatine	$_{85}$At	210	Cobalt	$_{27}$Co	58.933194
Barium	$_{56}$Ba	137.327	Copernicium	$_{112}$Cn	285
Berkelium	$_{97}$Bk	247	Copper	$_{29}$Cu	63.546
Beryllium	$_{4}$Be	9.0121831	Curium	$_{96}$Cm	247
Bismuth	$_{83}$Bi	208.98040	Darmstadtium	$_{110}$Ds	281
Bohrium	$_{107}$Bh	270	Dubnium	$_{105}$Db	268
Boron	$_{5}$B	10.81	Dysprosium	$_{66}$Dy	162.500
Bromine	$_{35}$Br	79.904	Einsteinium	$_{99}$Es	252
Cadmium	$_{48}$Cd	112.414	Erbium	$_{68}$Er	167.259
Calcium	$_{20}$Ca	40.078	Europium	$_{63}$Eu	151.964

1	2	3	1	2	3
Fermium	$_{100}Fm$	257	Phosphorus	$_{15}P$	30.9737620
Flerovium	$_{114}Fl$	289	Platinum	$_{78}Pt$	195.084
Fluorine	$_{9}F$	18.9984032	Plutonium	$_{94}Pu$	244
Francium	$_{87}Fr$	223	Polonium	$_{84}Po$	209
Gadolinium	$_{64}Gd$	157.25	Potassium	$_{19}K$	39.0983
Gallium	$_{31}Ga$	69.723	Praseodymium	$_{59}Pr$	140.90766
Germanium	$_{32}Ge$	72.630	Promethium	$_{61}Pm$	145
Gold	$_{79}Au$	196.966570	Protactinium	$_{91}Pa$	231.03588
Hafnium	$_{72}Hf$	178.49	Radium	$_{88}Ra$	226
Hassium	$_{108}Hs$	270	Radon	$_{86}Rn$	222
Helium	$_{2}He$	4.002602	Rhenium	$_{75}Re$	186.207
Holmium	$_{67}Ho$	164.930328	Rhodium	$_{45}Rh$	102.90549
Hydrogen	$_{1}H$	1.008	Roentgenuim	$_{111}Rg$	282
Indium	$_{49}In$	114.818	Rubidium	$_{37}Rb$	85.4678
Iodine	$_{53}I$	126.90447	Ruthenium	$_{44}Ru$	101.07
Iridium	$_{77}Ir$	192.217	Rutherfordium	$_{104}Rf$	267
Iron	$_{26}Fe$	55.845	Samarium	$_{62}Sm$	150.36
Krypton	$_{36}Kr$	83.798	Scandium	$_{21}Sc$	44.955908
Lanthanum	$_{57}La$	138.90547	Seaborgium	$_{106}Sg$	269
Lawrencium	$_{103}Lr$	266	Selenium	$_{34}Se$	78.971
Lead	$_{82}Pb$	207.2	Silicon	$_{14}Si$	28.085
Lithium	$_{3}Li$	6.94	Silver	$_{47}Ag$	107.8682
Livermorium	$_{116}Lv$	293	Sodium	$_{11}Na$	22.9897693
Lutetium	$_{71}Lu$	174.9668	Strontium	$_{38}Sr$	87.62
Magnesium	$_{12}Mg$	24.305	Sulfur	$_{16}S$	32.06
Manganese	$_{25}Mn$	54.938043	Tantalum	$_{73}Ta$	180.94788
Meitnerium	$_{109}Mt$	278	Technetium	$_{43}Tc$	98
Mendelevium	$_{101}Md$	258	Tellurium	$_{52}Te$	127.60
Mercury	$_{80}Hg$	200.592	Tennessine	$_{117}Ts$	294
Molybdenum	$_{42}Mo$	95.95	Terbium	$_{65}Tb$	158.925354
Moscovium	$_{115}Mc$	290	Thallium	$_{81}Tl$	204.38
Neodymium	$_{60}Nd$	144.242	Thorium	$_{90}Th$	232.0377
Neon	$_{10}Ne$	20.1797	Thulium	$_{69}Tm$	168.934218
Neptunium	$_{93}Np$	237	Tin	$_{50}Sn$	118.710
Nickel	$_{28}Ni$	58.6934	Titanium	$_{22}Ti$	47.867
Nihonium	$_{113}Nh$	286	Tungsten	$_{74}W$	183.84
Niobium	$_{41}Nb$	92.90637	Uranium	$_{92}U$	238.02891
Nitrogen	$_{7}N$	14.007	Vanadium	$_{23}V$	50.9415
Nobelium	$_{102}No$	259	Xenon	$_{54}Xe$	131.293
Oganesson	$_{118}Og$	294	Ytterbium	$_{70}Yb$	173.045
Osmium	$_{76}Os$	190.23	Yttrium	$_{39}Y$	88.90584
Oxygen	$_{8}O$	15.999	Zinc	$_{30}Zn$	65.38
Palladium	$_{46}Pd$	106.42	Zirconium	$_{40}Zr$	91.224

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