MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
ODESSA STATE ACADEMY OF CIVIL ENGINEERING AND ARCHITECTURE

Pysarenko A. N.

GUIDE TO PHYSICS PROBLEMS
ELECTROMAGNETISM

Odessa
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The manual «Guide to physics problems. Electromagnetism» contains basic theoretical information, methodological and reference materials, as well as control tasks for the section "Electricity and magnetism" of the general course of physics. The main attention is paid to solving problems on the following topics: electrostatics, conductors and dielectrics in an electrostatic field, direct current, magnetic field in vacuum and in matter, electromagnetic induction, magnetic field energy, electromagnetic oscillations and waves.
This textbook is intended for vocational training of students in the specialty 192 Construction and Civil Engineering for the Bachelor degree.

Reviewers:

Gotsulskiy V.Ya. – D.Sc. in Physics and Maths., Professor, Head of the Department of Physics and Astronomy of the Odessa National University named I. I. Mechnikov.
Masleeva N.V. – Ph.D. in Physics and Maths., Associate Professor of the Department of Experimental Physics of the Odessa National University named I. I. Mechnikov.

Responsible for edition

Head of the Department of Physics Associated Professor Pysarenko A. N.

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PREFACE

The section "Electromagnetism" of the general physics course, which is studied by students of technical institutes in the second semester, requires special attention. Firstly, this section includes a fairly complex mathematical apparatus, in particular, integral and differential calculus, vector and operator algebra. Secondly, the historical features of understanding electromagnetic phenomena have led to different approaches in describing the quantitative parameters of electric and magnetic fields (meaning the SI and CGS systems). Legislation and standards in many countries recommend the use of the International System of Units (SI) in science and education, which, in relation to electrodynamics, goes back to the system of "absolute practical units of measurement". However, the use of the SI system in electrodynamics still raises objections. Among physicists, the positions of the Gaussian (symmetric) CGS system are traditionally strong, which better takes into account the symmetries of electrodynamics, and is the standard for scientific publications and textbooks in theoretical physics. The appearance of the sets of formulas used in these approaches often differ significantly from each other even when describing the same characteristics of these fields. Thirdly, the course of electromagnetism is the base for further consideration of the experimental and theoretical aspects of wave, optical and even quantum phenomena.

Most of the educational material, which is associated with the ability to quantitatively and qualitatively solve problems in electromagnetism, falls on seminars. The development of skills, abilities and methods for solving a huge number of typical problems, of course, cannot be realized only at the expense of the hours allotted for seminars, and implies a lot of independent work by the student. In this regard, the problem of developing a general methodology for solving problems in the electromagnetism section becomes extremely urgent, which, on the one hand, would streamline and unify the plans for conducting the seminars themselves, and on the other hand, would become a guide for independent work of students.

It should be borne in mind that methodological recommendations for solving problems from different sections of the general physics course have both common, coinciding, and individual, differing points. A necessary condition for the successful solution of most of the problems related to the features of the quantitative and qualitative description of electric and magnetic fields is the strict observance of all methodological recommendations for their solution.

A detailed analysis of the initial data of the problem is the first stage of its solution. It is necessary to establish whether all the initial data necessary to solve the problem are available. Most of the tasks involve the construction of a detailed drawing that displays both the details of the equipment used or the electrical circuit, and the directions of vector quantities that describe the electromagnetic field. Each problem must be approached creatively. For example, it is not always advisable to complete the solution in a general, analytical form. Sometimes it is convenient to divide the problem into several blocks that involve theoretical and numerical solutions. The analysis of the formula obtained for any physical quantity must include a check of its dimension. An incorrect dimension is a sure sign of a wrong decision.

When starting calculations, it must be remembered that the numerical values of physical quantities are approximate. Therefore, calculations must be performed based on the rules of operations with approximate numbers. In particular, for the calculated value, it is necessary to save the last sign, the unit of which still exceeds the error of this value. All the following figures must be discarded. Further, for the obtained numerical values, a plausibility check must also be carried out. For example, the calculated speed of an electron in a betatron should not exceed the speed of light in vacuum.

The entire material of the manual is divided into six chapters. Qualitative and quantitative characteristics of the electric field are considered in the first three chapters. The tasks related to the calculation of the characteristics of magnetic fields are given in the last three chapters. Each chapter contains five paragraphs.

The first paragraph is for reference and contains definitions of the basic concepts and quantities used in the chapter, formulations of fundamental laws, and frequently used formulas. The second paragraph is devoted to guidelines, the main purpose of which is to focus on the features and difficulties of solving problems of this type. In the third paragraph, solutions to problems that are typical for this chapter are considered. The solution itself contains the full condition of the problem, a brief condition, derivation of the necessary formulas and calculation of numerical values, as well as a short answer. The fourth paragraph is focused on consolidating the educational material and contains twenty tasks for independent solution. In this section, tasks of the main types with answers are selected, the solution of which allows students to conduct self-control of the depth and correctness of mastering all the previous material. Finally, the fifth paragraph consists of short answers to the problems of the previous paragraph.

In addition, at the end of the tutorial, there are reference materials necessary for solving problems, as well as a list of references.

CHAPTER 1. ELECTROSTATICS

1.1. Basic formulas

The force of interaction of two point charges Q_1 and Q_2, placed at a distance r in a medium with a relative permittivity ε is determined using **Coulomb's law** (Charles-Augustin de Coulomb 1736 – 1806)

$$F = \frac{1}{4\pi\varepsilon\varepsilon_0}\frac{Q_1 Q_2}{r^2}, \qquad (1.1.1)$$

where ε_0 = 8.8541878128(13) F·m^{-1}.

A **point charge** is an electric charge placed on a body whose dimensions are small compared to the characteristic dimensions of this problem (for formula (1.1.1), the characteristic size is the value r). The force F has a positive sign in the case of repulsion (charges of the same sign) and a negative sign in the case of attraction (charges with opposite signs). The direction of vector $\vec{F}$ is directly away from the pdrticle if the value Q (Q_1 or Q_2) is positive and directly toward it if Q is negative [1, p. 619]. The value ε shows how many times the force of interaction between electric charges in a given medium is less than the force of interaction of the same charges in vacuum.

The **electric field** at a given point is numerically equal to the force F, which acts on a unit of positive charge Q at this point

$$E = \frac{F}{Q}. \qquad (1.1.2)$$

The electric field $\vec{E}$ is a vector quantity, the direction of which coincides with the direction of the force vector $\vec{F}$. The force that acts on a charge Q, can be determined using the formula

$$\vec{F} = Q\vec{E}. \qquad (1.1.3)$$

The electric field of a point charge and the field outside a uniformly charged sphere (for the case when r is greater than the radius of the sphere) is equal to

$$E = \frac{Q}{4\pi\varepsilon\varepsilon_0 r^2}. \qquad (1.1.4)$$

The electric field of a plane that is uniformly charged with an electric charge with a surface density σ at any distance from this plane is determined by the formula

$$E = \frac{\sigma}{2\varepsilon\varepsilon_0}. \tag{1.1.5}$$

The **surface density of the electric charge** of a uniformly charged conductor is numerically equal to the charge per unit surface of the conductor and is expressed by the ratio of the total charge to the surface area of the conductor S

$$\sigma = \frac{Q}{S}. \tag{1.1.6}$$

The electric field of an infinitely long charged thin wire at a distance r from it is

$$E = \frac{\tau}{2\pi\varepsilon\varepsilon_0 r}, \tag{1.1.7}$$

where τ is the linear charge density.

The **linear density of the electric charge** is numerically equal to the ratio of the total charge to the length of the thread L, on which this charge is placed

$$\tau = \frac{Q}{L}. \tag{1.1.8}$$

The electric field of a dipole at a distance r from it can be expressed by the formula

$$E = \frac{p}{4\pi\varepsilon\varepsilon_0 r^2}\sqrt{1+3cos^2\theta}, \tag{1.1.9}$$

where

$p = Ql$ is the dipole moment;

l is the arm of the dipole;

θ is the angle between the direction of the dipole axis and the direction of the radius vector drawn from the center of the dipole to the point where the value E is determined.

The electric field in flat, cylindrical and spherical capacitors is determined by the formulas

$$E = \frac{\sigma}{\varepsilon\varepsilon_0}, \quad E = \frac{\tau}{2\pi\varepsilon\varepsilon_0 r}, \quad E = \frac{Q}{4\pi\varepsilon\varepsilon_0 r^2}. \tag{1.1.10}$$

Electric potential φ at a given point M is numerically equal to the work that the field forces do when moving a unit of positive charge Q from infinity to a point M

$$\varphi = \frac{A_{\infty \to M}}{Q}. \tag{1.1.11}$$

The electric potential of a point charge and a uniformly charged sphere outside it is

$$\varphi = \frac{Q}{4\pi\varepsilon\varepsilon_0 r}. \tag{1.1.12}$$

Consider the case when the electrostatic field is formed by a system of n point charges. The electric potential of such a field at a fixed point M can be expressed by formula

$$\varphi = \sum_{i=1}^{n} \frac{Q_i}{4\pi\varepsilon\varepsilon_0 r_i}, \tag{1.1.13}$$

where r_i is the distance from the point M to the charge Q_i.

Electric induction $\vec{D}$ s an auxiliary quantity, which, along with the electric field E (main quantity), describes the characteristics of the electric field. The relationship between electrical induction and electric field is described by the formula

$$\vec{D} = \varepsilon\varepsilon_0 \vec{E}. \tag{1.1.14}$$

The flux of the electric induction vector $\vec{D}$ through an arbitrary surface is given by formula

$$N = \int_S D_n dS, \tag{1.1.15}$$

where D_n is the projection of the vector $\vec{D}$ onto the direction of the normal to the surface.

The integration is performed over the entire surface S. For the case of a closed surface, we can write

$$N = \oint D_n dS, \qquad (1.1.16)$$

Formula (1.2.16) assumes integration over the entire surface.

The flux of the electric induction vector through a closed surface that covers the charges Q_1, Q_2, … is determined according to the **Ostrogradsky-Gauss theorem** (Ostrogradsky M.V. 1801 – 1862; Johann Carl Friedrich Gauß 1777 – 1855)

$$N = \oint D_n dS = \sum_{i=1}^{n} Q_i, \qquad (1.1.17)$$

where n is the number of charges (taking into account the sign) that are inside the closed surface).

The electric potential created by the dipole at a distance r from the center of the dipole (provided that the value r is much greater than the arm of the dipole) is equal to

$$\varphi = \frac{p}{4\pi\varepsilon\varepsilon_0 r^2} cos\,\theta. \qquad (1.1.18)$$

The work done by the field when moving a point charge Q from point 1 to point 2 is given by

$$A = QU, \qquad (1.1.19)$$

where $U = \varphi_1 - \varphi_2$ is the potential difference (voltage) between points 1 and 2.

The electric potential difference in flat, cylindrical and spherical capacitors is given by equations

$$U = \frac{\sigma d}{\varepsilon\varepsilon_0}, \quad U = \frac{\tau}{2\pi\varepsilon\varepsilon_0} ln\frac{R_2}{R_1}, \quad U = \frac{Q}{4\pi\varepsilon\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \qquad (1.1.20)$$

where

d is the distance between the plates of a flat capacitor;

R_1, R_2 are the inner and outer radii of cylindrical and spherical capacitors.

The measure of the intensity of the potential change along the electric line of force of the field formed by a point charge is the derivative of the potential with respect to the distance to the field source. This value is called the **potential gradient** and is equal to the electric field with a negative sign. The gradient is a vector quantity, the direction of which coincides with the direction of increasing potential

$$E_l = -\frac{d\varphi}{dl}, \quad \text{or} \quad U = -\int_1^2 E_l dl, \tag{1.1.21}$$

where

E_l is the projection of the electric field E on the direction of movement;

dl is the amount of displacement.

1.2. Problem-solving framework

The most characteristic tasks of this section can be divided into four groups: 1) calculation of the interaction force of charged bodies; 2) determination of the electrtic field; 3) determination of the electric potential; 4) the distribution of a given charge over the surface of an insulated conductor whose dimensions and shape are known.

For each group of tasks, we can specify some general ways to solve them. These methods are based on certain physical concepts and laws.

1. The force of interaction between charged bodies of any shape with any distribution of charges on them can be calculated using Coulomb's law. To do this, it is necessary to geometrically add the forces with which each point charge on one body acts on each point charge on the second body. Such a calculation is quite simple for small charged balls and for charged bodies whose linear dimensions are much smaller than the distance between them. In all other cases, such a calculation is rather complicated, and it is better to calculate the force of interaction between bodies using the formula

$$F = EQ, \tag{1.2.1}$$

where E – is the electric field created by all the charges (except the charge Q) of the system in the place where the charge is located Q.

2. The main physical quantities in this case are the electric field and the electric potential. Problems in which it is necessary to determine the electric field formed by a system of charges can be solved in two ways: a) using the Ostrogradsky-Gauss theorem; b) using the formula for the electric field of a point charge.

It is convenient to use the Ostrogradsky-Gauss theorem in integral form when, due to symmetry considerations, it is possible to indicate in advance the direction of the lines of force and choose a surface of the correct symmetrical shape (cylinder, sphere), which covers all charges in the system. In this case, one must always remember that the magnitude of the electric field at each point of such a surface must be the same.

According to the second method, the electric charge located on the surface of the body or in a certain volume is divided into elementary charges of such a magnitude that the formula for the electric field of a point charge can be applied to each of them. In this case, we can use the superposition principle for the electric field. For example, the resulting electric field formed by charges Q_1 and Q_2 can be determined by adding the corresponding vectors $\vec{E}_1$ and $\vec{E}_2$ for the electric fields that are created by each charge separately

$$\vec{E} = \vec{E}_1 + \vec{E}_2. \tag{1.2.2}$$

The direction of the electric field vectors $\vec{E}_i$ is determined by the directions of the corresponding radius vectors that connect the charges Q_i with a fixed point. Integration of the electric fields of a large number of elementary charges makes it possible to calculate the electric field created by a charged body. It should always be remembered that the electric field is a vector quantity. This means that it is necessary to take into account the sum of only those components of the electric field of elementary charges that coincide with the direction of the resulting electric field vector, which is created by all charges.

3. The electric potential is another important characteristic of the electric field. Unlike electric induction $\vec{D}$ and electric field $\vec{E}$, which are vector quantities, electric potential is defined by a scalar function $\varphi(x,y,z)$, that depends only on the coordinates of a point. Consider the case of arbitrary placement of point charges Q_1, $Q_2 \ldots$ in some system. Select a point P in space. The distances from this point to the charges will be equal l_1, $l_2 \ldots$ The electric potential at point P, according to the principle of superposition, is equal to the algebraic sum of the electric potentials of individual charges

$$\varphi = \frac{1}{4\pi\varepsilon\varepsilon_0} \sum_{i=1}^{n} \frac{Q_i}{l_i}. \tag{1.2.3}$$

This method of calculating the potential can only be used in the case of a continuous potential distribution, for example, for a charged body. Consider the elementary volume dV of the body. This volume contains an elementary charge $dq = \rho(x,y,z)dV$. Then the potential at the point P can be calculated by summing the potentials of elementary charges of all elements of the volume

$$\varphi(P)=\frac{1}{4\pi\varepsilon\varepsilon_0}\iiint\frac{\rho(x,y,z)}{l}dV. \qquad (1.2.4)$$

This method of calculating the potential for points in a field created by an infinitely long uniformly charged filament or an infinite uniformly charged plate leads to an electric potential equal to infinity. In practice, we are not dealing with an infinite thread or an infinite plate. Therefore, the result obtained will not have a physical meaning. However, by integrating the expression $U=-\int E_l dl$, after substituting the value of the electric field into it, one can calculate the potential difference between field points that are at finite distances from the field source, for which the formulas for the electric field lead to accuracy sufficient for practice.

For the case when the formula for the electric field created by any system of charges is known, the potential can be calculated by integrating this formula within certain limits. To do this, you can use the relationship between the potential gradient and the electric field

$$E_l=-\frac{d\varphi}{dl}. \qquad (1.2.5)$$

4. **The condition of electrical equilibrium of charges** on conductors can be formulated in the form of two points: 1) equality to zero of the electric field inside the conductor; 2) perpendicularity of the electric field vector to the surface of the body at any point. The mathematical expression for these conditions is

$$\varphi=const. \qquad (1.2.6)$$

The value of the electric potential must be the same at all points of the conductor, while its surface must be equipotential.

The solution of problems concerning the calculation of the charge distribution on the surface of an insulated conductor with a given shape and dimensions is reduced to determining such a density σ distribution on the elements dS of the conductor surface, in which at all internal points M it had the same value

$$\varphi=\frac{1}{4\pi\varepsilon\varepsilon_0}\int_S\frac{\sigma\, dS}{l}. \qquad (1.2.7)$$

where l is the distance from the point M to the surface element dS.

1.3. Problem-solving examples

Problem 1.3.1

Problem description. Two electric charges $Q_1 = 2 \cdot 10^{-8}\, C$ and $Q_1 = 10^{-8}\, C$ are located in the air at a distance of $r_0 = 10\ cm$ from each other. At the initial moment of time, the charges are fixed motionless, and then the charge Q_2 is released and under the action of the repulsive force it begins to move away from the charge Q_1. Calculate the work done by the repulsive forces for two cases: a) charge Q_2 moves a distance of $r = 30\ cm$ from charge Q_1; b) charge Q_2 moves to infinity.

Known quantities: $Q_1 = 2 \cdot 10^{-8}\, C$, $Q_1 = 10^{-8}\, C$, $r_0 = 10\ cm$, $r = 30\ cm$.

Quantities to be calculated: A_1, A_2.

Problem solution. We will assume that charge Q_2 is at a distance of x from charge Q_1. The repulsive force on the length interval from x to $(x + dx)$ can be considered approximately constant. In this case, the work done by the force at a distance of dx is given by

$$dA = f(x)dx, \tag{1}$$

where $f(x)$ is the repulsive force, which, according to Coulomb's law, is equal to

$$f(x) = \frac{Q_1 Q_2}{4\pi\varepsilon\varepsilon_0 x^2}, \tag{2}$$

where
ε is the relative permittivity (for the case of air $\varepsilon \approx 1$);
ε_0 is the electric constant.

Substitute (2) into (1)

$$A = \int \frac{Q_1 Q_2}{4\pi\varepsilon\varepsilon_0 x^2} dx = -\frac{Q_1 Q_2}{4\pi\varepsilon\varepsilon_0} \frac{1}{x}. \tag{3}$$

The work done by the repulsive force in case a) is equal to

$$A_1 = -\frac{Q_1 Q_2}{4\pi\varepsilon\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_0}\right). \quad (4)$$

The work done by the repulsive force in case b) is equal to

$$A_1 = -\frac{Q_1 Q_2}{4\pi\varepsilon\varepsilon_0}\left(\frac{1}{\infty} - \frac{1}{r_0}\right). \quad (5)$$

We substitute numerical values in formulas (4) and (5)

$$A_1 = 1.2 \cdot 10^{-5}\ J, \qquad A_2 = 1.8 \cdot 10^{-5}\ J. \quad (6)$$

Answer. The work done by the repulsive force in cases a) and b) is: $A_1 = 1.2 \cdot 10^{-5}\ J$, $A_2 = 1.8 \cdot 10^{-5}\ J$ respectively.

Problem 1.3.2

Problem description. Two small balls are suspended on silk threads so that they are in contact. After the balls were charged, they repelled each other and their centers moved a distance of $d = 5\ cm$. Determine the charges of these balls, if the mass of each of them is $m = 0.1\ g$, and the length of the suspension threads is $l = 25\ cm$.

Known quantities: $d = 5\ cm$, $m = 0.1\ g$, $l = 25\ cm$.

Quantities to be calculated: Q.

Problem solution. Two forces act on the balls deviated from the vertical: the Coulomb repulsive force (directed horizontally)

$$F = \frac{Q^2}{4\pi\varepsilon\varepsilon_0 d^2}. \quad (1)$$

and gravity (directed vertically)

$$P = mg, \quad (2)$$

where
m is the mass of the ball;

g is the free fall acceleration.

The silk thread is directed along the resultant of these forces at an angle of α to the vertical. For a given angle, the following relations can be written

$$\frac{(d/2)}{l} = sin\alpha , \qquad \frac{F}{P} = tg\alpha . \tag{3}$$

We consider the case of a small angle of deviation of the thread from the vertical. In this case, the following relations are valid

$$sin\alpha \approx \alpha , \qquad \frac{F}{P} = \frac{d}{2l} . \tag{4}$$

We express the force F from formula (4)

$$F = \frac{d}{2l} P = \frac{mgd}{2l} . \tag{5}$$

Let us substitute (5) into (1) and write down the mathematical expression for the electric charge

$$Q = d\sqrt{\frac{4\pi\varepsilon\varepsilon_0 mgd}{2l}} = d\sqrt{\frac{2\pi\varepsilon\varepsilon_0 mgd}{l}} . \tag{6}$$

Substituting known values, we get

$$Q = 5.2 \cdot 10^{-9}\ C . \tag{7}$$

<u>Answer</u>. The electric charge of one ball is $Q = 5.2 \cdot 10^{-9}\ C$.

Problem 1.3.3

<u>Problem description</u>. The charge $Q_2 = 10^{-7}\ C$ is uniformly distributed along a straight conductor with a length of $L = 6\ cm$. Calculate the force with which this charge acts on another point charge $Q_1 = 2 \cdot 10^{-9}\ C$. The charge Q_1 is located on a straight line passing along the conductor, at a distance of $l = 5\ cm$ from the middle of the conductor.

<u>Known quantities</u>: $Q_2 = 10^{-7}\ C$, $L = 6\ cm$, $Q_1 = 2 \cdot 10^{-9}\ C$, $l = 5\ cm$.

<u>Quantities to be calculated</u>: F.

<u>Problem solution</u>. Charge Q_2 is not concentrated at one point, so it is not possible to use Coulomb's law. We will divide the conductor into a large number of small segments dx. An elementary segment will have a charge

$$dQ = \tau\, dx, \tag{1}$$

wherr $\tau = \dfrac{Q_2}{L}$ is the linear charge density.

Coulomb's law can now be used to describe an elementary charge. The force dF, with which the charge dQ acts on the charge Q_1, is given by

$$dF = \frac{Q_1 \tau\, dx}{4\pi\varepsilon\varepsilon_0 x^2}, \tag{2}$$

where x is the distance between charges.

In order to calculate the total force of interaction of charges Q_1 and Q_2, it is necessary to add geometrically all elementary forces that act on charge Q_1 from charge Q_2. Since the direction of the force does not change when moving from one element of the conductor to another, the problem is reduced to integrating equation (2)

$$F = \int_{l-L/2}^{l+L/2} \frac{Q_1 \tau\, dx}{4\pi\varepsilon\varepsilon_0 x^2} = -\frac{Q_1 \tau}{4\pi\varepsilon\varepsilon_0} \frac{1}{x}\Bigg|_{l-L/2}^{l+L/2} =$$

$$= \frac{Q_1 \tau}{4\pi\varepsilon\varepsilon_0}\left(\frac{1}{l - L/2} - \frac{1}{l + L/2}\right). \tag{3}$$

Next, we will substitute the mathematical expression for the quantity in equation (3)

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon\varepsilon_0 \left(l^2 - L^2/4\right)}. \tag{4}$$

Numerically

$$F \approx 1.125 \cdot 10^{-3}\ N. \tag{5}$$

Answer. The strength of interaction between charges is $F \approx 1.125 \cdot 10^{-3}\ N$.

Problem 1.3.4

Problem description. A large vertical plate is uniformly charged with a surface charge density $\sigma = 5 \cdot 10^{-4}\ C \cdot m^{-2}$. A thread is attached to the plate, on which a ball with a mass of $m = 1\ g$ is suspended. The electric charge of the ball has the same sign as the electric sign of the plate. Calculate the charge of the ball for the case when the thread is deviated from the vertical by an angle of $\alpha = 30°$.

Known quantities: $\sigma = 5 \cdot 10^{-4}\ C \cdot m^{-2}$, $m = 1\ g$, $\alpha = 30°$.

Quantities to be calculated: Q.

Problem solution. We will denote the gravity of the ball as P. The vector $\vec{P}$ can be represented as the sum of two components $\vec{P}_1$ and $\vec{P}_2$, directed along the thread and in the horizontal direction, respectively. Component $\vec{P}_1$ is balanced by the thread tension. Component $\vec{P}_2$ tends to deflect the ball towards the plate. However, this component is balanced by the force of electrostatic repulsion of the ball from the plate. For modules of vectors $\vec{P}$ and $\vec{P}_2$ we can write

$$p_2 = Ptg\alpha = mg\,tg\alpha, \tag{1}$$

where
m is the mass of the ball;
g is the free fall acceleration.

The electric field of a charged plate can be approximately expressed by the formula

$$E = \frac{\sigma}{2\varepsilon\varepsilon_0}, \tag{2}$$

where
σ is the surface charge density of the plate;

ε is the relative permittivity of the medium (we assume that the plate and the ball are in the air, for which $\varepsilon \approx 1$);
ε_0 is the electric constant.

The repulsive force of the ball from the plate is numerically equal to the product of the magnitude of the electric field E by the magnitude of the electric charge Q

$$F = EQ = \frac{Q\sigma}{2\varepsilon\varepsilon_0}. \qquad (3)$$

Vectors $\vec{P}_2$ and $\vec{F}$ have the same modulus, so

$$\frac{Q\sigma}{2\varepsilon\varepsilon_0} = mg\,tg\alpha. \qquad (4)$$

We write, using formula (4), the equation for the electric charge

$$Q = \frac{2\varepsilon\varepsilon_0\,mg\,tg\alpha}{\sigma}. \qquad (5)$$

Substituting the given data, we find

$$Q = 2\cdot 10^{-9}\ C. \qquad (6)$$

Answer. The electric charge of the ball is $Q = 2\cdot 10^{-9}\ C$.

Problem 1.3.5

Problem description. The diameter, density and electric charge of the ball are $d = 1\,cm$, $\rho = 1.5\cdot 10^3\ kg\cdot m^{-3}$ and $Q = 10^{-9}\ C$, respectively. The ball is immersed in oil, the density of which is $\rho_0 = 8\cdot 10^2\ kg\cdot m^{-3}$. The entire system is in an electric field that is directed vertically upwards. Calculate the strength of this electric field for the case when the ball floats in oil.

Known quantities: $d = 1\,cm$, $\rho = 1.5\cdot 10^3\ kg\cdot m^{-3}$, $Q = 10^{-9}\ C$, $\rho_0 = 8\cdot 10^2\ kg\cdot m^{-3}$.

<u>Quantities to be calculated</u>: E.

<u>Problem solution</u>. In order for a ball to float in oil, the resultant of all forces acting on it must be zero. We will consider the forces acting on the ball. Gravity is applied to the center of the ball and directed vertically downwards

$$P = \frac{1}{6}\pi d^3 \rho g, \tag{1}$$

where
d is the diameter of the ball;
ρ is the density of the ball;
g is the free fall acceleration.

The buoyant force is applied to the center of the ball and directed vertically upwards

$$P_1 = \frac{1}{6}\pi d^3 \rho_0 g, \tag{2}$$

where ρ_0 is the density of the oil.

The force with which the electrostatic field acts on the ball is applied to the center of the ball and is directed vertically upwards

$$F = EQ, \tag{3}$$

where
Q is the charge of the ball;
E is an electrostatic field.

The equilibrium condition for all these forces has the form

$$\frac{1}{6}\pi d^3 \rho g = \frac{1}{6}\pi d^3 \rho_0 g + EQ. \tag{4}$$

We determine the electric field using equation (4)

$$E = \frac{\pi d^3 g (\rho - \rho_0)}{6Q}. \tag{5}$$

Substituting known values, we get

$$E = 3.6 \cdot 10^4\ V \cdot m^{-1}. \tag{6}$$

Answer. The electric field is $E = 3.6 \cdot 10^4\ V \cdot m^{-1}$.

Problem 1.3.6

Problem description. Calculate the work done when two plates of a flat capacitor are moved apart by a distance of $d = 3\ cm$. The area and charges of the capacitor plates are $S = 200\ cm^2$, $Q_1 = +2 \cdot 10^{-7}\ C$, $Q_2 = -2 \cdot 10^{-7}\ C$ ($|Q_1| = |Q_2| = Q$).

Known quantities: $d = 3\ cm$, $S = 200\ cm^2$, $Q_1 = +2 \cdot 10^{-7}\ C$, $Q_2 = -2 \cdot 10^{-7}\ C$, $|Q_1| = |Q_2| = Q$.

Quantities to be calculated: A.

Problem solution. The force of interaction between the plates of the capacitor is

$$F = EQ, \tag{1}$$

where
E is the electric field that is created by the charge of one plate in the place where the other plate is located;
Q is the electric charge of one of the plates.

The uniform electric field between the plates of a capacitor is

$$E = \frac{\sigma}{2\varepsilon\varepsilon_0}, \tag{2}$$

where
σ is the surface charge density on the plates;
ε is the relative permittivity of the medium in which the plates are located ($\varepsilon \approx 1$);
ε_0 is the electric constant.

The surface charge density is given by

$$\sigma = \frac{Q}{S}, \tag{3}$$

where S is the area of one of the plates.

Substitute (2) and (3) in (1)

$$F = \frac{Q^2}{2\varepsilon\varepsilon_0 S}. \quad (4)$$

The magnitude of the force F does not depend on time. In this case, the work done when pushing the two plates apart is

$$A = Fd = \frac{Q^2 d}{2\varepsilon\varepsilon_0 S}. \quad (5)$$

Numerically

$$A = 3.4 \cdot 10^{-3}\ J. \quad (6)$$

Answer. The work done when moving apart two plates of a flat capacitor is equal to $A = 3.4 \cdot 10^{-3}\ J$.

Problem 1.3.7

Problem description. Calculate the force of interaction of two water vapor molecules, the dipoles of which are placed along one straight line. The electric moment of the water dipole is $p = 6.2 \cdot 10^{-30}\ C \cdot m$. The distance between molecules is $l = 10^{-7}\ m$.

Known quantities: $p = 6.2 \cdot 10^{-30}\ C \cdot m$, $l = 10^{-7}\ m$.

Quantities to be calculated: F.

Problem solution. We will consider the case when molecules (dipoles) of water vapor are directed relative to each other with their ends charged with opposite signs. The distance l is measured from the middle of the first molecule to the closest charge $-Q_2$ of the second molecule. The electric field E, formed by the first molecule in the place where the charge $-Q_2$ is located (under the condition $l >> d$, where d is the distance between the charges of one molecule) is equal to

$$E_1 = \frac{p}{2\pi\,\varepsilon\varepsilon_0\,l^3}, \tag{1}$$

where p is the electric moment of the water vapor molecule.

A force of F_1 acts on the charge $-Q_2$ of the second molecule

$$F_1 = -\frac{p}{2\pi\varepsilon\varepsilon_0 l^3} Q_2. \tag{2}$$

A force F_2 acts on the charge $+Q_2$ of the second molecule

$$F_2 = -\frac{p}{2\pi\varepsilon\varepsilon_0 (l+d)^3} Q_2. \tag{3}$$

The resulting force that acts on the second molecule is

$$F = F_1 + F_2 = -\frac{p}{2\pi\varepsilon\varepsilon_0 l^3} Q_2 - \frac{p}{2\pi\varepsilon\varepsilon_0 (l+d)^3} Q_2 =$$

$$= -\frac{pQ_2}{2\pi\varepsilon\varepsilon_0 l^3}\left[1 - \frac{1}{\left(1+\frac{d}{l}\right)^3}\right]. \tag{4}$$

We will consider the condition $l >> d$. For this condition, we can write

$$\frac{1}{\left(1+\frac{d}{l}\right)^3} \approx 1 - 3\frac{d}{l}. \tag{5}$$

Therefore, the force is

$$F = -\frac{3pdQ_2}{2\pi\varepsilon\varepsilon_0 l^4}. \tag{6}$$

The dipole moment of the second molecule is

$$p = Q_2 d\,. \tag{7}$$

Substitute (7) in (6)

$$F = -\frac{3p^2}{2\pi\varepsilon\varepsilon_0 l^4}\,. \tag{8}$$

Numerically,

$$F = -2.15 \cdot 19^{-12}\ N\,. \tag{9}$$

Answer. The force of interaction of two water vapor molecules is $F = -2.15 \cdot 19^{-12}\ N$.

Problem 1.3.8

Problem description. Determine the electrostatic pressure on the surface of a charged ball whose surface charge density is $\sigma = 2 \cdot 10^{-7}\ C \cdot m^{-2}$.

Known quantities: $\sigma = 2 \cdot 10^{-7}\ C \cdot m^{-2}$.

Quantities to be calculated: P.

Problem solution. We will consider a ball with a radius of R, on which a charge of Q is located. The surface charge density on this conductor will be

$$\sigma = \frac{Q}{4\pi R^2}\,. \tag{1}$$

The electric field strength in the area near the surface of the ball will be determined by the formula

$$E = \frac{Q}{4\pi\varepsilon\varepsilon_0 R^2} = \frac{\sigma}{\varepsilon\varepsilon_0}\,, \tag{2}$$

where
ε is the relative permittivity of the medium;
ε_0 is the electric constant.

We select an infinitesimal area element dS on the surface of the ball. The electric field E_1, formed by this charge near the selected element can be determined under the assumption that the surface element is flat. In this case, the electric field

$$E_2 = E - E_1 = \frac{\sigma}{\varepsilon\varepsilon_0} - \frac{\sigma}{2\varepsilon\varepsilon_0} = \frac{\sigma}{2\varepsilon\varepsilon_0} \qquad (3)$$

formed by the remaining charges located on the surface of the ball. These charges collectively affect the charge σdS with a force

$$df = \frac{\sigma}{2\varepsilon\varepsilon_0}\sigma dS = \frac{\sigma^2 dS}{2\varepsilon\varepsilon_0}. \qquad (4)$$

This force is directed along the outer normal to the surface of the conductor. From equation (4) it follows that the surface of the charged ball is under electrostatic pressure

$$P = \frac{df}{dS} = \frac{\sigma^2}{2\varepsilon\varepsilon_0}. \qquad (5)$$

Substituting known values, we find

$$P = 2.3\cdot 10^{-3}\ N\cdot m^{-2}. \qquad (6)$$

<u>Answer</u>. The electrostatic pressure acting on the ball is $P = 2.3\cdot 10^{-3}\ N\cdot m^{-2}$.

Problem 1.3.9

<u>Problem description</u>. A thin straight conductor is uniformly charged with a charge of $Q = 2\cdot 10^{-8}\ C$. Calculate the field strength at a point that is at a distance of $l = 20\ cm$ from the ends of the conductor and at a distance of $l_0 = 5\ cm$ from the middle of the conductor.

<u>Known quantities</u>: $Q = 2\cdot 10^{-8}\ C$, $l = 20\ cm$, $l_0 = 5\ cm$.

<u>Quantities to be calculated</u>: E.

Problem solution. We will denote the length of the conductor as L. At a distance of x from the middle of the conductor, an infinitesimal element of length dx an be distinguished. The charge of this element is

$$dQ = \frac{Q}{L} dx, \tag{1}$$

where
Q is the charge of the entire conductor;
L is the length of the conductor.

The electric field created by an elementary charge at the point indicated in the condition of the problem (we will refer to it as point C), can be calculated using the formula for the strength of a point charge

$$dE = \frac{dQ}{4\pi\varepsilon\varepsilon_0 d^2} = \frac{Qdx}{4\pi\varepsilon\varepsilon_0 d^2 L}, \tag{2}$$

where
d is the distance from the elementary charge dQ to the point C;
ε is the relative permittivity;
ε_0 is the electric constant.

The resulting electric field at a point C is equal to the algebraic sum of the components dE per direction of the normal passing through the middle of the conductor

$$E = \int \frac{Qdx}{4\pi\varepsilon\varepsilon_0 d^2 L} cos\,\beta, \tag{3}$$

where β is the angle between the normal and the direction of the electric field component $d\vec{E}$.

In this case, we can write the following geometric relations

$$d = \frac{l_0}{cos\,\beta}, \qquad x = l_0\, tg\beta, \qquad dx = \frac{l_0 d\beta}{cos^2 \beta}, \tag{4}$$

where l_0 is the distance from point C to the edge of the conductor.

The resulting electric field, taking into account formulas (4), is equal to

$$E = \frac{Q}{4\pi\varepsilon\varepsilon_0 L}\int \frac{l_0 \cos\beta \cos^2\beta\, d\beta}{l_0^2 \cos^2\beta} =$$

$$= \frac{Q}{4\pi\varepsilon\varepsilon_0 l_0 L}\int_{-\alpha}^{+\alpha} \cos\beta\, d\beta =$$

$$= \frac{Q}{4\pi\varepsilon\varepsilon_0 l_0 L} 2\sin\alpha . \qquad (5)$$

The value $\sin\alpha$ can be determined as a function of the ratio of the values L and l, namely $\sin\alpha = L/(2l)$. We use this functional relationship to calculate the resulting electric field

$$E = \frac{2QL}{4\pi\varepsilon\varepsilon_0 l_0 2lL} = \frac{Q}{4\pi\varepsilon\varepsilon_0 l_0 l}. \qquad (6)$$

Substituting known values, we find

$$E = 2\cdot 10^4\ V\cdot m^{-1}. \qquad (7)$$

Answer. The electric field at a point C is $E = 2\cdot 10^4\ V\cdot m^{-1}$.

Problem 1.3.10

Problem description. A ring of thin wire has a radius of $R = 10\ cm$. This ring is uniformly charged with a negative electric charge of $Q = -5\cdot 10^{-9}\ C$. Calculate the strength of the electrostatic field on the axis of the ring at point C, located at a distance of $l = 10\ cm$ from the center of the ring. Determine the distance l_0 from the center of the ring for which the electrostatic field strength will be maximum.

Known quantities: $R = 10\ cm$, $Q = -5\cdot 10^{-9}\ C$, $l = 10\ cm$.

Quantities to be calculated: E, l_0.

Problem solution. We will consider the element dx, of the ring, on which the electric charge dQ is located. The electric field at point C, specified in the condition of the problem, created by the charge dQ, is equal to

$$dE = \frac{dQ}{4\pi\varepsilon\varepsilon_0 x^2}, \quad (1)$$

where
ε is the relative permittivity;
ε_0 is the electric constant;
x is the distance from point C to element dx.

The electric field vector $d\vec{E}$ is directed along the line x, that connects the ring element dx with the point C. According to the principle of superposition, the electric field created by all the charge on the ring is equal to the vector sum of the vectors $d\vec{E}$. The vector $d\vec{E}$ can be decomposed into two components: dE_τ (tangential component, parallel to the direction from the point C to the center of the ring) and dE_n (normal component, perpendicular to the tangential component). Components dE_n from every two diametrically opposed elements dx cancel each other out, therefore

$$E = \int dE_\tau . \quad (2)$$

The relationship between quantities dE and dE_τ has the form

$$dE_\tau = dE \cos\alpha , \quad (3)$$

where α is the angle between the direction from point C to the center of the ring and to element dx.

The tangential component, taking into account (1) - (3), can be represented as

$$dE_\tau = \frac{dQ \cdot l}{4\pi\varepsilon\varepsilon_0 x^3}, \quad (4)$$

where
ε is the relative permittivity;
ε_0 is the electric constant;

l is the distance from the point C to the center of the ring (in addition, formula (4) takes into account that $cos\,\alpha = l/x$).

We will rewrite equation (2) taking into account (4)

$$E = \frac{l}{4\pi\varepsilon\varepsilon_0 x^3}\int dQ = \frac{Ql}{4\pi\varepsilon\varepsilon_0 x^3}. \tag{5}$$

The relationship between the radius R of the ring and the quantities x and l has the form

$$x = \sqrt{R^2 + l^2}. \tag{6}$$

Let us rewrite formula (5) taking into account formula (6)

$$E = \frac{Ql}{4\pi\varepsilon\varepsilon_0\left(R^2 + l^2\right)^{3/2}}. \tag{7}$$

Substituting known values, we get

$$E = 1.6\cdot 10^3\ V\cdot m^{-1}. \tag{8}$$

We express explicitly the functional dependencies $R = f_1(x,\alpha)$ and $l = f_2(x,\alpha)$

$$R = x\,sin\,\alpha, \qquad l = x\,cos\,\alpha. \tag{9}$$

In this case, the resulting electric field represented by formula (7) can be rewritten as follow

$$E = \frac{Q}{4\pi\varepsilon\varepsilon_0 R^2}\,sin^2\alpha\cdot cos\,\alpha. \tag{10}$$

In order to find the maximum value E, it is necessary to take the derivative with respect to the angle α and equate this derivative to zero

$$\frac{dE}{d\alpha} = \frac{Q}{4\pi\varepsilon\varepsilon_0 R^2}\left(cos^2\alpha\cdot 2\,sin\,\alpha - sin^3\alpha\right) = 0. \tag{11}$$

Formula (11) can be rewritten as

$$tg^2\alpha = 2. \tag{12}$$

Therefore, the fulfillment of condition $tg\alpha = \sqrt{2}$ leads to the fact that the distance l_0 between the point C and the center of the ring, at which the electric field created by the ring will be maximum, is equal to

$$l_0 = \frac{R}{tg\alpha} = \frac{R}{\sqrt{2}}. \tag{13}$$

Substituting known values, we obtain

$$l_0 = 7.1 \cdot 10^{-2}\ m. \tag{14}$$

Answer. The electric field on the axis of the ring at a point C is $E = 1.6 \cdot 10^3\ V \cdot m^{-1}$. The distance between the point C and the center of the ring, at which the electric field will be maximum, is $l_0 = 7.1 \cdot 10^{-2}\ m$.

Problem 1.3.11

Problem description. A ring of thin wire is uniformly charged with a charge of $Q = 2 \cdot 10^{-8}\ C$. The radius of the ring is $R = 5\ cm$. Determine the electric potential at the center of the ring and at a perpendicular to the plane of the ring at a point that is at a distance of $h = 10\ cm$ from the plane of the ring.

Known quantities: $Q = 2 \cdot 10^{-8}\ C$, $R = 5\ cm$, $h = 10\ cm$.

Quantities to be calculated: φ, φ_0.

Problem solution. A ring made of a conductor is charged, so there is an electric field around the ring, at each point of which the electric potential can be determined. We consider the point B, which is perpendicular to the plane of the ring at a distance h from its plane. The field of a charged ring can be considered as a field formed by the superposition of fields from the charges of individual points of the ring. In this case, the ring can be divided into infinitesimal segments dx. The ring is made of thin wire, so it makes sense to consider the linear charge density τ. The ring is uniformly charged, so the linear charge density is

$$\tau = \frac{Q}{2\pi R}, \tag{1}$$

where
R is the radius of the ring;
Q is an electrical charge placed on the ring.

The charge of an elementary segment dx is equal to

$$dQ = \tau\, dx. \tag{2}$$

Since the elementary charge is concentrated on a very small segment of the length dx, of the conductor, it can be considered as a point charge. We will denote the potential at point B of the electric field formed by the the point charge dQ, with the symbol $d\varphi$. The elementary potential is

$$d\varphi = \frac{dQ}{4\pi\varepsilon\varepsilon_0 l}, \tag{3}$$

where
l s the distance from the point B to the elementary segment dx,
ε is the relative permittivity of the environment (for the conditions of the problem, we can write $\varepsilon \approx 1$);
ε_0 is the electric constant.

Joint consideration of equations (1) – (3) allows us to write

$$d\varphi = \frac{Qdx}{8\pi^2\varepsilon\varepsilon_0 lR}. \tag{4}$$

Electric potential is a scalar quantity. According to the principle of superposition for potentials, the potential of the resulting electric field is equal to the algebraic sum of the potentials formed at a given point by individual point charges. Hence

$$\varphi = \frac{Q}{8\pi^2\varepsilon\varepsilon_0 lR}\int_0^{2\pi R} dx. \tag{5}$$

After integrating and performing some algebraic transformations, we can write the following formula for the potential

$$\varphi = \frac{Q}{4\pi\varepsilon\varepsilon_0 l}. \quad (6)$$

Functional dependency $l = f(R,h)$ can be written in the following form

$$l = \sqrt{R^2 + h^2}. \quad (7)$$

Let's rewrite (6) taking into account (7)

$$\varphi = \frac{Q}{4\pi\varepsilon\varepsilon_0 \sqrt{R^2 + h^2}}. \quad (8)$$

Substituting known values, we have

$$\varphi = 1.608 \cdot 10^3 \ V. \quad (9)$$

The electric potential φ_0 at a point O (the center of the ring) can be calculated using formula (8), provided that the value h is zero ($h = 0$)

$$\varphi_0 = \frac{Q}{4\pi\varepsilon\varepsilon_0 R}. \quad (10)$$

Substituting known values, we get

$$\varphi_0 = 3.6 \cdot 10^3 \ V. \quad (11)$$

Answer. The electric potential at a point perpendicular to the plane of the ring and the electric potential at the center of the ring are equal, respectively $\varphi = 1.608 \cdot 10^3 \ V$ and $\varphi_0 = 3.6 \cdot 10^3 \ V$.

Problem 1.3.12

Problem description. An electron without an initial velocity passed through a potential difference of $U_0 = 10 \ kV$ and flew into the space between the plates of a flat capacitor charged to a potential difference of $U = 100 \ V$, along a line parallel to the plates. The distance between the plates is $d = 2 \ cm$. The length of the capacitor plates in the direction of electron motion is $l = 20\ cm$. Calculate the distance X, that the electron will move from its original direction when it hits the screen. The screen is located perpendicular to the initial direction of electron motion at a distance of $L = 1\ m$ from the edge of the capacitor plates.

Known quantities: $U_0 = 10\ kV$, $U = 100\ V$, $d = 2\ cm$, $l = 20\ cm$, $L = 1\ m$.

Quantities to be calculated: X.

Problem solution. The curvilinear motion of an electron inside a capacitor can be decomposed into two simple movements: 1) movement with a constant speed υ_0 by inertia along a line parallel to the plates of the capacitor (the electron received the speed υ_0 under the action of a potential difference U_0, having passed it to the capacitor); 2) accelerated movement in the vertical direction to a positively charged plate under the action of a constant force that acts on an electron in the electrostatic field of a capacitor.

Outside the capacitor, the electron moves uniformly at a constant speed υ. For the desired distance X we can write the following equality

$$X = h_1 + h_2, \qquad (1)$$

where

h_1 this is the distance, counted in the vertical direction, by which the electron will move during its movement in the capacitor;

h_2 is the distance between the point on the screen where the electron would hit, moving in the direction of its original velocity υ_0, after leaving the capacitor, and the point on the screen where the electron actually hits.

We use the uniformly accelerated path formula to calculate the magnitude h_1

$$h_1 = \frac{at^2}{2}, \qquad (2)$$

where

a is the acceleration that an electron has received under the action of a force from an electrostatic field;

t is the flight time of an electron in a capacitor.

According to Newton's second law, we can write

$$a = \frac{F}{m}, \qquad (3)$$

where
F is the force with which an electrostatic field acts on an electron;
m is the mass of the electron.

In addition, for a force F we can write the following equation

$$F = eE = e\frac{U}{d}, \tag{4}$$

where
e is the charge of the electron;
U is the potential difference between the plates of a capacitor;
d is the distance between the plates of the capacitor;
E is the electrostatic field strength of the capacitor.

We can calculate the flight time t of an electron inside a capacitor using the uniform motion formula

$$l = \upsilon_0 t, \tag{5}$$

or

$$t = \frac{l}{\upsilon_0}, \tag{6}$$

where l is the length of the capacitor in the direction of the electron's initial motion.

We will determine the initial speed of the electron from the condition of equality of the work on moving the electron, performed by the electrostatic field of the capacitor, and the kinetic energy received by the electron

$$eU_0 = \frac{m\upsilon_0^2}{2}, \tag{7}$$

where U_0 is the accelerating potential difference.

In this case

$$\upsilon_0^2 = \frac{2eU_0}{m}. \tag{8}$$

Substituting formulas (3), (4), (6) and (8) into formula (2), we obtain an equation for the quantity h_1

$$h_1 = \frac{Ul^2}{4dU_0}. \tag{9}$$

The features of the electron trajectory can be described using the following geometric relation for the similarity of the triangle of velocities and the triangle of distances

$$\frac{\upsilon_1}{\upsilon_0} = \frac{h_2}{L}, \tag{10}$$

where

υ_1 is the velocity of the electron in the vertical direction at the moment of departure from the capacitor (perpendicular to the plates of the capacitor);

L is the distance from the edge of the capacitor plates to the screen.

We will rewrite equation (10)

$$h_2 = \frac{\upsilon_1 L}{\upsilon_0}. \tag{11}$$

Speed υ_1 can be calculated using equation

$$\upsilon_1 = at, \tag{12}$$

then, using equations (4), (6) and (7), we obtain

$$\upsilon_1 = \frac{eUl}{md\upsilon_0}. \tag{13}$$

We will write the equation for h_2, using formulas (8), (11) and (13)

$$h_2 = \frac{lUL}{2dU_0}. \tag{14}$$

Then equation (1) can be rewritten in the following form

$$X = \frac{Ul^2}{4dU_0} + \frac{ULl}{2dU_0} = \frac{Ul}{2dU_0}\left(\frac{l}{2} + L\right). \tag{15}$$

Let's substitute numerical values

$$X = 5.5 \cdot 10^{-2}\ m. \quad (16)$$

Answer. The distance the electron will move from its original direction is $X = 5.5 \cdot 10^{-2}\ m$.

1.4. Level 1 problems

1.4.1. Two charged balls of the same radius and weight, suspended on threads of the same length, are lowered into kerosene. The angle of divergence of the balls in air and in kerosene is the same. Calculate the density of the material of the balls.

1.4.2. Two spherical dust particles each have one extra electron. The electrostatic repulsion force of dust particles balances the force of their mutual gravitational attraction. The volume of one grain of dust is 4 times the volume of another grain of dust. The density of the dust particle material is $\rho = 2700\ kg \cdot m^{-3}$. Determine the radii of the dust particles.

1.4.3. Calculate the electric field strength at a point that is located in the middle between point charges $q_1 = 8 \cdot 10^{-9}\ C$ and $q_2 = -6 \cdot 10^{-9}\ C$. The distance between charges is $r = 10\ cm$. The relative permittivity of the medium is $\varepsilon = 1$.

1.4.4. At the vertices of a square with a side equal to a are the same charges q. A charge q' of opposite sign is placed in the center of the square so that the force acting on any charge would be equal to zero. Calculate the amount of charge q'.

1.4.5. In the elementary theory of the hydrogen atom, it is assumed that the electron revolves around the nucleus in a circular orbit. Determine the speed of the electron if the radius of its orbit is $r = 53 pm$, as well as the frequency of rotation of the electron.

1.4.6. Point charges q, $2q$, $3q$, $4q$, $5q$, $6q$ ($q = 1 \cdot 10^{-7}\ C$) are located at the vertices of a regular hexagon with a side of $a = 10\ cm$. Calculate the force that acts on a charge q, located in the plane of the hexagon and equidistant from its vertices.

1.4.7. Two identical conducting charged balls are at a distance of $r = 60\ cm$. The repulsion force of the balls is $F_1 = 70\ \mu N$. After the balls were brought into contact and moved away from each other by the same distance, the repulsive force increased and became equal to $F_2 = 160\ \mu N$. Calculate the charges q_1 and q_2, that were on the balls before they touched. Consider the diameter of the balls to be much smaller than the distance between them.

1.4.8. A thin filament with a length of $l = 20\ cm$ is uniformly charged with a linear density of $\tau = 10\ nC \cdot m^{-1}$. At a distance of $a = 10\ cm$ from the thread, against its middle, there is a point charge of $q = 1\ nC$. Calculate the force acting on this charge from the side of the charged thread.

1.4.9. A thin endless thread is bent at an angle of 90^0. The electric charge is evenly distributed along the filament with a linear density of $\tau = 1\ \mu C \cdot m^{-1}$. Determine the force acting on a point charge $q = 0.1\ \mu C$, located on the continuation of one of the sides and removed from the top of the corner at a distance of $a = 50\ cm$.

1.4.10. A thin ring with a radius of $R = 10\ cm$ carries a uniformly distributed charge of $q = 0.1\ \mu C$. On the perpendicular to the plane of the ring, drawn from its middle, there is a point charge $q_1 = 10\ nC$. Calculate the force that acts on a point charge q from the side of a charged ring, provided that this point charge is removed from the center of the ring by a distance: 1) $l_1 = 20\ cm$; $l_2 = 2\ m$.

1.4.11. A charge with a linear density of $\tau = 1\ nC \cdot m^{-1}$ is uniformly distributed along a thin ring with a radius of $R = 10\ cm$. In the center of the ring is a charge equal to $q = 0.4\ \mu C$. Determine the force that stretches the ring. Ignore the interaction of the charges of the ring.

1.4.12. The electric field is created by two point charges $q_1 = 10\ nC$ and $q_2 = -20\ nC$, located at a distance of $d = 20\ cm$ from each other. Calculate the electric field at point A. Point A is removed from the first charge at a distance of $r_1 = 30\ cm$ and from the second charge at a distance of $r_2 = 50\ cm$.

1.4.13. A thin rod with a length of $l = 12\ cm$ is charged with a linear density of $\tau = 200\ nC \cdot m^{-1}$. Calculate the electric field at point C. Point C is at a distance of $r = 5\ cm$ from the rod against its middle.

1.4.14. A charge with a linear density of $\tau = 10\ nC \cdot m^{-1}$ is uniformly distributed on a segment of a thin straight conductor. Calculate the potential created by this charge at a point located on the axis of the conductor, and remote from the nearest end of the segment at a distance equal to the length of this segment.

1.4.15. A metal ball with a radius of $R = 10\ cm$ is charged to a potential of $\varphi_1 = 300\ V$. Determine the potential of this ball in two cases: 1) after it is surrounded by a spherical conductive shell with a radius of $R_2 = 15\ cm$ and for a short time connected to it by a conductor; 2) after the ball is surrounded by a spherical conductive grounded shell with a radius of $R_2 = 15\ cm$.

1.4.16. Two infinite parallel planes are at a distance of $d = 1\ cm$ from each other. The planes carry charges uniformly distributed over the surfaces with densities equal to $\sigma_1 = 0.2\ \mu C \cdot m^{-2}$ and $\sigma_2 = 0.5\ \mu C \cdot m^{-2}$, respectively. Calculate the potential difference between these planes.

1.4.17 A metal ball with a diameter of $d = 2\ cm$ is negatively charged up to a potential of $\varphi = 150\ V$. Determine how many electrons are on the surface of the ball.

1.4.18 The electric field is created by an infinite uniformly charged plane with a surface charge density of $\sigma = 2\ \mu C \cdot m^{-2}$. A point electric charge $q = 10\ nC$ moves in this field at a distance of $l = 20\ cm$ along a straight line making an angle of $\alpha = 60^0$ with the plane. Determine the work done by the field forces to move this charge.

1.4.19 The potential difference between the cathode and anode of the electron tube is $U = 90\ V$. The distance between the cathode and the anode is $r = 1\ mm$. Determine the acceleration with which the electron moves from the cathode to the anode. Calculate the speed of the electron at the moment of impact on the anode. Calculate the time it takes for an electron to travel the distance from the cathode to the anode. The electric field between the cathode and the anode is considered to be approximately uniform.

1.4.20 A positively charged particle, the charge of which is equal to the elementary charge, has passed the accelerating potential difference, which is equal to

$U = 60\ kV$, and moves to the nucleus of the lithium atom, the charge of which is equal to three elementary charges. Calculate the minimum distance that an electron can approach the nucleus. The initial distance from the electron to the nucleus can be considered almost infinitely large, and the mass of the particle can be assumed to be negligible compared to the mass of the nucleus.

1.5. Answers to problems

1.4.1. $\rho = 1.6 \cdot 10^3\ kg \cdot m^{-3}$.

1.4.2. $r_1 = 4 \cdot 10^{-5}\ m$; $r_2 = 9 \cdot 10^{-5}\ m$.

1.4.3. $E = 5.04 \cdot 10^4\ V \cdot m^{-1}$.

1.4.4. $q' \approx 0.96q$.

1.4.5. $\upsilon = 2.19 \cdot 10^5\ m \cdot s^{-1}$; $n = 6.59 \cdot 10^{14}\ s^{-1}$.

1.4.6. $F = 5.4 \cdot 10^{-2}\ N$.

1.4.7. $q_1 = 1.4 \cdot 10\ C$; $q_2 = 2 \cdot 10^{-8}\ C$.

1.4.8. $F = 1.27 \cdot 10^{-6}\ N$.

1.4.9. $F = 4.03 \cdot 10^{-3}\ N$.

1.4.10. $F_1 = 1.6 \cdot 10^{-4}\ N$; $F_2 = 2.25 \cdot 10^{-6}\ N$.

1.4.11. $F = 3.5 \cdot 10^{-5}\ N$.

1.4.12. $E = 2.8 \cdot 10^2\ V \cdot m^{-1}$.

1.4.13. $E = 5.57 \cdot 10^4\ V \cdot m^{-1}$.

1.4.14. $\varphi = 6.24 \cdot 10^1\ V$.

1.4.15. 1) $\varphi_2 = 2 \cdot 10^2\ V$; 2) $\varphi_2 = 1 \cdot 10^2\ V$.

1.4.16. $\Delta\varphi = 1.7 \cdot 10^2\ V$.

1.4.17. $N = 1.04 \cdot 10^9$.

1.4.18. $A = 1.96 \cdot 10^{-6}\ J$.

1.4.19. $a = 1.58 \cdot 10^{16}\ m \cdot s^{-2}$; $\upsilon = 5.63 \cdot 10^6\ m \cdot s^{-1}$; $t = 3.56 \cdot 10^{-10}\ s$.

1.4.20. $r_{min} = 7.2 \cdot 10^{-14}\ m$.

CHAPTER 2. CONDUCTORS AND DIELECTRICS IN ELECTROSTATIC FIELD

2.1. Basic formulas

On the surface of an uncharged conductor placed in an electrostatic field, charges of both signs are induced in equal amounts, and the electric field inside the conductor is zero

$$E_0 = 0. \tag{2.1.1}$$

Boundary conditions on the conductor surface:
a) tangential components of the electric field E_τ and electric induction D_τ are equal to zero

$$D_\tau = \varepsilon\varepsilon_0 E_\tau = 0, \tag{2.1.2}$$

where
ε is the relative permittivity of the medium in which the conductor is located;
ε_0 is the electric constant;
b) the components of the electric field and electric induction normal to the surface of the conductor are equal to the surface charge density σ

$$D_n = \varepsilon\varepsilon_0 E_n = \sigma. \tag{2.1.3}$$

Dielectrics in an electrostatic field are polarized in this field. The electrical properties of a dielectric are characterized by **relative permittivity** ε и and absolute dielectric susceptibility κ. The concept of permittivity was introduced by the English experimental physicist and chemist Faraday (Michael Faraday 1791 – 1867). The relationship between these quantities is illustrated by the formula

$$1 + \frac{\kappa}{\varepsilon_0} = 1 + k_e, \tag{2.1.4}$$

where $k_e = \kappa / \varepsilon_0$ is the **relative dielectric susceptibility**.

The relationship between the surface density σ' of bound charges at the conductor-dielectric interface and the surface density σ of free charges on the conductor has the form

$$\sigma' = \sigma \frac{\varepsilon - 1}{\varepsilon}. \tag{2.1.5}$$

The surface density of bound charges at the interface between two dielectrics with relative permittivities ε_1 and ε_2 is

$$\sigma'' = \sigma\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right). \qquad (2.1.6)$$

The **law of refraction of field lines** at the interface between two isotropic dielectrics has the form

$$\frac{tg\alpha}{tg\beta} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{E_{n2}}{E_{n1}}. \qquad (2.1.7)$$

Relation (2.1.7) can be formulated as follows: the ratio of the tangents of the angles of incidence and refraction is equal to the ratio of the relative dielectric permittivities of the dielectric, from where the lines of force come out and the dielectric, where the lines of force enter.

The electric **capacitance** (capacitance) **of conductor** is a physical quantity that is measured by the amount of charge Q (in air without other conductors) that must be imparted to the conductor in order to change its potential per unit of potential. The term capacitance was introduced by the Italian physicist, chemist and physiologist Allesandro Volta (Alessandro Giuseppe Antonio Anastasio Volta 1745 – 1827). Electric capacity C can be expressed as the ratio of charge Q to potential φ

$$C = \frac{Q}{\varphi}. \qquad (2.1.8)$$

The electric capacitance of conductors does not depend on the type of substance from which the conductor is made, but significantly depends on the size and shape of the conductor. In addition, the electrical capacity depends on the electrical properties of the environment and the influence of other conductors that are nearby.

The **capacitance of a ball** with a radius R hat is far from other conductors and is in a medium with a relative permittivity ε, is equal to

$$C = 4\pi\varepsilon\varepsilon_0 R. \qquad (2.1.9)$$

Capacitance of flat, cylindrical and spherical capacitors

$$C = \frac{\varepsilon\varepsilon_0 S}{d}, \qquad C = \frac{2\pi\varepsilon\varepsilon_0 l}{ln\left(\frac{R_2}{R_1}\right)}, \qquad C = \frac{4\pi\varepsilon\varepsilon_0}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}, \tag{2.1.10}$$

where
S is the area of the plate;
d is the distance between the plates of a flat capacitor;
R_1 is the radius of the inner wall of the spherical capacitor;
R_2 is the radius of the outer wall of the spherical capacitor;
l is the length of the cylindrical capacitor.

Practical capacitors, with values ranging from 10^{-2} F to 10^{+5} F, often consist of two long strips of metal foil, separated by long strips of dielectric, rolled up like a “Swiss roll”.

Batteries of parallel and series-connected capacitors have the **electric capacity**, respectivly

$$C = \sum_{i=1}^{n} C_i ; \qquad C = \frac{1}{\sum_{i=1}^{n} C_i}. \tag{2.1.11}$$

The energy W of a point charge in electrostatic field, is equal to the product of the charge Q and the field potential φ at the point where the charge is located

$$W = Q\varphi . \tag{2.1.12}$$

The energy of the charge Q, which is in the field of other point charges Q_1, Q_2, … , is equal to the product of the charge Q and the potential of the resulting field of all other charges at the point where the charge Q is located

$$W = Q\left(\frac{Q_1}{4\pi\varepsilon\varepsilon_0 l_1} + \frac{Q_2}{4\pi\varepsilon\varepsilon_0 l_2} + ...\right), \tag{2.1.13}$$

where l_i is the distance from charge Q to charge Q_i.

The energy of n charges is given by equation

$$W = \frac{1}{2}\sum_{k=1}^{n} W_k , \tag{2.1.14}$$

where W_k is the energy of charge Q_k in the field of all other charges with indices: $k = 1, 2, \ldots, (k-1), (k+1), \ldots, n$.

Charged insulated conductor energy is

$$W = \frac{Q^2}{2C} = \frac{C\varphi^2}{2} = \frac{Q\varphi}{2}, \tag{2.1.15}$$

where Q, φ, C are the charge, potential and capacitance of the conductor, respectively.

Charged capacitor energy is [2, p. 654]

$$W = \frac{Q^2}{2C} = \frac{CU^2}{2} = \frac{QU}{2}, \tag{2.1.16}$$

where Q, U, C are the charge, voltage and capacitance of the capacitor, respectively.

Volumetric energy density ω, i.e. the energy of the electrostatic field, which falls on a unit volume, is equal to

$$\omega = \frac{\varepsilon\varepsilon_0 E^2}{2} = \frac{D^2}{2\varepsilon\varepsilon_0}, \tag{2.1.17}$$

where
E is the electric field in a medium with permittivity ε;
D is the electrical induction of the electrostatic field.

2.2. Problem-solving framework

First of all, we will consider the case when a conductor is placed in the field formed by a charged body. Under the action of field forces, at the end of such a conductor, which is closer to the electrified body, a charge of the opposite sign will appear, and at the other end of the conductor, a charge of the same name will appear. The phenomenon of redistribution of charges in a conductor under the action of an external field is called **electrostatic induction**.

The electric field inside an uncharged conductor placed in an electrostatic field is zero

$$E = 0. \tag{2.2.1}$$

It follows that the potential of all points of the conductor and the space inside it, if the conductor is hollow, will be the same. The equality to zero of the electric field inside the conductor is used in solving problems to determine the distribution of the induced charge on the surface of the conductor.

Due to electrostatic induction, conductors attract free electric charges that are located nearby. In most problems where it is necessary to calculate the force of such an attraction, it is not necessary to determine the exact distribution of induced charges. The fact is that the charges induced on the conductor form outside the conductor the same field that would be formed by the "electric image" of the free charge in the conductor.

"**Electric image**" of a given system of charges A relative to some conductive surface S is called a system of charges B, which is located on the other side of the surface S. he action of the system B is identical to the action of charges on the surface S, which are induced by the system A. The "electric image" of a charge or a system of charges in a conductor can be found by writing down the condition of equipotentiality of the conductor surface. However, to determine both the magnitude and position of the charge equivalent to the charge induced on the surface of the conductor, the equipotentiality condition alone is often not enough. Therefore, based on the specific conditions of the problem, it is necessary to use other additional conditions.

The solution of this type of problem should be accompanied by a separate consideration of two cases: a) the conductor is grounded; b) the conductor is insulated.

We will assume that in case a) the charge and potential of the conductor are Q and $\varphi = 0$, respectively, and in case b) the charge of the conductor and its potential are: $Q = 0$, φ. For the electrical capacity of the conductor, we will use the symbol C, and for the surface charge density in areas 1 and 2 – the symbols σ_1 and σ_2, respectively. Our goal is to determine the relationship between the problems of these types.

Suppose that the problem of type a) is solved and the values Q and σ_1 are determined, in addition, the potential is equal to zero $\varphi = 0$. The solution of problem b) can be obtained in the case when the charge Q, distributed over the conductor according to condition $\varphi = 0$, is added to the charge $-Q$, distributed over the conductor as if the conductor were isolated and an external electrostatic field would not act on it. In this case, the surface charge density in different areas of the conductor surface is σ. Then the desired surface charge density is

$$\sigma_2 = \sigma_1 + \sigma . \qquad (2.2.2)$$

The total charge of the conductor in this case is equal to zero $Q = 0$, and the potential of the conductor is equal to

$$\varphi = -\frac{Q}{C} . \qquad (2.2.3)$$

Now suppose that problem b) has been solved. In this case, the surface charge density σ_2 and potential φ are known quantities, and the condition

$$\iint \sigma \, dS = 0 . \qquad (2.2.4)$$

In this case, it is possible to determine the magnitude of the charge $Q = \varphi C$ and produce a conditional charge distribution $-Q$. We assume that the surface charge density in this case will be equal to $-\sigma$. hen the potential is zero $\varphi = 0$, and for the surface charge density σ_1 we can write

$$\sigma_1 = \sigma_2 - \sigma . \qquad (2.2.5)$$

When calculating the electrical capacity of a certain conductor A, which is located close to several other conductors B, C, D, …, the concept of partial electrical capacity is used. **Partial capacitance** between conductors A and B is the absolute value of the ratio of the charge on the conductor A to the potential difference between the conductors A and B provided that all other conductors C, D, … have the same potential. Information on the partial capacitance of each pair of conductors of the entire system allows us to calculate the true capacitance of any conductor by solving a system of linear equations.

When calculating the capacitance of capacitors or a system formed by two conductors, it is convenient to use a formula that expresses the relationship between the potential gradient and the electric field at some point in space

$$E = -\frac{d\varphi}{dl} . \qquad (2.2.6)$$

The electric field must be represented as a function of distance and charges, which are the sources of the field. After that, using formula (2.2.6), we can find the potential difference between the indicated conductors. Then the electrical capacity is equal to the ratio of the charge of the conductor to the potential difference.

2.3. Problem-solving examples

Problem 2.3.1

Problem description. The point charge $Q = 2 \cdot 10^{-9}\ C$ is located at a distance $l = 3\ cm$ from the metal plate, which is connected to the ground. Calculate the surface charge density on the plate at two points: 1) at the point that is located at the minimum distance from the charge Q; 2) at a point that is located at a distance $l_1 = 5\ cm$ from the charge Q. In addition, it is necessary to determine the total charge induced on the surface of the plate.

Known quantities: $Q = 2 \cdot 10^{-9}\ C$, $l = 3\ cm$, $l_1 = 5\ cm$.

Quantities to be calculated: σ_1, σ_2, Q'.

Problem solution. The charge Q induces negative charges on the surface of the conductor, the surface density of which decreases symmetrically, starting from the maximum value at a point as close as possible to a point charge Q. At any point near the surface of the plate, the field strength is equal to the sum of the field strengths formed by the charge Q and the charge on the surface of the conductor near the point we have chosen. The electric field inside the conductor is zero.

1) We will denote the surface density of the induced charge at the point closest to the charge Q, as σ_1. In this case, the following equation can be written

$$\frac{Q}{4\pi\varepsilon\varepsilon_0 l^2} + \frac{\sigma_1}{2\pi\varepsilon\varepsilon_0} = 0, \tag{1}$$

where
ε is the relative permittivity;
ε_0 is the electric constant;
l is the distance from the charge Q to the metal plate.

Let's rewrite equation (1)

$$\sigma_1 = -\frac{Q}{2\pi l^2}. \tag{2}$$

Numerically,

$$\sigma_1 = -3 \cdot 10^{-6}\ C \cdot m^{-2}. \tag{3}$$

2) We will reason in the same way as in case 1) and for a point on the surface of the plate, located at a distance of l_1 from the charge Q we write the equation

$$\frac{Q}{4\pi\varepsilon\varepsilon_0 l_1^2}\cos\alpha + \frac{\sigma_2}{2\varepsilon\varepsilon_0} = \frac{Ql}{4\pi\varepsilon\varepsilon_0 l_1^3} + \frac{\sigma_2}{2\varepsilon\varepsilon_0} = 0, \tag{4}$$

where

$\cos\alpha = \frac{l}{l_1}$;

σ_2 is the surface density of the induced charge at point A (the distance between point A and the charge Q is l_1).

Let's rewrite equation (4)

$$\sigma_2 = -\frac{Ql}{2\pi l_1^3}. \tag{5}$$

Substituting known values, we get

$$\sigma_2 = -7.64\cdot 10^{-8}\ C\cdot m^{-2}. \tag{6}$$

Now we will calculate the total charge induced on the surface of the plate. To do this, we consider an infinitely narrow ring of radius x and width dx centered at a point on the surface of the plate, which is at a minimum distance from the charge Q. The area of this ring is

$$dS = 2\pi x dx. \tag{7}$$

The induced charge distributed in this ring is

$$dQ' = \sigma\, dS = -\frac{Qlxdx}{\sqrt{\left(l^2 + x^2\right)^3}}. \tag{8}$$

Then the total induced charge of the entire metal plate is determined by the equation

$$Q' = -\frac{Ql}{2}\int_0^\infty \frac{2xdx}{\sqrt{\left(l^2 + x^2\right)^3}} = -Q. \tag{9}$$

Substituting known values, we find

$$Q' = -2 \cdot 10^{-9}\ C. \tag{10}$$

Answer. The surface charge densities for cases 1) and 2) are, respectively $\sigma_1 = -3 \cdot 10^{-6}\ C \cdot m^{-2}$ and $\sigma_2 = -7.64 \cdot 10^{-8}\ C \cdot m^{-2}$. The total induced charge of the entire metal plate is $Q' = -2 \cdot 10^{-9}\ C$.

Problem 2.3.2

Problem description. Calculate the force with which the charge $Q = 3 \cdot 10^{-8}\ C$ is attracted to a conducting ball with radius $R = 0.5\ cm$. The ball is connected to the ground. The charge Q is located at a distance of $l = 70\ cm$ from the center of the ball.

Known quantities: $Q = 3 \cdot 10^{-8}\ C$, $R = 0.5\ cm$, $l = 70\ cm$.

Quantities to be calculated: F.

Problem solution. Due to electrostatic induction, negative charges appear on the surface of the conducting ball. First of all, we will find the "electrical image" of the charge Q in the conducting ball, i.e. we determine the magnitude and position of the charge equivalent to the charges induced on the surface of the ball. To do this, connect the point A, where the charge Q is located with the center C of the ball. The corresponding segment will be denoted by the symbol AC. A consequence of symmetry is the fact that the charge equivalent to the charges induced on the surface of the ball is located on a straight line AC, for example, at some point A'. We will denote the magnitude of this charge by the symbol Q'.

Since the surface of the ball is an equipotential surface with a potential that is zero (the ball is connected to the ground), then for any point B on the surface of the ball the following relation is true

$$\frac{Q}{4\pi\varepsilon\varepsilon_0 r} + \frac{Q'}{4\pi\varepsilon\varepsilon_0 r'} = 0, \tag{1}$$

where $r = AB$, $r' = A'B$.

Due to the fact that the ball is connected to the ground, the potential at all points inside the ball (and, therefore, in the center) is equal to zero. This potential is equal to the algebraic sum of the potentials due to the free charge Q and the charges induced on the surface of the ball. Therefore, for the center of the ball, we can write the following relation

$$\frac{Q}{4\pi\varepsilon\varepsilon_0 l} + \int \frac{\sigma\, dS}{4\pi\varepsilon\varepsilon_0 R} = 0, \tag{2}$$

where
ε is the relative permittivity;
ε_0 is the electric constant;
σ is the surface density of the induced charge on the surface element of the ball dS;
l is the distance from the charge Q to the center of the ball C;
R is the radius of the sphere.

In this case, the magnitude of the induced charge

$$\int\limits_{(S)} \sigma\, dS = -\frac{R}{l} Q \tag{3}$$

equal to the equivalent charge Q'

$$Q' = -\frac{R}{l} Q. \tag{4}$$

Analysis of equations (1) and (4) allows us to write the following relationship

$$\frac{Q}{4\pi\varepsilon\varepsilon_0 r} - \frac{\left(\frac{R}{l} Q\right)}{4\pi\varepsilon\varepsilon_0 r'} = 0 \tag{5}$$

or

$$\frac{r}{r'} = \frac{R}{l}. \tag{6}$$

A surface that satisfies this condition is the surface of a ball centered on a line AC. Indeed, we draw a certain plane through AC and arrange the coordinate axes XY in it so that the origin coincides with the point A, and the axis X is directed along the straight line AC. Then we can write the following system of equations

$$r^2 = x^2 + y^2 \tag{7}$$

$$r'^2 = (x-a)^2 + y^2, \tag{8}$$

where
x and y are the coordinates of the point B;
$a = AA'$.

Substituting equations (7) and (8) into equation (6), we obtain

$$\frac{(x-a)^2 + y^2}{x^2 + y^2} = \frac{R^2}{l^2}. \tag{9}$$

And equation (9) is the equation of a circle centered on the axis X.

If we choose a point A' so that $a = R^2/l$, i.e. between points A and C at a distance of R^2/l from the center of the ball, then equation (9) will be the equation of a circle with a radius R centered at the point C.

Thus, the equipotential surface with a potential equal to zero, from the charges Q and $Q' = -RQ/l$ (which are placed at points A and A', moreover $A'C = R^2/l$), coincides with the given surface of the ball.

Therefore, the value of the charge $Q' = -RQ/l$ at point A' will be the "electric image" of the free charge Q in the conducting ball. Now the force of interaction between a free charge Q and a conducting ball can be easily calculated using Coulomb's law

$$F = -\frac{Q \cdot \left(\frac{R}{l} Q\right)}{4\pi\varepsilon\varepsilon_0 \left(l - \frac{R^2}{l}\right)^2} = -\frac{Q^2 R l}{4\pi\varepsilon\varepsilon_0 \left(l^2 - R^2\right)^2}. \tag{10}$$

If the radius of the ball satisfies the inequality $R << l$, then equation (10) can be simplified

$$F = -\frac{Q^2 R}{4\pi\varepsilon\varepsilon_0 l^3}. \tag{11}$$

Substituting known values, we have

$$F = 1.2 \cdot 10^{-7}\ N. \tag{12}$$

Answer. The force with which the charge is attracted to the conducting ball is $F = 1.2 \cdot 10^{-7}\ N$.

Problem 2.3.3

Problem description. Near the glass surface ($\varepsilon_2 = 7$) the electric field in air is $E_1 = 2 \cdot 10^4\ V \cdot m^{-1}$ and forms an angle of $\alpha_1 = 30^0$ with the direction of the normal to the surface. Calculate: 1) angle between the electric field and the normal in the glass; 2) electric field vector modulus in glass; 3) density of bound charges at the interface between air and glass.

Known quantities: $\varepsilon_2 = 7$, $E_1 = 2 \cdot 10^4\ V \cdot m^{-1}$, $\alpha_1 = 30^0$.

Quantities to be calculated: α_2, E_2, σ_2'.

Problem solution. When passing from one dielectric to another, the electric field lines are refracted so that

$$\frac{tg\alpha_2}{tg\alpha_1} = \frac{\varepsilon_2}{\varepsilon_1}, \qquad (1)$$

where

ε_1 and ε_2 are the relative permittivities of the first and second dielectric, respectively;

α_1 and α_2 are the angles that the vectors $\vec{E}_1$ and $\vec{E}_2$ of the electrostatic field in the first and second dielectrics, respectively, make with the normals to the interface.

Let us rewrite equation (1) in the form $f_1(\alpha_2) = f_2(\alpha_2, \varepsilon_1, \varepsilon_2)$

$$tg\alpha_2 = \frac{\varepsilon_2}{\varepsilon_1} tg\alpha_1. \qquad (2)$$

Numerically

$$tg\alpha_2 = 4.0418, \qquad \alpha_2 = 76°06'. \qquad (3)$$

The tangential components of the electrostatic field are the same on both sides of the dielectric interface. Therefore, the electrostatic field in the glass will be equal to

$$E_2 = \frac{E_{2\tau}}{\sin\alpha_2} = \frac{E_{1\tau}}{\sin\alpha_2} = \frac{E_1 \sin\alpha_1}{\sin\alpha_2}, \qquad (4)$$

where $E_{1\tau}$ and $E_{2\tau}$ are the tangential components of the electrostatic field in the first and second dielectrics, respectively.

Substitute numerical values in equation (4)

$$E_2 = 1.03 \cdot 10^4 \ V \cdot m^{-1}. \qquad (5)$$

The surface charge density is numerically equal to the normal component of the dielectric polarization. The polarization of the glass is

$$\vec{P}_2 = \kappa \vec{E}_2 = \varepsilon_0(\varepsilon_2 - 1)\vec{E}_2, \qquad (6)$$

where

κ is the dielectric susceptibility of the second dielectric;

ε_0 is the electric constant;

The normal component of the polarization vector for the second dielectric is

$$P_{2n} = P_2 \cos\alpha_2. \qquad (7)$$

In this case, for the density of bound charges, we can write the following expression

$$\sigma_2' = P_{2n} = \varepsilon_0(\varepsilon_2 - 1)E_2 \cos\alpha_2 = \frac{\varepsilon_0(\varepsilon_2 - 1)E_1 \sin\alpha_1 \cos\alpha_2}{\sin\alpha_2} =$$

$$= \frac{\varepsilon_0(\varepsilon_2 - 1)E_1 tg\alpha_1 \cos\alpha_1}{tga_2} = \frac{\varepsilon_0(\varepsilon_2 - 1)E_1\varepsilon_1 \cos\alpha_1}{\varepsilon_2}. \qquad (8)$$

Substituting known values, we get

$$\sigma_2' = 3.8 \cdot 10^{-7} \ C \cdot m^{-2}. \qquad (9)$$

Answer. The angle between the electric field strength and the normal in the glass is $\alpha_2 = 76°06'$. The modulus of the electric field strength vector in glass is $E_2 = 1.03 \cdot 10^4 \ V \cdot m^{-1}$. The density of bound charges at the interface between air and glass is $\sigma_2' = 3.8 \cdot 10^{-7} \ C \cdot m^{-2}$.

Problem 2.3.4

Problem description. In kerosene, at a depth of $h = 3\ m$ below the free surface, there is a point charge $Q = +2 \cdot 10^{-8}\ C$. Calculate the density of charges on the surface of kerosene in the following cases: 1) above the charge; 2) at a distance of $l = 5\ cm$ from the charge. Calculate the total charge on the entire surface of the kerosene.

Known quantities: $h = 3\ m$, $Q = +2 \cdot 10^{-8}\ C$, $l = 5\ cm$.

Quantities to be calculated: σ_1, σ_2, Q'.

Problem solution. First of all, we will write down the formula for the equality of the normal components of the electric displacement vector on both sides of the kerosene-air interface

$$\varepsilon_0 \varepsilon_1 E_{1n} = \varepsilon_0 \varepsilon_2 E_{2n}, \tag{1}$$

where
ε_0 is the electric constant;
ε_1 and ε_2 are the relative permittivities of the first and second dielectric, respectively;
E_{1n} and E_{2n} are the normal components of the electrostatic field strength in the first and second dielectric, respectively.

The electric field $\vec{E}$ is determined by both free and bound charges.

We will first consider the first case described in the problem statement. The electrostatic field E_1 created at point 1 on the surface of kerosene directly above the point charge Q, is $\dfrac{Q}{4\pi\varepsilon_0 h^2}$ (where h is the distance from the charge Q to point 1). 1). The electric field vector $\vec{E}_1$ is directed perpendicular to the interface upwards. The electric field of bound charges (in fact, this is the electric field of a charged plane) is $\dfrac{\sigma_1}{2\varepsilon_0}$ (where σ_1 is the surface density of bound charges for the first case). The electric field of bound charges is also directed perpendicular to the interface: in kerosene - down, and in air - up. Therefore, for the moduli of the electrostatic field in the first dielectric (E_1) and in the second dielectric (E_2) we can write

$$E_1 = \frac{Q}{4\pi\varepsilon_0 h^2} + \frac{\sigma_1}{2\varepsilon_0}, \qquad E_2 = \frac{Q}{4\pi\varepsilon_0 h^2} - \frac{\sigma_1}{2\varepsilon_0}. \tag{2}$$

Due to the perpendicularity of the electrostatic field to the dielectric interface, we can write the following relationships

$$E_{1n} = E_1, \qquad E_{2n} = E_2. \tag{3}$$

Therefore, the equality of the normal components of the electric displacement vector is described by the equation

$$\left(\frac{Q}{4\pi\varepsilon_0 h^2} + \frac{\sigma_1}{2\varepsilon_0}\right)\varepsilon_0\varepsilon_1 = \left(\frac{Q}{4\pi\varepsilon_0 h^2} - \frac{\sigma_1}{2\varepsilon_0}\right)\varepsilon_0\varepsilon_2 \tag{4}$$

or

$$\frac{\varepsilon_1\sigma_1}{2} + \frac{\varepsilon_2\sigma_1}{2} = \frac{Q}{4\pi\varepsilon_0 h^2}\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right). \tag{5}$$

Then

$$\sigma_1 = \frac{Q}{2\pi h^2} \cdot \frac{(\varepsilon_2 - \varepsilon_1)}{\varepsilon_2 + \varepsilon_1}. \tag{6}$$

Substituting known values, we get

$$\sigma_1 = 1.18 \cdot 10^{-6}\ C \cdot m^{-2}. \tag{7}$$

Now we will consider the second case described in the problem statement. In this case, point 2 on the surface of kerosene is at a distance of l from the point charge Q. The strength of the electrostatic field at point 2 is $\frac{Q}{4\pi\varepsilon_0 l^2}$. This electrostatic field is not directed perpendicular to the interface, but at an angle of α to the normal, and $cos\,\alpha = \frac{h}{l}$. Therefore, the normal component of this electrostatic field is equal to $\frac{Qh}{4\pi\varepsilon_0 l^3}$.

The electric field of bound charges in the second case is equal to $\frac{\sigma_2}{2\varepsilon_0}$ (where σ_2 is the surface density of bound charges for the second case). In addition, the electric field of bound charges is perpendicular to the dielectric interface and is directed upwards in air, and downwards in kerosene. Therefore, for the normal components of the electrostatic field in the first and second dielectrics in this case, we can write

$$E_{1n} = \frac{Qh}{4\pi\varepsilon_0 l^3} + \frac{\sigma_2}{2\varepsilon_0}, \qquad E_{2n} = \frac{Qh}{4\pi\varepsilon_0 l^3} - \frac{\sigma_2}{2\varepsilon_0}. \qquad (8)$$

We write the equality of the normal components of the electric displacement vector as an equation

$$\left(\frac{Qh}{4\pi\varepsilon_0 l^3} + \frac{\sigma_2}{2\varepsilon_0}\right)\varepsilon_0\varepsilon_1 = \left(\frac{Qh}{4\pi\varepsilon_0 l^3} - \frac{\sigma_2}{2\varepsilon_0}\right)\varepsilon_0\varepsilon_2 \qquad (9)$$

or

$$\frac{\varepsilon_1\sigma_2}{2} + \frac{\varepsilon_2\sigma_2}{2} = \frac{Qh(\varepsilon_2 - \varepsilon_1)}{4\pi\, l^3}. \qquad (10)$$

Then, for the surface density of bound charges in the second case, we can write

$$\sigma_2 = \frac{Qh(\varepsilon_2 - \varepsilon_1)}{2\pi\, l^3(\varepsilon_2 + \varepsilon_1)}. \qquad (11)$$

Numerically

$$\sigma_2 = 8 \cdot 10^{-7}\ C \cdot m^{-2}. \qquad (12)$$

In order to determine the total bound charge on the surface of kerosene, we consider an infinitely narrow ring with a radius of x, a width of dx and centered at point 1. The area of this ring is

$$dS = 2\pi\, x dx. \qquad (13)$$

The charge placed on this ring is

$$dQ' = \sigma_2 dS = \frac{Qh2\pi\, xdx}{2\pi\sqrt{\left(h^2 + x^2\right)^3}} \cdot \frac{\left(\varepsilon_2 - \varepsilon_1\right)}{\left(\varepsilon_2 + \varepsilon_1\right)}. \qquad (14)$$

Then we will integrate equation (14) in the range from 0 to ∞

$$Q' = \frac{Qh}{2} \cdot \frac{\left(\varepsilon_2 - \varepsilon_1\right)}{\left(\varepsilon_2 + \varepsilon_1\right)} \int_0^\infty \frac{2xdx}{\sqrt{\left(h^2 + x^2\right)^3}} = Q\frac{\left(\varepsilon_2 - \varepsilon_1\right)}{\left(\varepsilon_2 + \varepsilon_1\right)}. \qquad (15)$$

Substituting the given data, we find

$$Q' = 6.7 \cdot 10^{-9}\ C. \qquad (16)$$

Answer. The density of bound charges on the surface of kerosene in the first case is $\sigma_1 = 1.18 \cdot 10^{-6}\ C \cdot m^{-2}$. The density of bound charges on the surface of kerosene in the second case is $\sigma_2 = 8 \cdot 10^{-7}\ C \cdot m^{-2}$. The total bound charge on the surface of kerosene is $Q' = 6.7 \cdot 10^{-9}\ C$.

Problem 2.3.5

Problem description. The space between the plates of a flat capacitor is filled with a dielectric ($\varepsilon = 6$). The distance between the plates of the capacitor is $d = 4\ mm$. A voltage of $U = 1200\,V$ is applied to the plates. Calculate the following quantities: 1) field strength in a dielectric; 2) surface charge density on the capacitor plates; 3) surface density of bound charges on the dielectric; 4) dielectric susceptibility.

Known quantities: $\varepsilon = 6$, $d = 4\ mm$, $U = 1200\,V$.

Quantities to be calculated: E, σ_d, σ_z, κ.

Problem solution. The electric field strength in the dielectric is determined only by the potential difference on the capacitor plates and the distance between them

$$E = \frac{U}{d}, \qquad (1)$$

where

U is the potential difference between the plates of a capacitor;
d is the distance between the plates of the capacitor.

Substituting known values, we get

$$E = 3 \cdot 10^5 \; V \cdot m^{-1}. \tag{2}$$

To further solve the problem, we introduce the following notation: σ_0 is the surface charge density on the capacitor plates without a dielectric; σ_d is the surface density on the capacitor plates in the presence of a dielectric; σ_z is the surface density of bound charges on the dielectric.

The combined effect of charges with surface densities σ_d and σ_z can be described by a model according to which there are charges distributed at the interface between the conductor and the dielectric, with a density

$$\sigma' = \sigma_d - \sigma_z, \tag{3}$$

where σ' is the surface density of the "effective" charges that determine the resulting electric field in the dielectric.

Then the electric field in a capacitor without a dielectric is given by

$$E_0 = \frac{\sigma_0}{\varepsilon_0} = \frac{U}{d}, \tag{4}$$

where ε_0 is the electric constant.

In turn, the resulting electric field in the dielectric is

$$E = \frac{\sigma_d}{\varepsilon\varepsilon_0} = \frac{\sigma'}{\varepsilon_0} = \frac{U}{d}, \tag{5}$$

where ε is the relative permittivity of the dielectric.

From here we can write the following equation for the surface charge density on the capacitor plates

$$\sigma_d = \frac{\varepsilon\varepsilon_0 U}{d}. \tag{6}$$

Substituting known values, we have

$$\sigma_d = 1.6 \cdot 10^{-5} \; C \cdot m^{-2}. \tag{7}$$

Now we will rewrite equation (3)

$$\sigma_z = \sigma_d - \sigma'. \tag{8}$$

Then, using equation (6), we obtain an expression for the surface density of bound charges on the dielectric

$$\sigma_z = \varepsilon\varepsilon_0 E - \varepsilon_0 E = \varepsilon_0(\varepsilon - 1)E = \varepsilon_0(\varepsilon - 1)\frac{U}{d}. \tag{9}$$

Substituting known values, we get

$$\sigma_z = 1.33 \cdot 10^{-5}\ C \cdot m^{-2}. \tag{10}$$

The normal component of the polarization vector $\vec{P}$ is numerically equal to the surface density σ_z of bound charges. In addition, the modulus of the polarization vector is proportional to the electric field in the dielectric

$$P_n = P = \sigma_z, \qquad P = \kappa E, \tag{11}$$

where κ is the absolute dielectric susceptibility.

The value κ can be determined from equations (11) and (9)

$$\kappa = \frac{\sigma_z}{E} = \frac{\varepsilon_0(\varepsilon - 1)E}{E} = \varepsilon_0(\varepsilon - 1). \tag{12}$$

Numerically

$$\kappa = 4.44 \cdot 10^{-11}\ F \cdot m^{-1}. \tag{13}$$

Answer. The electric field in a dielectric is $E = 3 \cdot 10^5\ V \cdot m^{-1}$. The surface charge density on the capacitor plates is $\sigma_d = 1.6 \cdot 10^{-5}\ C \cdot m^{-2}$. The surface density of bound charges on a dielectric is $\sigma_z = 1.33 \cdot 10^{-5}\ C \cdot m^{-2}$. The dielectric susceptibility is $\kappa = 4.44 \cdot 10^{-11}\ F \cdot m^{-1}$.

Problem 2.3.6

Problem description. Two horizontal plates are charged with charges $Q=+2\cdot10^{-7}\ C$ and $Q=-2\cdot10^{-7}\ C$. The plates are so close to each other that the electric field between them can be considered uniform. The lower plate is in a liquid dielectric ($\varepsilon=3$). The area of the plate is $S=300\ cm^2$. Calculate the forces that act on each of the plates and on the surface of the liquid.

Known quantities: $Q=+2\cdot10^{-7}\ C$, $Q=-2\cdot10^{-7}\ C$, $\varepsilon=3$, $S=300\ cm^2$.

Quantities to be calculated: $F_1,\ F_2,\ F_3$.

Problem solution. The resulting force that acts on the top plate is equal to the sum of two forces: the first force is due to the interaction of charges on the top and bottom plates; the second force is determined by the interaction of charges on the upper plate and polarization charges on the surface of the liquid dielectric.

The density of polarization charges (problem 2.3.4) is

$$\sigma_1=\frac{Q(\varepsilon-\varepsilon_1)}{S(\varepsilon+\varepsilon_1)}, \qquad (1)$$

where

Q is the charge module on the capacitor plates;

S is the area of the capacitor plate;

ε is the relative permittivity of the liquid dielectric;

ε_1 is the relative permittivity of air.

The force of attraction between the charges on the plates is given by

$$F_1'=QE_1, \qquad (2)$$

where E_1 is the electric field formed by the charges of the lower plate in the place where the upper plate is located.

Therefore, for the quantity F_1' we can write

$$F_1'=\frac{Q^2}{2\varepsilon_0\varepsilon_1 S}, \qquad (3)$$

where ε_0 is the electric constant.

The force of interaction between the charges of the upper plate and the charges on the surface of the liquid dielectric is

$$F_2' = \frac{Q^2}{2\varepsilon_0\varepsilon_1 S} \cdot \frac{(\varepsilon - \varepsilon_1)}{(\varepsilon + \varepsilon_1)}. \quad (4)$$

Therefore, the resulting force that acts on the top plate is given by

$$F_1 = F_1' + F_2' = \frac{Q^2}{2\varepsilon_0\varepsilon_1 S} - \frac{Q^2}{2\varepsilon_0\varepsilon_1 S} \cdot \frac{(\varepsilon - \varepsilon_1)}{(\varepsilon + \varepsilon_1)} = \frac{Q^2}{\varepsilon_0 S} \cdot \frac{\varepsilon}{\varepsilon_1(\varepsilon + \varepsilon_1)}. \quad (5)$$

Substituting known values, we have

$$F_1 = 0.113\ N. \quad (6)$$

Similarly, we define the force that acts on the bottom plate

$$F_2 = \frac{Q^2}{2\varepsilon\varepsilon_0 S} - \frac{Q^2}{2\varepsilon\varepsilon_0 S} \cdot \frac{(\varepsilon - \varepsilon_1)}{(\varepsilon + \varepsilon_1)} = \frac{Q^2}{\varepsilon_0 S} \cdot \frac{\varepsilon_1}{\varepsilon(\varepsilon + \varepsilon_1)}. \quad (7)$$

Numerically

$$F_2 = 0.013\ N. \quad (8)$$

The force that acts on the surface of the liquid dielectric is also equal to the sum of two forces, namely: the force of interaction of charges on the surface of the liquid dielectric with the charges of the upper plate, as well as the force of interaction of charges on the surface of the liquid dielectric with the charges of the lower plate

$$F_3 = Q\frac{(\varepsilon - \varepsilon_1)}{(\varepsilon + \varepsilon_1)} \cdot \left(\frac{Q}{\varepsilon_0\varepsilon_1 S} + \frac{Q}{\varepsilon_0\varepsilon\, S}\right) =$$

$$= \frac{Q^2}{\varepsilon_0 S} \cdot \frac{(\varepsilon - \varepsilon_1)}{(\varepsilon + \varepsilon_1)}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon}\right) = \frac{Q^2}{\varepsilon_0 S}\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon}\right). \quad (9)$$

Substituting known values, we get

$$F_3 = 0.1\ N. \quad (10)$$

Answer. The force acting on the top plate is $F_1 = 0.113\ N$. The force acting on the bottom plate is $F_2 = 0.013\ N$. The force acting on the surface of a liquid dielectric is $F_3 = 0.1\ N$.

Problem 2.3.7

Problem description. The flat capacitor is filled with three layers of dielectrics: glass with a thickness of $d_1 = 0.35\,cm$ and relative dielectric permittivity $\varepsilon_1 = 7$; paraffin with a thickness of $d_2 = 0.21\,cm$ and relative dielectric permittivity $\varepsilon_2 = 2.1$ and porcelain with a thickness of $d_3 = 0.9\ cm$ and relative dielectric permittivity $\varepsilon_3 = 4.5$. Calculate the electric field in each layer if a potential difference of $U = 10\ kV$ is applied to the capacitor .

Known quantities: $d_1 = 0.35\,cm$, $\varepsilon_1 = 7$, $d_2 = 0.21\,cm$, $\varepsilon_2 = 2.1$, $d_3 = 0.9\ cm$, $\varepsilon_3 = 4.5$, $U = 10\ kV$.

Quantities to be calculated: $E_1,\ E_2,\ E_3$.

Problem solution. We will denote the electric field strength in the three layers by the symbols: $E_1,\ E_2$ and E_3. In addition, for the potential difference of these three layers, it is convenient to use the symbols: U_1, U_2, U_3.

The potential difference applied to the capacitor is

$$U = U_1 + U_2 + U_3. \tag{1}$$

The functional dependences of the potential difference, the thicknesses of the dielectric layers, and the corresponding electric fields have the form

$$U_1 = E_1 d_1,\quad U_2 = E_2 d_2,\quad U_3 = E_3 d_3, \tag{2}$$

where $d_1,\ d_2,\ d_3$ are the thicknesses of the first, second and third dielectrics, respectively.

Let us rewrite formula (1) taking into account formula (2)

$$U = E_1 d_1 + E_2 d_2 + E_3 d_3. \tag{3}$$

The electric field E_1 can be represented as a function of the electric fields E_2 and E_3. To do this, we take into account that the electrical displacement D (electrical induction) of the field in any dielectric layer is the same. Hence

$$D = \varepsilon_1 \varepsilon_0 E_1 = \varepsilon_2 \varepsilon_0 E_2 = \varepsilon_3 \varepsilon_0 E_3, \tag{4}$$

где

ε_0 is the electric constant;

ε_1, ε_2, ε_3 are the relative permittivities of the first, second, and third dielectrics, respectively.

Now we can write functional dependencies $E_2 = f(\varepsilon_1, \varepsilon_2, E_1)$ and $E_3 = f(\varepsilon_1, \varepsilon_3, E_1)$

$$E_2 == \frac{\varepsilon_1}{\varepsilon_2} E_1, \qquad E_3 = \frac{\varepsilon_1}{\varepsilon_3} E_1. \tag{5}$$

In order to determine E, it is necessary to substitute equation (5) into equation (3)

$$U = E_1 d_1 + \frac{\varepsilon_1}{\varepsilon_2} E_1 d_2 + \frac{\varepsilon_1}{\varepsilon_3} E_1 d_3. \tag{6}$$

Then

$$E_1 = \frac{U}{\left(d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2 + \frac{\varepsilon_1}{\varepsilon_3} d_3 \right)}. \tag{7}$$

Substituting known values, we get

$$E_1 = 4.1 \cdot 10^5 \ V \cdot m^{-1}. \tag{8}$$

The electrostatic field in the second dielectric is

$$E_2 = \frac{\varepsilon_1 U}{\varepsilon_2 \left(d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2 + \frac{\varepsilon_1}{\varepsilon_3} d_3 \right)}. \tag{9}$$

Numerically

$$E_2 = 1.35 \cdot 10^6 \ V \cdot m^{-1}. \tag{10}$$

The electrostatic field in the third dielectric is

$$E_3 = \frac{\varepsilon_1 U}{\varepsilon_3 \left(d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2 + \frac{\varepsilon_1}{\varepsilon_3} d_3 \right)}. \tag{11}$$

Let us now insert the given data

$$E_3 = 6.3 \cdot 10^5 \ V \cdot m^{-1}. \tag{12}$$

Answer. The electrostatic field in the first dielectric is $E_1 = 4.1 \cdot 10^5 \ V \cdot m^{-1}$. The electrostatic field in the second dielectric is $E_2 = 1.35 \cdot 10^6 \ V \cdot m^{-1}$. The electrostatic field in the third dielectric is $E_3 = 6.3 \cdot 10^5 \ V \cdot m^{-1}$.

Problem 2.3.8

Problem description. Calculate the capacitance of a cylindrical capacitor, the length of which is $l = 5 \ cm$, and the radii of the outer and inner plates are $R_2 = 1.5 \ cm$ and $R_1 = 0.3 \ cm$, respectively. The space between the plates is filled with paraffin.

Known quantities: $l = 5 \ cm$, $R_1 = 0.3 \ cm$, $R_2 = 1.5 \ cm$.

Quantities to be calculated: C.

Problem solution. The electrostatic field at any point between the capacitor plates can be determined by the Ostrogradsky-Gauss theorem

$$2\pi x l \varepsilon \varepsilon_0 E_x = \sigma \cdot 2\pi R_1 l, \tag{1}$$

where
ε is the relative permittivity of the dielectric between the plates of the capacitor;
ε_0 is the electric constant;
l is the length of the cylindrical capacitor;

x is the distance from the axis of the cylindrical capacitor to the point where the electric field is E_x;
σ is the surface charge density on the capacitor plate;
R_1 is the radius of the inner lining of the capacitor.

Now we can express the value E_x from formula (1)

$$E_x = \frac{\sigma R_1}{\varepsilon\varepsilon_0 x}. \qquad (2)$$

The potential gradient φ between the capacitor plates is related to the electric field E by the following formula

$$E = -\frac{d\varphi}{dx} = \frac{\sigma R_1}{\varepsilon\varepsilon_0 x}. \qquad (3)$$

Therefore, the elementary potential of the field between the plates of the capacitor is given by

$$d\varphi = -\frac{\sigma R_1}{\varepsilon\varepsilon_0 x} dx. \qquad (4)$$

Now we can integrate equation (4)

$$\int_{\varphi_1}^{\varphi_2} d\varphi = -\int_{R_1}^{R_2} \frac{\sigma R_1}{\varepsilon\varepsilon_0 x} dx, \qquad (5)$$

where R_2 is the radius of the outer plate of the capacitor.

Therefore, the potential difference between the capacitor plates is

$$U = \varphi_1 - \varphi_2 = \frac{\sigma R_1}{\varepsilon\varepsilon_0} ln \frac{R_2}{R_1}. \qquad (6)$$

The charge on the inner lining of the capacitor is given by

$$Q = 2\pi R_1 \sigma l. \qquad (7)$$

Formula (7), taking into account (6), can be rewritten as follows

$$Q = \frac{2\pi\,\varepsilon\varepsilon_0 l}{ln\left(\frac{R_2}{R_1}\right)}U. \tag{8}$$

The ratio of the charge to the potential difference is equal to the electric capacitance of a cylindrical capacitor

$$C = \frac{Q}{U} = \frac{2\pi\,\varepsilon\varepsilon_0 l}{ln\left(\frac{R_2}{R_1}\right)}. \tag{9}$$

Numerically

$$C = 3.5\cdot 10^{-12}\ F. \tag{10}$$

Answer. The electrical capacity of a cylindrical capacitor, the parameters of which are given in the condition of this problem, is $C = 3.5\cdot 10^{-12}\ F$.

Problem 2.3.9

Problem description. A metal ball with a radius of $R_1 = 5\ cm$ is surrounded by a spherical dielectric layer ($\varepsilon = 7$) with a thickness of $d = 1\ cm$ and another metal surface with a radius of $R_2 = 7\ cm$, which is concentric with respect to the first. Calculate the capacitance of such a capacitor.

Known quantities: $R_1 = 5\ cm$, $R_2 = 7\ cm$, $\varepsilon = 7$, $d = 1\ cm$.

Quantities to be calculated: C.

Problem solution. The system described in the problem statement can be considered as two capacitors connected in series. The electrical capacity of a battery of two capacitors connected in series is

$$C = \frac{C_1 C_2}{C_1 + C_2}. \tag{1}$$

The capacitance of a spherical capacitor can be determined using the relation

$$C = \frac{4\pi\,\varepsilon\varepsilon_0 R_1 R_2}{R_2 - R_1}, \tag{2}$$

where
ε is the relative permittivity of the dielectric between the plates of the capacitor;
ε_0 is the electric constant;
R_1 and R_2 are the radii of the inner and outer plates of the capacitor, respectively.

We will write down the formula for the electrical capacitance of the first capacitor, the plates of which are the surface of a metal sphere and the surface of a spherical dielectric layer

$$C_1 = \frac{4\pi\,\varepsilon\varepsilon_0 R_1 (R_1 + d)}{(R_1 + d) - R_1}, \tag{3}$$

where d is the thickness of the dielectric.

Substituting known values, we get

$$C_1 = 2.3 \cdot 10^{-10}\ F. \tag{4}$$

The capacitance of the second capacitor, the plates of which are the surface of the spherical dielectric layer and the spherical metal surface, is given by

$$C_2 = \frac{4\pi\,\varepsilon\varepsilon_0 (R_1 + d) R_2}{R_2 - (R_1 + d)}. \tag{5}$$

Numerically

$$C_2 = 4.6 \cdot 10^{-11}\ F. \tag{6}$$

Substituting the numerical values of the electrical capacities C_1 and C_2 into the formula (1) for the electrical capacity of a battery of series-connected capacitors, we determine the electrical capacity of the capacitor, the characteristics of which are given in the conditions of the problem

$$C = 3.6 \cdot 10^{-11}\ F. \tag{7}$$

Answer. The capacitance of the capacitor is $C = 3.6 \cdot 10^{-11}\ F$.

Problem 2.3.10

Problem description. A flat air capacitor was charged to a potential difference $U = 60\ V$ and the EMF source was turned off. The area of the capacitor plate is $S = 200\ cm^2$. The distance between the capacitor plates is $d = 0.5\ cm$. Capacitor plates are arranged vertically. A vessel with a non-conductive liquid ($\varepsilon = 2$) is brought from below so that it fills half of the capacitor. Calculate the capacitance of a capacitor. Determine the electric field in the air gap between the plates and in the gap between the plates filled with liquid. Calculate the change in the energy of the electric field of the capacitor. The liquid-air interface is flat. In addition, all electrical quantities of this system change in jumps.

Known quantities: $U = 60\ V$, $S = 200\ cm^2$, $d = 0.5\ cm$, $\varepsilon = 2$.

Quantities to be calculated: C, E_1, E_2, ΔW.

Problem solution. The system consists of two capacitors connected in parallel. The electrical capacity of a battery of two capacitors connected in parallel is determined by

$$C = C_1 + C_2, \qquad (1)$$

where C_1, C_2 are the capacitances of the first and second capacitors, respectively.

Therefore, the capacitance of an equivalent capacitor is

$$C = \frac{\varepsilon_0 \varepsilon_1 S}{2d} + \frac{\varepsilon_0 \varepsilon_2 S}{2d} = \frac{\varepsilon_0 S \left(\varepsilon_1 + \varepsilon_2\right)}{2d}, \qquad (2)$$

where

ε_0 is the electric constant;

ε_1 and ε_2 are the relative permittivities of the dielectrics between the plates of the first and second capacitors, respectively;

S is half the area of the capacitor plate (according to the condition of the problem);

d is the distance between the plates of the capacitor.

Substituting known values, we get

$$C = 5.3 \cdot 10^{-11}\ F. \qquad (3)$$

The electric field in the air part of the capacitor is

$$E_1 = \frac{\sigma_1}{\varepsilon_0\varepsilon_1} = \frac{2Q_1}{\varepsilon_0\varepsilon_1 S}, \tag{4}$$

where Q_1 is the charge on that part of the capacitor plate that is in the air.

The electric field of that part of the capacitor that is immersed in the liquid is

$$E_2 = \frac{\sigma_2}{\varepsilon_0\varepsilon_2} = \frac{2Q_2}{\varepsilon_0\varepsilon_2 S}, \tag{5}$$

where Q_2 is the charge on that part of the capacitor plate that is immersed in the liquid.

According to the law of conservation of charge, we get

$$Q = Q_1 + Q_2. \tag{6}$$

In addition, we can write the following relations

$$\frac{Q_1}{Q_2} = \frac{C_1}{C_2} = \frac{\varepsilon_1}{\varepsilon_2}. \tag{7}$$

In formula (7), we took into account the fact that $Q = CU$ (where U is the potential difference between the capacitor plates, which is the same for both capacitors). For an air capacitor charged to a potential difference U, we get

$$E = \frac{Q}{\varepsilon_0\varepsilon_1 S} = \frac{U}{d}. \tag{8}$$

Now we will rewrite equation (8) taking into account equations (6) and (7)

$$Q_1 = \frac{\varepsilon_0\varepsilon_1^2 US}{d(\varepsilon_1+\varepsilon_2)}, \qquad Q_2 = \frac{\varepsilon_0\varepsilon_2^2 US}{d(\varepsilon_1+\varepsilon_2)}. \tag{9}$$

The electric fields in both capacitors are equal (taking into account (9), (4) and (5))

$$E_1 = \frac{2\varepsilon_0\varepsilon_1^2 US}{\varepsilon_0\varepsilon_1(\varepsilon_1+\varepsilon_2)Sd} = \frac{2\varepsilon_1 U}{d(\varepsilon_1+\varepsilon_2)}, \tag{10}$$

$$E_2 = \frac{2\varepsilon_0\varepsilon_1\varepsilon_2 US}{\varepsilon_0\varepsilon_2(\varepsilon_1+\varepsilon_2)Sd} = \frac{2\varepsilon_1 U}{d(\varepsilon_1+\varepsilon_2)}. \quad (11)$$

Numerically

$$E_1 = E_2 = 8\cdot 10^3\ V\cdot m^{-1}. \quad (12)$$

The electric fields are the same in both parts of the capacitor.

The energy of the capacitor before its half was immersed in the liquid was

$$W_1 = \frac{\varepsilon_0\varepsilon_1 E^2}{2} Sd = \frac{\varepsilon_0\varepsilon_1 U^2 S}{2d}, \quad (13)$$

where E is the electric field between the plates of a capacitor before being immersed in a liquid.

After half of the capacitor is immersed in the liquid, its energy is equal to the sum of the energies of the two capacitors

$$W_2 = \frac{4\varepsilon_0\varepsilon_1^3 U^2}{2d^2(\varepsilon_1+\varepsilon_2)} Sd + \frac{4\varepsilon_0\varepsilon_1^2\varepsilon_2 U^2}{2d^2(\varepsilon_1+\varepsilon_2)} Sd = \frac{2\varepsilon_0\varepsilon_1^2 SU^2}{d(\varepsilon_1+\varepsilon_2)}. \quad (14)$$

The change in energy of the capacitor is

$$\Delta W = W_1 - W_2 = \frac{\varepsilon_0\varepsilon_1 U^2 S}{2d} - \frac{2\varepsilon_0\varepsilon_1^2 U^2 S}{d(\varepsilon_1+\varepsilon_2)} =$$

$$= \frac{\varepsilon_0\varepsilon_1 U^2 S}{d}\left(\frac{1}{2} - \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}\right) = \frac{\varepsilon_0\varepsilon_1 U^2 S(\varepsilon_2-\varepsilon_1)}{2d(\varepsilon_1+\varepsilon_2)}. \quad (15)$$

Numerically

$$\Delta W = 2\cdot 10^{-7}\ J. \quad (16)$$

Answer. The capacitance of the capacitor is $C = 5.3\cdot 10^{-11}\ F$. The electric field in the air gap between the capacitor plates is $E_1 = 8\cdot 10^3\ V\cdot m^{-1}$. The electric field in the gap between the capacitor plates filled with liquid is $E_2 = 8\cdot 10^3\ V\cdot m^{-1}$. The change in energy of the capacitor is $\Delta W = 2\cdot 10^{-7}\ J$.

Problem 2.3.11

Problem description. Inside a flat capacitor with a plate area of $S = 200\ cm^2$ and a distance between them equal to $d = 0.1\ cm$, there is a glass plate ($\varepsilon = 5$), which completely fills the space between the capacitor plates. Calculate the change in energy after the glass plate has been removed from the capacitor. Consider two cases: 1) the capacitor was always connected to the battery with EMF $E' = 300\ V$; 2) the capacitor was first attached to the same battery, and then it was disconnected from the battery, and only after that the glass plate was removed from the capacitor. Calculate the mechanical work required to remove the plate in both cases.

Known quantities: $S = 200\ cm^2$, $d = 0.1\ cm$, $\varepsilon = 5$, $E' = 300\ V$.

Quantities to be calculated: ΔW_1, ΔW_2, A_1, A_2.

Problem solution. The condition of the problem suggests that in the first case, the potential difference between the plates of the capacitor remains constant, while the charge changes. Therefore, the change in the energy of the electric field of the capacitor must be expressed through the potential difference and other constants.

The energy of the electric field of a capacitor with a dielectric is given by

$$W_1 = \frac{C_1 U^2}{2}, \tag{1}$$

where
C_1 is the capacitance of the capacitor before the glass plate is removed;
U is the potential difference between the plates of a capacitor.

The energy of the electric field of a capacitor without a dielectric is

$$W_2 = \frac{C_2 U^2}{2}, \tag{2}$$

where C_2 is the capacitance of the capacitor after removing the glass plate.

We can write the following relations for electric capacities

$$C_1 = \frac{\varepsilon \varepsilon_0 S}{d}, \tag{3}$$

where
ε_0 is the electric constant;

ε is the relative permittivity of the glass plate;
d is the distance between the plates of the capacitor;
S is the area of the capacitor plate;

$$C_2 = \frac{\varepsilon_0 \varepsilon_1 S}{d}, \tag{4}$$

where ε_1 is the relative permittivity of air.

The change in the energy of the capacitor in the first case is

$$\Delta W_1 = W_1 - W_2 = \frac{\varepsilon_0 \varepsilon U^2 S}{2d} - \frac{\varepsilon_0 \varepsilon_1 U^2 S}{2d} = \frac{\varepsilon_0 (\varepsilon - \varepsilon_1) U^2 S}{2d}. \tag{5}$$

Substituting known values, we get

$$\Delta W_1 = 3.18 \cdot 10^{-5}\ J. \tag{6}$$

The charge on the plates of the capacitor in the second case remains unchanged. Therefore, the change in the energy of the electric field must be expressed in terms of charge. The energy of the electric field of a capacitor with a dielectric is

$$W_1 = \frac{Q^2}{2C_1}. \tag{7}$$

The energy of the electric field of a capacitor without a dielectric is

$$W_2 = \frac{Q^2}{2C_2}. \tag{8}$$

The change in the energy of the capacitor in the second case is given by

$$\Delta W_2 = W_1 - W_2 = \frac{Q^2}{2C_1} - \frac{Q^2}{2C_2}. \tag{9}$$

We can write the following relationship for the charge

$$Q = \frac{\varepsilon \varepsilon_0 U S}{d}. \tag{10}$$

Therefore, for formula (9) we have

$$\Delta W_2 = \frac{\varepsilon_0^2\varepsilon^2 U^2 S^2 d}{2\varepsilon_0\varepsilon S d^2} - \frac{\varepsilon_0^2\varepsilon_1^2 U^2 S^2 d}{2\varepsilon_0\varepsilon_1 S d^2} = \frac{\varepsilon_0\varepsilon U^2 S}{2d}\cdot\frac{(\varepsilon-\varepsilon_1)}{\varepsilon_1}. \tag{11}$$

Numerically

$$\Delta W_2 = 1.6\cdot 10^{-4}\ J. \tag{12}$$

In both cases, not only the energy of the capacitor changes, but work is done to remove the plate. This work is greater in the second case due to the fact that when the plate is removed, the electric field of the capacitor increases, while in the first case this electric field remains constant. In the first case, when the plate is removed, not only mechanical work is performed, but the energy of the capacitor decreases and the energy of the EMF source increases. The work done against the EMF source is equal to

$$A = \Delta Q U, \tag{13}$$

where ΔQ is the change in the charge of the capacitor after the plate is removed.

The change in the charge of the capacitor is

$$\Delta Q = (\varepsilon - \varepsilon_1) C U. \tag{14}$$

Then the work is

$$A = (\varepsilon - \varepsilon_1) C U^2. \tag{15}$$

The work done when removing the plate is determined by the relation

$$A_1 = A + \Delta W_1 = \frac{\varepsilon_0(\varepsilon - \varepsilon_1) U^2 S}{2d}. \tag{16}$$

Numerically

$$A_1 = 3.18\cdot 10^{-5}\ J. \tag{17}$$

In the second case, when the plate is removed, mechanical work is performed, which is equal to the increase in the energy of the capacitor

$$A_2 = \Delta W_2 = 1.6\cdot 10^{-4}\ J. \tag{17}$$

Answer. The change in the energy of the electric field of the capacitor in the first case is $\Delta W_1 = 3.18 \cdot 10^{-5}\ J$. The change in the energy of the electric field of the capacitor in the second case is $\Delta W_2 = 1.6 \cdot 10^{-4}\ J$. The mechanical work that must be expended to remove the plate in the first case is $A_1 = 3.18 \cdot 10^{-5}\ J$. The mechanical work that must be spent to remove the plate in the second case is $A_2 = 1.6 \cdot 10^{-4}\ J$.

Problem 2.3.12

Problem description. A battery of two Leyden jars connected in series with an electrical capacity of $C_1 = 5 \cdot 10^{-10}\ F$ and $C_2 = 1.3 \cdot 10^{-9}\ F$ is charged to a potential difference of $U = 1800\,V$. Then the capacitors, without discharging, are disconnected from the current source and connected in parallel. Calculate the work done in this discharge.

Known quantities: $C_1 = 5 \cdot 10^{-10}\ F$, $C_2 = 1.3 \cdot 10^{-9}\ F$, $U = 1800\,V$.

Quantities to be calculated: A.

Problem solution. In problems of this type, it is understood that when switching capacitors, plates charged with the same charge are connected to each other. In this case, each pair of interconnected plates will have a charge $2Q_0$, where Q_0 is the charge that was on each plate of capacitors connected in series.

We define the discharge work as the energy difference W_1 (before the capacitors are switched) and W_2 (after the capacitors are switched). For magnitude W_1 we get

$$W_1 = \frac{C'U^2}{2}, \tag{1}$$

where U is the potential difference applied across the capacitor bank.

The value C' is

$$C' = \frac{C_1 C_2}{C_1 + C_2}, \tag{2}$$

where C_1 and C_2 are the capacitances of the first and second capacitors, respectively.

In this case, the energy before the connection is

$$W_1 = \frac{C_1 C_2}{2(C_1 + C_2)} U^2. \tag{3}$$

We will express the energy of parallel-connected capacitors in terms of the charge on their plates

$$W_2 = \frac{Q^2}{2C''}, \tag{4}$$

where
$C'' = C_1 + C_2$ is the capacitance of a system of two capacitors connected in parallel;
$Q = 2Q_0$ is the charge on the capacitor plates after a new connection.

The charge Q_0 can be determined from the following relation

$$Q = C'U \tag{5}$$

or, taking into account (2)

$$Q_0 = \frac{C_1 C_2}{C_1 + C_2} U. \tag{6}$$

Now we can write the final equation for the energy of capacitors connected in parallel

$$W_2 = \frac{2C_1^2 C_2^2}{(C_1 + C_2)^2} U^2. \tag{7}$$

In this case, the discharge work is equal to

$$A = W_1 - W_2 = \frac{C_1 C_2 (C_1 - C_2)^2}{2(C_1 + C_2)} U^2. \tag{8}$$

Numerically

$$A = 4.3 \cdot 10^{-22}\ J. \tag{9}$$

Answer. The work that is done with the discharge of capacitors specified in the condition of the problem is $A = 4.3 \cdot 10^{-22}\ J$.

2.4. Level 1 problems

2.4.1. Two metal spheres with radii $R_1 = 2\ cm$ and $R_2 = 6\ cm$ are connected by a conductor whose capacitance can be neglected. The charge of spheres is $Q = 1\ nC$. Calculate the surface charge density on the spheres.

2.4.2. A ball with a radius of $R = 6\ cm$ is charged to a potential $\varphi_1 = 300\ V$. The second ball with a radius of $R_2 = 4\ cm$ is charged to a potential $\varphi_2 = 500\ V$. Determine the potential of the balls after they are connected by a metal conductor. The capacitance of the connecting conductor under these conditions can be neglected.

2.4.3. The distance between the plates of a flat capacitor is $d = 1.33\ m$. The area of the capacitor plates is $S = 20\ cm^2$. In the space between the capacitor plates there are two layers of dielectrics: mica with a thickness of $d_1 = 0.7\ m$ and ebonite with a thickness of $d_2 = 0.3\ m$. Calculate the capacitance of such a capacitor.

2.4.4. The capacitance of a flat capacitor is $C = 1.5\ \mu F$. The distance between the capacitor plates is $d = 5\ mm$. Calculate the capacitance of the capacitor for the case when there is a sheet of ebonite with a thickness of $d_1 = 3\ mm$.

2.4.5. Between the plates of a flat capacitor is a glass plate tightly adjacent to them. The capacitor is charged to a potential difference of $U_1 = 100\ V$. Calculate the potential difference U_2 for the case when the glass plate is pulled out of the capacitor.

2.4.6. The capacitor consists of two concentric spheres. The radius of the inner sphere is $R_1 = 10\ cm$. The radius of the outer sphere is $R_2 = 10.2\ cm$. The gap between the spheres is filled with paraffin. The inner sphere has a charge $Q = 5\ \mu C$. Calculate the potential difference between the spheres.

2.4.7. Between the plates of the first and second capacitors are air and porcelain, respectively. The first capacitor was charged to a potential difference of $U = 600\ V$ and disconnected from the voltage source. The second capacitor has the same size and shape. Then the first and second capacitors were connected in parallel.

Determine the relative permittivity of porcelain if, after connecting the second capacitor, the potential difference has decreased to a value of $U_1 = 100\,V$.

2.4.8. A capacitor with an electrical capacity $C_1 = 0{,}2\ \mu F$ was charged to potential difference $U_1 = 320\,V$. After it was connected in parallel with a second capacitor charged to a potential difference of $U_2 = 450\,V$, the first capacitor voltage changed to a value of $U = 400\,V$. Calculate the capacitance of the second capacitor.

2.4.9. Three identical flat capacitors are connected in series. The capacitance of such a capacitor bank is $C = 89\ pF$. The area of each plate is $S = 100\ cm^2$. The dielectrics in each capacitor are identical glass plates. Calculate the thickness of the glass plate.

2.4.10. Two capacitors with capacitances of $C_1 = 3\ \mu F$ and $C_2 = 6\ \mu F$, respectively, are connected to each other and connected to the EMF $E = 120\,V$. etermine the charges of capacitors and the potential difference between their plates. Consider two cases: 1) capacitors connected in parallel; 2) capacitors connected in series.

2.4.11. The distance between the plates of a flat capacitor is $d = 2\ cm$. The potential difference between the capacitor plates is $U = 6\ kV$. The charge of each plate is $Q = 10\ nC$. Calculate the energy of the electric field of the capacitor and the force of mutual attraction of the plates.

2.4.12. Calculate the amount of heat that will be released during the discharge of a flat capacitor if the potential difference between the plates is 0.65 kV, and the distance between the plates is $d = 1\ mm$. Between the plates of the capacitor there is a dielectric (mica). The area of each plate is $S = 300\ cm^2$.

2.4.13. The force of attraction between the plates of a flat air capacitor is $F = 50\ mN$. The area of each plate is $S = 200\ cm^2$. Calculate the energy density of the electric field of the capacitor.

2.4.14. The flat air condenser consists of two round plates. The radius of each plate is $r = 10\ cm$. The distance between the plates is $d_1 = 1\ cm$. The capacitor was charged to a potential difference of $U = 1.2\ kV$ and disconnected from the power source. Calculate the work that needs to be done to increase the distance between the plates to $d_2 = 3.5\ cm$.

2.4.15. The capacitance of the capacitor is $C = 666\ pF$. The capacitor was charged to a potential difference of $U = 1.5\ kV$ and disconnected from the current source. Then a second uncharged capacitor was connected in parallel to the capacitor, the electrical capacity of which was equal to $C_2 = 444\ pF$. Determine the energy that was released during the formation of a spark that appeared when the capacitors were connected.

2.4.16. The space between the plates of a flat capacitor is filled with a dielectric (porcelain), the volume of which is $V = 100\ cm^3$. The surface charge density on the capacitor plates is $\sigma = 8.85\ nC \cdot m^{-2}$. Calculate the work that must be done to remove the dielectric from the capacitor. The friction of the dielectric on the capacitor plates can be neglected.

2.4.17. An ebonite plate with a thickness of $d = 2\ mm$ and an area of $S = 300\ cm^2$ was placed in a uniform electric field $E = 12\ kV \cdot m^{-1}$ so that the lines of force of the electric field were perpendicular to the surfaces of the plate. Calculate the density of bound charges on the plate surface and the energy of the electric field between the plates.

2.4.18. A solitary metal sphere with an electrical capacity of $C = 10\ pF$ is charged to a potential of $\varphi = 3\ kV$. Determine the energy of the field contained in a spherical layer bounded by a sphere and a spherical surface concentric with it, the radius of which is three times the radius of the sphere.

2.4.19. An electric field is created by a charged sphere. The sphere's charge and radius are $Q = 1 \cdot 10^{-7}\ C$ and $R = 10\ cm$, respectively. Calculate the energy of the electric field enclosed in a volume bounded by a sphere and a spherical surface concentric with it. The radius of spherical surface is twice the radius of sphere.

2.4.20. A solid paraffin ball is charged uniformly over its volume. The radius of the sphere and its volumetric charge density are $R = 10\ cm$ and $\rho = 10\ nC \cdot m^{-3}$, respectivly. Calculate the energy of the electric field, concentrated both in the ball itself and outside it.

2.5. Answers to problems

2.4.1. $\sigma_1 = 49.8\ nC \cdot m^{-2}$; $\sigma_2 = 16.6\ nC \cdot m^{-2}$.

2.4.2. $\varphi = 380\ V$.

2.4.3. $C = 35.4\ pF$.

2.4.4. $C = 2.5\ \mu F$.

2.4.5. $U_2 = 700\,V$.

2.4.6. $\Delta\varphi = 4.41\,kV$.

2.4.7. $\varepsilon = 5$.

2.4.8. $C_2 = 0.32\ \mu F$.

2.4.9. $d = 2.32\ mm$.

2.4.10. $Q_1 = 360\ \mu C$; $Q_1' = 720\ \mu C$; $\Delta\varphi_1 = 120\,V$; $Q_2 = 240\ \mu C$;

$Q_2' = 80\ \mu C$; $\Delta\varphi_2 = 80\ V$.

2.4.11. $W = 30\ \mu J$; $F = 15\ mN$.

2.4.12. $Q = 0.209\ J$.

2.4.13. $\omega = 2.5\ J \cdot m^{-3}$.

2.4.14. $A = 50\ \mu J$.

2.4.15. $W = 0.3\ mJ$.

2.4.16. $A = 63.5\ nJ$.

2.4.17. $\sigma = 5.9\ nC \cdot m^{-2}$; $W = 88.5\ pJ$.

2.4.18. $W = 30\ \mu J$.

2.4.19. $W = 225\ \mu J$.

2.4.20. $W_1 = 7.88\ nJ$; $W_2 = 78.8\ nJ$.

CHAPTER 3. DIRECT CURRENT

3.1. Basic formulas

The **electric current** is determined by the amount of electricity Q, that passes through a fixed section of the conductor per unit of time t

$$I = \frac{dQ}{dt} \tag{3.1.1}$$

or for direct current

$$I = \frac{Q}{t}. \tag{3.1.2}$$

The term electric current was introduced by the French physicist, mathematician and naturalist Ampère (André-Marie Ampére 1775 – 1836).

The **current density** is, by definition, the ratio of the current strength to the cross-sectional area S of the conductor perpendicular to the electric field in it

$$j = \frac{dI}{dS}. \tag{3.1.3}$$

or for direct current

$$j = \frac{I}{S}. \tag{3.1.4}$$

The **resistance of a cylindrical conductor** is is proportional to the length of the conductor and inversely proportional to its cross-sectional area

$$R = \rho \frac{l}{S}, \tag{3.1.5}$$

where
l is the length of the cylindrical conductor;
ρ is the resistivity of the material from which the conductor is made.

The dependence of resistivity on temperature can be expressed by the formula

$$\rho = \rho_0 (1 + \alpha t), \tag{3.1.6}$$

where
α is the **temperature coefficient of resistance**;
ρ_0 is the resistivity at temperature $0°C$;
t is the temperature measured in $°C$.

Ohm's law for a circuit section has the form

$$I = \frac{U}{R}, \tag{3.1.7}$$

where U is the potential difference applied to the conductor.

The law that determines the relationship of electrical voltage with current and conductor resistance was established by the German physicist Georg Ohm (Georg Simon Ohm 1789 – 1854).

Ohm's law for a closed circuit can be formulated using the relation

$$I = \frac{E}{R + r}, \tag{3.1.8}$$

where
E is the EMF of the current source;
r is the current source internal resistance.

The current that passes through parallel connection of n identical elements (with EMF E and internal resistance r) is

$$I = \frac{E}{R + \left(\frac{r}{n}\right)}. \tag{3.1.9}$$

The current that passes through a series connection of identical elements is

$$I = \frac{nE}{R + nr}. \tag{3.1.10}$$

The current that passes through a mixed connection of k parallel rows with m series-connected elements ($mk = n$) is

$$I = \frac{nE}{kR + mr}. \tag{3.1.11}$$

Equivalent resistance of an electrical circuit, which consists of sections connected in series with resistances R_1, R_2, … is given by

$$R = R_1 + R_2 + ... = \sum_{i=1}^{n} R_i . \qquad (3.1.12)$$

The equivalent resistance of an electrical circuit, which consists of sections connected in parallel with resistances $R_1,\ R_2, \ldots$ is

$$R = \frac{1}{\sum_{i=1}^{n} R_i} . \qquad (3.1.13)$$

Kirchhoff's first law can be formulated as follows: the algebraic sum of all currents converging at a branch point is equal to zero

$$\sum_{i=1}^{n} I_i = 0 . \qquad (3.1.14)$$

Kirchhoff's second law can be formulated as follows: in any closed circuit of an electric circuit, the algebraic sum of all voltages is equal to the algebraic sum of all EMFs that are present in this circuit

$$\sum_{i=1}^{n} I_i R_i = \sum_{k=1}^{m} E_k . \qquad (3.1.15)$$

Relations (3.1.14) and (3.1.15), which are valid between currents and voltages in sections of any electrical circuit, were established by the German physicist Kirchhoff (Gustav Robert Kirchhoff 1824 – 1887).

The work of electric forces in a section of an electric circuit with a resistance R is

$$A = QU = IUt = I^2 Rt = \frac{U^2}{R} t . \qquad (3.1.16)$$

The full work of the current source in the entire closed electrical circuit is proportional to the EMF

$$A = EIt . \qquad (3.1.17)$$

The amount of heat that is released in an electrical circuit, provided that all the work is converted into heat, is given by

$$Q' = IUt = \frac{U^2}{R} t . \tag{3.1.18}$$

If due to the energy of an electric current, mechanical work is performed or chemical reactions occur, then the amount of heat that is released in the electrical circuit is determined by the **Joule-Lenz formula** (James Prescott Joule 1818 – 1889, Heinrich Friedrich Lenz 1804 – 1865)

$$Q' = I^2 Rt . \tag{3.1.19}$$

The efficiency of the current source is

$$\eta = \frac{IUt}{IEt} = \frac{U}{E} = \frac{IR}{I(R+r)} = \frac{R}{R+r} . \tag{3.1.20}$$

Efficiency of the power transmission line from the station to the consumer is given by

$$\eta_l = \frac{U_1}{U_2} = \frac{R_1}{R_1 + R_2} , \tag{3.1.21}$$

where
U_1 and U_2 are the voltages at the consumer level and at the station level, respectively;
R_1 and R_2 are consumer resistances and transmission line resistances, respectively.

3.2. Problem-solving framework

The electrons do not move in straight lines along the conductor. Instead, they collide repeatedly with metal atoms, and their resultant motion is complicated and zigzag. For instance, electrons traveling with a drift speed of $2.22 \cdot 10^{-4}$ m·s^{-1} would take about 75 min to travel 1 m [3, p. 835].

Ohm's law defines current from voltage or voltage from current for the simplest, unbranched electrical circuit. Calculations of currents in more complex branched electrical circuits are performed using Kirchhoff's laws. Problem solving skills using Kirchhoff's laws can only be acquired by solving a large number of problems. Before compiling equations according to Kirchhoff's laws, it is necessary to arbitrarily choose the directions of currents in each branch of a branched electrical circuit and indicate these directions with arrows. In addition, it is necessary to arbitrarily choose the direction of bypassing the contour (this is necessary only for compiling equations according to the second Kirchhoff law). When, it should be remembered that the bypass direction arbitrarily chosen for a given task must be the same for all circuits of a branched circuit.

According to Kirchhoff's first law, one less equation should be written than the number of nodes in a given electrical circuit, because the equation for the last node will not be independent, but only a consequence of the rest of the equations (it can be determined by adding these equations).

When compiling equations according to Kirchhoff's first law, one should adhere to the sign rule: the current that enters the node is indicated by the "+" sign in the equation, and the current that leaves the node is indicated by the "-" sign.

The number of independent equations that can be composed according to Kirchhoff's second law is also less than the number of closed contours. For these equations, it is necessary to choose closed contours so that each new contour contains at least one branch that would not be used in previous closed contours.

The sign rules for the second Kirchhoff law can be formulated as two points: 1) if the direction of the current coincides with the direction of bypassing the circuits, then the corresponding product of the current strength and the resistance enters the equation with the “+” sign, otherwise this resistance enters the equation with the sign "-"; 2) EMF values are used with a “+” sign if, when bypassing a closed contour in a positive direction, the first electrode will be negative, and the second electrode will be positive (regardless of where the current of the corresponding section of the electrical circuit is directed).

The total number of equations for nodes and closed contours should be equal to the number of unknown quantities in the problem statement. Next, it is necessary to solve the system of equations for the unknown quantities of this problem. To do this, it is convenient to use the method of determinants, which makes it possible to directly find unknown quantities.

As a result of solving the system of equations, the numerical values of unknown quantities can be negative. If currents are determined, then a negative value indicates that the real direction of the current in this section of the electrical circuit is opposite to the direction chosen at the beginning. If resistances are specified, then a negative value indicates an incorrect result (since electrical resistance is always positive by definition). In this case, it is necessary to change the direction of the current for this section of the electrical circuit and solve the problem again.

3.3. Problem-solving examples

Problem 3.3.1

Problem description. The voltage at the terminals of the electrical circuit was originally $U_0 = 120\,V$. Subsequently, the voltage decreases uniformly at a rate of $dU/dt = 0.01\,V \cdot s^{-1}$. At the same time, a resistance with a speed of $dR/dt = 0.1\,\Omega \cdot s^{-1}$ is placed in the electrical circuit. In addition, the electrical circuit contains a constant resistance $R_0 = 12\ \Omega$. Calculate the amount of charge that will pass through the electric circuit in time $\tau = 180\ s$.

Known quantities: $U_0 = 120\,V$, $dU/dt = 0.01\,V \cdot s^{-1}$, $dR/dt = 0.1\,\Omega \cdot s^{-1}$, $R_0 = 12\ \Omega$, $\tau = 180\ s$.

Quantities to be calculated: Q.

Problem solution. According to the condition of the problem, the rate of voltage change with time is $dU/dt = 0.01$. Therefore, the voltage at the terminals of the electrical circuit at time t will be determined by the equation

$$\int_{U_0}^{U} dU = -0.01 \int_0^t dt \tag{1}$$

or

$$U = U_0 - 0.01t, \tag{2}$$

where U_0 is the voltage at the time $t = 0$.

The rate of change of resistance, according to the condition of the problem, at time t is $dR/dt = 0.1$. Therefore, the resistance value at time t is given by

$$R = 0.1t. \tag{3}$$

For any moment in time, the current in the electrical circuit is determined according to Ohm's law

$$I = \frac{U}{R + R_0}, \tag{4}$$

where R_0 is the resistance at time $t = 0$.

The defining formula for the current has the form

$$I = \frac{dQ}{dt}. \tag{5}$$

Therefore, equation (4) can be rewritten in the following form

$$dQ = \frac{U_0 - 0.01t}{R_0 + 0.1t} dt. \tag{6}$$

Now we can integrate equation (6) over time from 0 to τ

$$Q = \int_0^{\tau} \frac{U_0 - 0.01t}{R_0 + 0.1t} dt \tag{7}$$

or

$$Q = (10U_0 + R_0) ln\left(\frac{R_0 + 0.1\tau}{R_0} - \frac{0.01}{0.1}\tau\right). \tag{8}$$

Substituting known values, we get

$$Q = 1.092 \cdot 10^3\ C. \tag{9}$$

Answer. The amount of charge that will pass through the electric circuit in time τ, is $Q = 1.092 \cdot 10^3\ C$.

Problem 3.3.2

Problem description. A flat capacitor with plates $a \times b$ ($a = 20\ cm$), the distance between which is b, is connected to a battery of batteries with EMF $E = 100\,V$ and internal resistance $R = 5\ \Omega$. A galvanometer whose resistance can be neglected is connected to an electric circuit. A glass plate with thickness and relative dielectric constant equal to $d = 1\ cm$ and $\varepsilon = 5$, respectively, is placed in the capacitor. The plate movement speed from side b is equal to $\upsilon = 1\ m \cdot s^{-1}$. Calculate the current that the galvanometer will record when the glass plate is inserted into the capacitor.

Known quantities: $a = 20\ cm$, $E = 100\,V$, $R = 5\ \Omega$, $d = 1\ cm$, $\varepsilon = 5$, $\upsilon = 1\ m \cdot s^{-1}$.

Quantities to be calculated: I.

<u>Problem solution</u>. During the placement of a glass plate in a capacitor, an additional charge will transfer to its plate

$$dQ = UdC, \tag{1}$$

where
U is the voltage across the plates of the capacitor;
C is the capacitance of the capacitor.

The change in the capacitance of the capacitor is due to the gradual filling of the space between the plates of the capacitor with a dielectric (glass)

$$dC = \left(\frac{\varepsilon \varepsilon_0 dS}{d} - \frac{\varepsilon' \varepsilon_0 dS}{d} \right) = \frac{(\varepsilon - \varepsilon')\varepsilon_0 dS}{d}, \tag{2}$$

where
ε_0 is the electric constant;
ε is the relative permittivity of the dielectric;
ε' is the relative permittivity of air;
d is the dielectric thickness;
dS is the area covered by the glass plate in time dt.

The value dS when the glass plate moves uniformly along side b of the capacitor plate is

$$dS = a\upsilon dt, \tag{3}$$

where
a is the width of the second side of the capacitor plate;
υ is the speed at which the dielectric is placed between the plates of the capacitor.

Now we can write an expression for the current, taking into account equations (1), (2) and (3)

$$dI = \frac{dQ}{dt} = \frac{(\varepsilon - \varepsilon')\varepsilon_0 a\upsilon U}{d}. \tag{4}$$

According to Ohm's law, the current is

$$I = \frac{E - U}{R}, \tag{5}$$

where E, R are the electromotive force and the internal resistance of the battery pack, respectively.

We will rewrite equation (5) taking into account equations (1) - (4)

$$I = \frac{(\varepsilon - \varepsilon')\varepsilon_0 E a \upsilon}{d + R(\varepsilon - \varepsilon')\varepsilon_0 a \upsilon}. \tag{6}$$

Numerically

$$I = 7 \cdot 10^{-8}\ A. \tag{7}$$

Answer. The current strength that the galvanometer will fix during the introduction of a glass plate into the capacitor is $I = 7 \cdot 10^{-8}\ A$.

Problem 3.3.3

Problem description. The internal resistance of the first voltmeter is $R_1 = 4500\ \Omega$. The indications of the first voltmeter when measuring the voltage on the anode battery were equal to $U_1 = 82\ V$. The internal resistance of the second voltmeter is $R_2 = 4500\ \Omega$. The indications of the second voltmeter when measuring the voltage on the anode battery were equal to $U_2 = 82\ V$. Calculate the EMF of the anode battery.

Known quantities: $R_1 = 4500\ \Omega$, $U_1 = 82\ V$, $R_2 = 4500\ \Omega$, $U_2 = 82\ V$.

Quantities to be calculated: E.

Problem solution. In the first case, the current in the circuit is determined by the ratio of voltage to resistance

$$I_1 = \frac{U_1}{R_1}, \tag{1}$$

where
U_1 is the indication of the first voltmeter;
R_1 is the internal resistance of the first voltmeter.

The internal voltage drop in the anode battery is determined by the difference between the EMF and voltage

$$I_1 r = E - U_1, \tag{2}$$

where
E is the electromotive force of the anode battery;
r is the internal resistance of the anode battery.

We can write a similar equation for measuring voltage with a second voltmeter

$$I_2 r = E - U_2, \tag{3}$$

where
I_2 is the current in the circuit when measured with a second voltmeter;
U_2 is the indication of the second voltmeter.

Solving the system of equations (1) - (3) with respect to the unknown quantity E, we get

$$E = \frac{U_1 U_2 (R_2 - R_1)}{R_2 U_1 - R_1 U_2}. \tag{4}$$

Substituting known values, we have

$$E = 92\ V. \tag{5}$$

Answer. The electromotive force of the second battery is $E = 92\ V$.

Problem 3.3.4

Problem description. The electrical circuit consists of copper wire with a cross-sectional area of $S_1 = 3\ mm^2$. A lead fuse with a cross-sectional area of $S_2 = 1\ mm^2$ is connected to an electrical circuit. Determine the short-circuit temperature rise for which this fuse is rated. All the heat that is released during a short circuit is spent on heating the wires. The initial temperature of the fuse is $t_0 = 17°C$.

Known quantities: $S_1 = 3\ mm^2$, $S_2 = 1\ mm^2$, $t_0 = 17°C$.

Quantities to be calculated: Δt.

Problem solution. The amount of heat that is released in a copper wire during a short circuit is proportional to the temperature difference

$$Q_1 = m_1 c_1 \Delta t = D_1 l_1 S_1 c_1 \Delta t, \tag{1}$$

where
D_1 is the density of copper;
l_1 is the length of the wire;
S_1 is the cross-sectional area;
c_1 is the specific heat capacity of copper;
$\varDelta t$ is the temperature rise of the wire;
m_1 is the mass of copper wire.

The amount of heat that is released in the lead wire is proportional to the melting point of the lead

$$Q_2 = D_2 l_2 S_2 \left(c_2 \varDelta t_1 + r\right), \tag{2}$$

where
r is the specific heat of fusion of lead;
$\varDelta t_1 = t_m - t_0$;
t_m is the melting point of lead;
t_0 is the initial temperature of the lead wire (room temperature);
D_2 is the density of lead;
l_2 is the length of the fuse;
S_2 is the cross-sectional area of the fuse;
c_2 is the specific heat capacity of lead.

Since both wires are connected in series in the electrical circuit, the current passing through them is the same. Given this fact, we can write the following relation

$$\frac{Q_1}{Q_2} = \frac{R_1}{R_2} = \frac{l_1 S_2 \rho_1}{l_2 S_1 \rho_2}, \tag{3}$$

where ρ_1 and ρ_2 are the resistivities of copper and lead, respectively.

Analysis of equations (1) - (3) leads us to the following relationship

$$\frac{D_1 l_1 S_1 c_1 \varDelta t}{D_2 l_2 S_2 \left(c_2 \varDelta t_1 + r\right)} = \frac{l_1 S_2 \rho_1}{l_2 S_1 \rho_2}. \tag{4}$$

Hence, the desired temperature increase is equal to

$$\Delta t = \frac{\rho_1 D_2 S_2^2 (c_2 \Delta t_1 + r)}{\rho_2 D_1 S_1^2 c_1}. \tag{5}$$

Substituting known values, we get

$$\Delta t = 1.8\ °C. \tag{6}$$

Answer. The change in temperature during a short circuit is $\Delta t = 1.8\ °C$.

Problem 3.3.5

Problem description. Calculate the resistance of an iron rod ($\rho_0 = 1.2 \cdot 10^{-5}\ \Omega \cdot cm$), whose end temperatures are $t_1 = 0°C$ and $t_2 = 800°C$, respectively. The rod length is $L = 5\ cm$. The cross-sectional area of the rod is $S = 1\ cm^2$. The temperature coefficient of resistance for iron is $\alpha = 6 \cdot 10^{-3}\ K^{-1}$. No heat is removed from the side surface of the rod. The dependence of the thermal conductivity coefficient on temperature should be neglected.

Known quantities: $\rho_0 = 1.2 \cdot 10^{-5}\ \Omega \cdot cm$, $t_1 = 0°C$, $t_2 = 800°C$, $L = 5\ cm$, $S = 1\ cm^2$, $\alpha = 6 \cdot 10^{-3}\ K^{-1}$.

Quantities to be calculated: R.

Problem solution. Thermal conduction is the mechanism that transfers heat in a rod. Since heat is not removed from the side surface of the rod, but is transferred only along the rod, then according to the Fourier law

$$Q_1 = -\kappa \frac{dT}{dl} S = -\kappa \frac{dt}{dl} S, \tag{1}$$

where

Q_1 is the amount of heat that is transferred through the cross section of the rod with area S;

κ is the thermal conductivity;

$dT / dl = dt / dl$ is the temperature gradient;

$d\,l$ is the elementary length counted along the rod.

Since the temperatures of the ends of the rod are kept constant, the heat conduction process is stationary, and the value Q_1 is the same for all cross sections of the rod. The values k and S are also not functions of temperature. Given the above arguments, we can write

$$\frac{dt}{dl} = const = \frac{t_2 - t_1}{L}, \tag{2}$$

where
L is the length of the rod;
t_1 and t_2 are the temperatures of the ends of the rod.

The law of temperature change along the rod can be determined by integrating equation (2)

$$\int_{t_1}^{t_2} dt = \int_0^L \frac{t_2 - t_1}{L} dl, \tag{3}$$

then

$$t_l = t_1 + \frac{t_2 - t_1}{L} l, \tag{4}$$

where l is the distance from the end of the bar to the fixed cross section.

Resistivity ρ at temperature t_l is

$$\rho = \rho_0 (1 + \alpha t_1) =$$

$$= \rho_0 \left\{ 1 + \alpha \left[t_1 + (t_2 - t_1) \frac{l}{L} \right] \right\} = \rho_0 (a + bl), \tag{5}$$

where $a = 1 + \alpha t_1$ and $b = (t_2 - t_1)\frac{\alpha}{L}$ are constants.

The resistance of the rod is given by

$$R = \int_0^L \rho \frac{dl}{S} = \frac{\rho_0}{S} \int_0^L (a + bl) dl =$$

$$= \frac{\rho_0}{S} \left(aL + \frac{bL^2}{2} \right). \tag{6}$$

Now we will substitute the values a and b into equation (6)

$$R=\frac{\rho_0 L}{S}\left[\left(1+\alpha t_1\right)+\frac{\alpha}{2}\left(t_2-t_1\right)\right]=$$

$$=\frac{\rho_0 L}{S}\left(1+\alpha\frac{t_2+t_1}{2}\right). \quad (7)$$

Numerically

$$R=2.04\cdot 10^{-2}\,\Omega. \quad (8)$$

Answer. The resistance of an iron rod is $R=2.04\cdot 10^{-2}\,\Omega$.

Problem 3.3.6

Problem description. A battery of $N=400$ cells (EMF and internal resistance of each element are $E=2\,V$ and $r=0.1\,\Omega$, respectively) is connected to an electrical circuit with an external resistance $R=10\,\Omega$. It is necessary to make a mixed battery from such a number (n_1) of parallel groups with n_2 cells connected in series in order to obtain the maximum current. Calculate the values n_1 and n_2, as well as the current in the resistance R in each element.

Known quantities: $N=400$, $E=2\,V$, $r=0.1\,\Omega$, $R=10\,\Omega$.

Quantities to be calculated: n_1, n_2, I, I_1.

Problem solution. We will apply Kirchhoff's laws for the electrical circuit that is mentioned in the problem statement (see Fig. 3.1).

The electrical circuit shown in Figure 1 has two nodes. Therefore, according to Kirchhoff's first law, we can write one equation for the node A

$$I=n_1 I_1, \quad (1)$$

where

n_1 is the number of parallel EMF groups;

I_1 is the current in each of these groups.

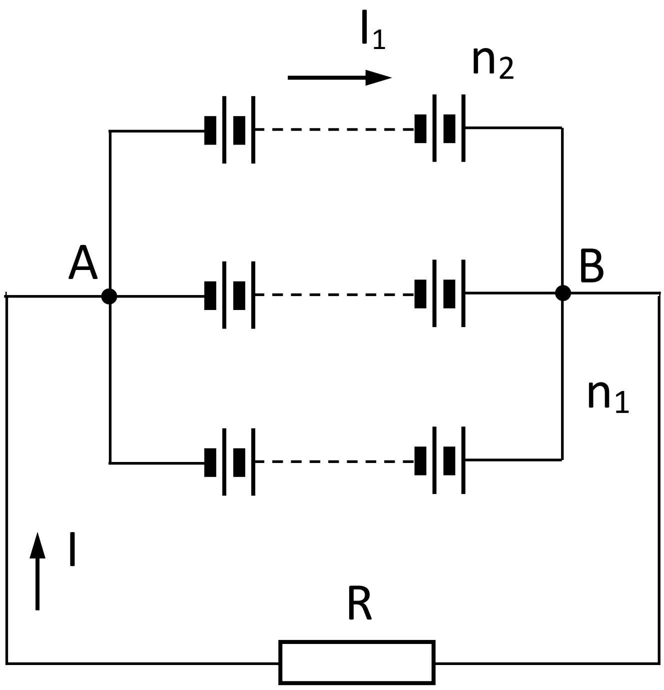

Figure 3.1. Problem 3.3.6.

Now we will write the second Kirchhoff's law for a closed loop that contains resistance R and one of the branches between the points A and B

$$IR + r\,n_2 I_1 = n_2 E\,, \tag{2}$$

where
r is the internal resistance of the EMF;
n_2 is the number of EMFs connected in series in each parallel branch;
E is the electromotive force of each element.

Besides

$$N = n_1 n_2\,, \tag{3}$$

where N is the number of all elements with electromotive forces equal to E, in an electric circuit.

The current in the electrical circuit can be determined from equations (1) - (3)

$$I = \frac{n_2 E}{R + r\left(\dfrac{n_2^2}{N}\right)}. \tag{4}$$

The current in the electrical circuit reaches its maximum value, provided that the external resistance is equal to the internal resistance of the EMF

$$R = r\frac{n_2^2}{N}, \tag{5}$$

and

$$n_2 = \sqrt{\frac{R}{r}N}. \tag{6}$$

Substituting known values, we get

$$n_2 = 200. \tag{7}$$

In this case, from equations (3) and (7) we can write the following relation for the number of parallel groups

$$n_1 = \sqrt{\frac{r}{R}N}. \tag{8}$$

Numerically

$$n_1 = 2. \tag{9}$$

Therefore, the maximum force at the values n_1 and n_2, given by equations (6) and (8) is

$$I = \frac{E\sqrt{N}}{2\sqrt{Rr}}. \tag{10}$$

Substituting known values, we get

$$I = 20\ A. \tag{11}$$

The current in individual cells can be determined according to equation (1)

$$I_1 = \frac{I}{n} = \frac{E\sqrt{RN}}{2\sqrt{RNr^2}}. \tag{12}$$

Substituting known values, we find

$$I_1 = 10\ A. \tag{13}$$

Answer. The number of parallel EMF groups is $n_1 = 2$. The number of series-connected elements is $n_2 = 200$. The current in the electrical circuit is $I = 20\ A$. The current in each of the elements is $I_1 = 10\ A$.

Problem 3.3.7

Problem description. Determine the resistance R between the points A and B for the electrical circuit shown in Fig. 3.2. The resistance of individual sections of the circuit (r, $2r$) is indicated in the figure.

Known quantities: r, $2r$.

Quantities to be calculated: R.

Problem solution. We will connect resistance R in an electrical circuit with EMF E and use Kirchhoff's laws.

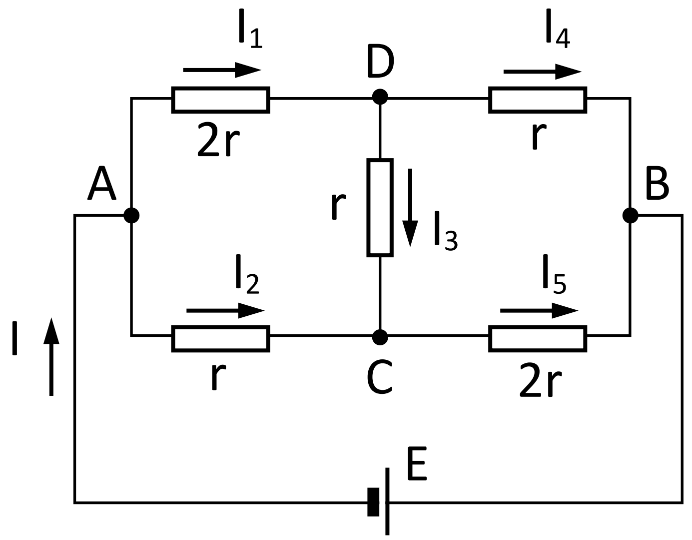

Figure 3.2. Problem 3.3.7.

The directions of currents in individual parts of the electrical circuit are indicated by arrows, and the magnitudes of the currents are indicated by a symbol I

with the corresponding index. The resistance values are also indicated in Fig. 2 with symbols r and $2r$. Based on Kirchhoff's first law for node A we can write

$$I_1 + I_3 = I. \tag{1}$$

respectively, for node C

$$I_1 = I_3 + I_4, \tag{2}$$

and for node D

$$I_2 + I_3 - I_5 = 0. \tag{3}$$

According to Kirchhoff's second law, for a closed circuit $ACDA$ we get

$$2r\,I_1 + r\,I_3 - r\,I_2 = 0, \tag{4}$$

respectively, for circuit $CBDC$

$$r\,I_4 - 2r\,I_5 - r\,I_3 = 0, \tag{5}$$

and for circuit $ADBEA$

$$r\,I_2 - 2r\,I_5 = E. \tag{6}$$

When writing the last equation, we neglected the internal resistance of the EMF E.

Now we solve the system of equations (1) - (6). Functional dependence $I = f(E,r)$ can be considered as a solution to the system of equations (1) - (6)

$$I = \frac{E}{\left(\frac{7}{5}r\right)}. \tag{7}$$

Ohm's law for a circuit section

$$I = \frac{E}{R}. \tag{8}$$

By comparing equations (7) and (8) we can determine the resistance R between the points A and B

$$R = \frac{7}{5} r . \tag{9}$$

Answer. Resistance between points A and B is $R = \frac{7}{5} r$.

Problem 3.3.8

Problem description. Determine the resistance R between points A and B for an electrical circuit consisting of five resistors R_1, R_2, R_3, R_4 and R_5, connected in a bridge circuit.

Known quantities: R_1, R_2, R_3, R_4, R_5.

Quantities to be calculated: R.

Problem solution. We introduce the following notation: I_1, I_2, I_3, I_4 and I_5 are the currents passing through the resistors R_1, R_2, R_3, R_4 and R_5, respectively (see Fig. 3.3).

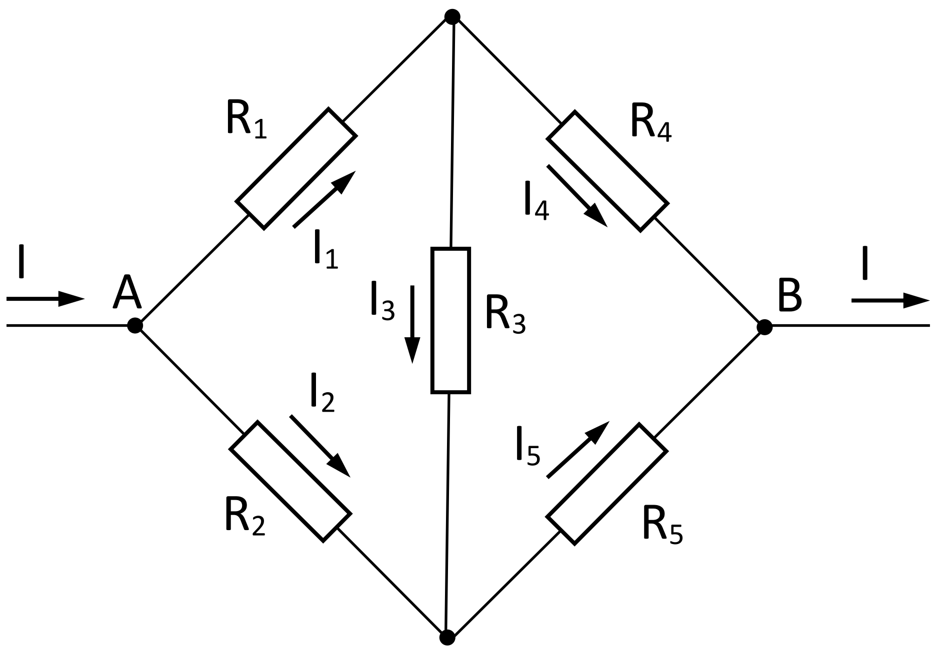

Figure 3.3. Problem 3.3.8.

The equivalent resistance of the entire electrical circuit between points A and B can be calculated according to Ohm's law by dividing the voltage between points A and B by the total current.

The voltage between the points A and B is equal to the sum of the voltages across the resistances R_1 and R_4 (or R_2 and R_5)

$$U_{AB} = I_1 R_1 + I_4 R_4. \tag{1}$$

The total current in the electrical circuit is

$$I = I_1 + I_2, \tag{2}$$

or

$$I = I_4 + I_5. \tag{3}$$

Therefore, we can express the total resistance as follows

$$R = \frac{U_{AB}}{I} = \frac{I_1 R_1 + I_4 R_4}{I_1 + I_2}. \tag{4}$$

To determine the currents, we will write, according to the first and second Kirchhoff laws, the following equations

$$I_4 = I_1 - I_3, \tag{5}$$

$$I_5 = I_2 + I_3, \tag{6}$$

$$I_1 R_1 + I_3 R_3 = I_2 R_2, \tag{7}$$

$$I_4 R_4 = I_3 R_3 + I_5 R_5. \tag{8}$$

We will rewrite equation (4) using equation (5)

$$R = \frac{I_1 R_1 + I_1 R_4 - I_3 R_4}{I_1 + I_2}. \tag{9}$$

or, taking into account the system of equations (5) - (8)

$$R = \frac{R_1 R_2 (R_3 + R_4 + R_5) + R_4 R_5 (R_1 + R_2 + R_3)}{(R_1 + R_2)(R_3 + R_4 + R_5) + R_3 (R_4 + R_5)} +$$

$$+ \frac{R_3 (R_1 R_5 + R_2 R_4)}{(R_1 + R_2)(R_3 + R_4 + R_5) + R_3 (R_4 + R_5)}. \qquad (10)$$

Answer. Resistance between points A and B is

$$R = \frac{R_1 R_2 (R_3 + R_4 + R_5) + R_4 R_5 (R_1 + R_2 + R_3)}{(R_1 + R_2)(R_3 + R_4 + R_5) + R_3 (R_4 + R_5)} +$$

$$+ \frac{R_3 (R_1 R_5 + R_2 R_4)}{(R_1 + R_2)(R_3 + R_4 + R_5) + R_3 (R_4 + R_5)}.$$

Problem 3.3.9

Problem description. Determine the strength of the currents that pass through the resistance $R_1 = R_4 = 2\ \Omega$ and $R_2 = R_3 = 4\ \Omega$ (Fig. 4). The magnitudes of the electromotive forces of current sources connected to the electrical circuit are $E_1 = 10\,V$ and $E_2 = 4\,V$. The internal resistance of current sources can be neglected.

Known quantities: $R_1 = R_4 = 2\ \Omega$, $R_2 = R_3 = 4\ \Omega$, $E_1 = 10\,V$, $E_2 = 4\,V$.

Quantities to be calculated: I_1, I_2, I_3, I_4.

Problem solution. The electrical circuit shown in Fig. 3.4 is a branched chain. Therefore, in order to determine the currents, we need to use Kirchhoff's laws.

To solve the problem, it is necessary to compose four equations (according to the number of unknown quantities - currents). We will arbitrarily choose the directions of the currents in each section of the electrical circuit (these directions are indicated by arrows in the figure).

We will also choose the direction of bypassing closed contours arbitrarily, namely, in the clockwise direction. In addition, to solve the problem, we introduce the following notation: I_1, I_2, I_3, I_4 are the currents passing through the resistances R_1, R_2, R_3, R_4, respectively; E_1, E_2 are the electromotive forces of current sources connected to an electrical circuit.

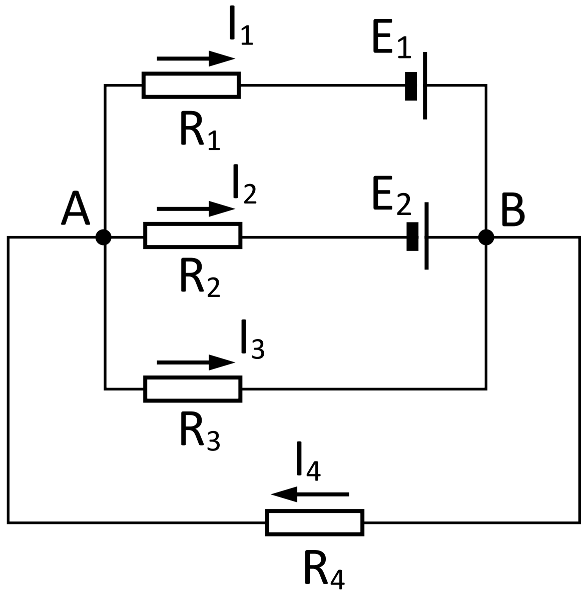

Figure 3.4. Problem 3.3.9.

According to Kirchhoff's first law, for node B we can write

$$I_1 + I_2 + I_3 - I_4 = 0. \quad (1)$$

We write the second Kirchhoff law for a closed sircuit AR_1BR_2A

$$I_1R_1 - I_2R_2 = E_1 - E_2, \quad (2)$$

for sircuit AR_1BR_3A

$$I_1R_1 - I_3R_3 = E_1, \quad (3)$$

for sircuit AR_3BR_4A

$$I_3R_3 + I_4R_4 = 0. \quad (4)$$

Now, to simplify the solution, we will substitute in equations (1) - (4) the numerical values of resistance and EMF

$$I_1 + I_2 + I_3 - I_4 = 0, \quad (5)$$

$$2I_1 - 4I_2 = 6, \quad (6)$$

$$2I_1 - 4I_3 = 10, \tag{7}$$

$$4I_3 + 2I_4 = 0. \tag{8}$$

The solution of this system is the following values of currents

$$I_1 = 3A, \qquad I_2 = 0, \qquad I_3 = -1A, \qquad I_4 = 2A. \tag{9}$$

The "–" sign in the numerical value of the current I_3 indicates that with an arbitrary choice of the directions of the currents (those indicated in Fig. 3.4), we made a mistake. In reality, the current I_3 has the opposite direction: i.e. current is directed from node B to node A.

Answer. The currents that pass through the resistances are: $I_1 = 3A$, $I_2 = 0$, $I_3 = -1A$, $I_4 = 2A$.

Problem 3.3.10

Problem description. The low-resistance winding for speaker magnetization is connected by a transmission line to the battery with EMF $E = 8\,V$. The transmission line resistance is $R = 1\,\Omega$. Calculate the resistance of winding, provided that the power of winding magnetizing is $P = 8\,W$.

Known quantities: $E = 8\,V$, $R = 1\,\Omega$, $P = 8\,W$.

Quantities to be calculated: R_x.

Problem solution. We will denote the resistance of the magnetization winding with the symbol R_x. The current in the electrical circuit is

$$I = \frac{E}{R + R_x}, \tag{1}$$

where
E is the EMF of the battery;
R s the resistance of the transmission line.

The power that is released in the speaker coil can be determined as follows

$$P = I^2 R_x = \left(\frac{E}{R + R_x} \right)^2 R_x . \qquad (2)$$

Given numerical values, we can form a quadratic equation for the unknown quantity R_x

$$R_x^2 - 6R_x + 1 = 0 . \qquad (3)$$

Therefore, the problem has two solutions.

1. The resistance of the magnetizing winding is $R'_x = 5.8\ \Omega$. In this case, the current is $I_1 = 1.17\ A$; the voltage at the winding terminals is $U_1 = I_1 R_x = 6.8\ V$; the line voltage loss is $U_2 = I_1 R = 1.17\ V$; the line power loss is $P_1 = I^2 R = 1.37\ W$.

2. The resistance of the magnetizing winding is $R''_x = 0.17\ \Omega$. In this case, the current is $I_2 = 6.8\ A$; the voltage at the winding terminals is $U'_1 = 1.17\ V$; the line voltage loss is $U'_2 = 6.8\ V$; the line power loss is $P_2 = 47\ W$.

The analysis of the presented numerical data allows us to conclude that the first option is practically more convenient than the second one. Consequently, $R_x = R'_x = 5.8\ \Omega$.

Answer. The resistance of winding is $R_x = 5.8\ \Omega$.

Problem 3.3.11

Problem description. The electrical circuit is energized at $U = 110\ V$ and contains an electrical furnace connected in series with a resistance $R = 5\ \Omega$. Calculate the resistance of the furnace, provided that its power is $P = 200\ W$.

Known quantities: $U = 110\ V$, $R = 5\ \Omega$, $P = 200\ W$.

Quantities to be calculated: R_x.

Problem solution. According to Ohm's law, we can write the following equation

$$U = I(R + R_x), \qquad (1)$$

where

U is the voltage in the circuit;
R s the resistance that is connected to the electrical circuit;
R_x is the resistance of the electric furnace.

The power of the electric current in the furnace is

$$P = I^2 R_x . \qquad (2)$$

Taking into account equation (2), we can write equation (1) in the following form

$$U = I\left(R + \frac{PI}{I^2} \right) = IR + PI , \qquad (3)$$

or

$$I^2 R - IU + P = 0 . \qquad (4)$$

The solution to equation (4) is the mathematical expression for the current

$$I = \frac{U \pm \sqrt{U^2 - 4RP}}{2R} . \qquad (5)$$

In this case, we obtain two solutions for the furnace resistance

$$R_{x,i} = \frac{P}{I_i^2} , \quad i = 1,\ 2 , \qquad (6)$$

where I_i is the solution of equation (5).

Substituting known values, we get
for $I_1 = 20\ A$

$$R_{x,1} = 0.5\ \Omega . \qquad (7)$$

for $I_2 = 2\ A$

$$R_{x,2} = 50\ \Omega . \qquad (8)$$

Both solutions correspond to the condition of the problem. However, these solutions have the following features. The first solution $R_{x,1} = 0.5\ \Omega$ corresponds to the power that is consumed from the source (the calculation was carried out according to formula (2)). The second solution $R_{x,2} = 50\ \Omega$ corresponds to the power $P_2 = 220\ W$ that is consumed from the source. Therefore, it is necessary to use a furnace that consumes less power and has resistance $R_x = R_{x,2} = 50\ \Omega$.

Answer. Furnace resistance is $R_x = 50\ \Omega$.

Problem 3.3.12

Problem description. A generator with EMF $E = 140\ V$ and internal resistance $r = 0.2\ \Omega$ produces a current of $I = 100\ A$. The resistance of the external electrical circuit is $R = 1.2\ \Omega$. Calculate the total and useful power of the generator, as well as electrical losses and efficiency. Write a power balance equation.

Known quantities: $E = 140\ V$, $r = 0.2\ \Omega$, $I = 100\ A$, $R = 1.2\ \Omega$.

Quantities to be calculated: P_0, P, ΔP, η, power balance equation.

Problem solution. The total power that the generator develops is determined by the equation

$$P_0 = EI, \tag{1}$$

where
E is the EMF of the generator;
I is the current in the generator circuit.

Substituting known values, we get

$$P_0 = 1.4 \cdot 10^4\ W. \tag{2}$$

The useful power that is released in the external electrical circuit is given by

$$P = IU, \tag{3}$$

где U is the potential difference at the ends of an external electrical circuit.

We can write the following equation for this potential difference

$$U = E - Ir\,, \tag{4}$$

where r is the internal resistance of the generator.

In this case, equation (3) can be rewritten as follows

$$P = (E - Ir)I\,. \tag{5}$$

Numerically

$$P = 1.2 \cdot 10^4 \; W\,. \tag{6}$$

Power losses (for heating) in the external electrical circuit are equal to

$$\Delta P = P_0 - P\,. \tag{7}$$

Substituting known values, we have

$$\Delta P = 2 \cdot 10^3 \; W\,. \tag{8}$$

The efficiency of the generator is determined by the following relationship

$$\eta = \frac{P}{P_0}\,. \tag{9}$$

Substituting known values, we get

$$\eta = 85.7\;\%\,. \tag{10}$$

Let's write the power balance equation

$$IE = I^2 r + I^2 R\,. \tag{11}$$

Numerically

$$140 \cdot 100 = 10\,000 \cdot 0.2 + 10\,000 \cdot 1.2\,. \tag{12}$$

<u>Answer</u>. The total power is $P_0 = 1.4 \cdot 10^4 \; W$. The useful power is $P = 1.2 \cdot 10^4 \; W$. The power losses in the external electrical circuit are $\Delta P = 2 \cdot 10^3 \; W$. The generator efficiency is $\eta = 85.7\;\%$. The power balance

equation has an analytical form: $IE = I^2 r + I^2 R$ and, accordingly, the numerical form: $140 \cdot 100 = 10\ 000 \cdot 0.2 + 10\ 000 \cdot 1.2$.

3.4. Level 1 problems

3.4.1. The current in the conductor increases uniformly from $I_0 = 0$ to $I = 3\ A$ during the time $t = 10\ s$. Determine the charge that has passed through the conductor during this time.

3.4.2. Calculate the current density in an iron conductor of length $l = 10\ m$ provided that the wire is energized with $U = 6\ V$.

3.4.3. The voltage on the busbars of the power plant is $U = 6.6\ kV$. The consumer of electrical energy is located at a distance of $l = 10\ km$ from the power plant. Determine the cross-sectional area of the copper wire that must be used for the device of a two-wire transmission line, if the current in the line is $I = 20\ A$ and the voltage loss in the wires should not exceed 3%.

3.4.4. Calculate the resistance of a graphite conductor made in the form of a right circular truncated cone with a height of $h = 20\ cm$ and base radii of $r_1 = 12\ mm$ and $r_2 = 8\ mm$. The temperature of the conductor is $t = 20\ °C$.

3.4.5. At one end of a cylindrical copper conductor with a resistance of $R_0 = 10\ \Omega$ (at $0\ °C$) a temperature of $t_1 = 20\ °C$ is maintained, and at the other end a temperature of $t_{23} = 400\ °C$ is maintained. Calculate the resistance of the conductor assuming that the temperature gradient does not change along the axis of the conductor.

3.4.6. The resistance R and ammeter are connected in series and connected to a current source. A voltmeter with a resistance of $R_1 = 1\ k\Omega$ is connected to the terminals of the inductor. Voltmeter and ammeter readings are $U = 100\ V$ and $I = 0.5\ A$, respectively. Determine the amount of resistance R. Calculate the relative error of the resistance R, due to the resistance of the voltmeter.

3.4.7. A resistance $R = 0.1\ \Omega$ was connected to a current source with an EMF of $E = 1.5\ V$. The ammeter reading was $I_1 = 0.5\ A$. When another current source with the same EMF was connected in series to the current source, the current passing through the resistance became equal to $I_2 = 0.4\ A$. Determine the internal resistances of the first and second current source.

3.4.8. Two groups of three series-connected elements are connected in parallel. The EMF of each element is $E = 1.2\ V$, and the internal resistance is $r = 0.2\ \Omega$. The resulting circuit is connected to an external resistance $R = 1.5\ \Omega$. Calculate the current in the external electrical circuit.

3.4.9. Twelve identical elements are characterized by an EMF and an internal resistance of $r = 0.4\ \Omega$. Determine such a way of connecting these elements so that the maximum current passes in an external electrical circuit with a resistance of $R = 0.3\ \Omega$. Calculate this maximum current.

3.4.10. Two identical current sources with EMF $E = 1.2\ V$ and internal resistance $R = 0.4\ \Omega$ are connected by poles with the same and opposite signs. Calculate the current in the electrical circuit, as well as the potential difference between points located on opposite sides of one of the current sources in the first and second cases.

3.4.11. The two elements of an electrical circuit have the following characteristics: $E_1 = 1.2\ V$, $r_1 = 0.1\ \Omega$, $E_2 = 0.9\ V$, $r_2 = 0.3\ \Omega$. These elements are connected by poles with the same signs. The resistance of the connecting wires is $R = 0.2\ \Omega$. Determine the current in the circuit.

3.4.12. Three electric batteries with EMF $E_1 = 12\ V$, $E_2 = 5\ V$, $E_3 = 10\ V$ have the same internal resistance $r = 1\ \Omega$. Electric batteries are interconnected by the same poles. The resistance of the connecting wires is negligible. Calculate the currents that pass through each electric battery.

3.4.13. A light bulb and a rheostat are connected in series and connected to a current source. The voltage across the bulb is $U = 40\ V$. The resistance of the rheostat is $R = 10\ \Omega$. The external electrical circuit consumes a power equal to $P = 120\ W$. Determine the current in this electrical circuit.

3.4.14. The EMF of the battery is $E = 12\ V$. The short circuit current is $I = 5\ A$. Calculate the maximum power in the external electrical circuit connected to such a battery.

3.4.15. The battery EMF is $E = 20\ V$. The resistance of the external electrical circuit is $R = 2\ \Omega$, and the current in it is $I = 4\ A$. Determine the efficiency of the battery. Calculate the value of external resistance at which the efficiency will be equal to $\eta = 99\ \%$.

3.4.16. A heater is connected to the battery terminals. The battery EMF is $E = 24\,V$, and the internal resistance is $r = 1\,\Omega$. The heater, when connected to an electrical circuit, consumes power $P = 80\,W$. Calculate the current in the circuit and the efficiency of the heater.

3.4.17. The winding of the electric boiler has two sections. If only the first section is turned on, then the water boils after a time of $t\ 1 = 15\ min$. If only the second section is turned on, then the water boils after a time of $t_2 = 30\ min$. Determine the boiling time of water for two cases: 1) both sections are connected in series, 2) both sections are connected in parallel.

3.4.18. The current in the external electrical circuit of the battery is $I_1 = 3\ A$. The power of the battery pack is $P_1 = 18\,W$. If a current $I_2 = 1\ A$ passes through the external electrical circuit, then power $P_2 = 10\ W$ is released. Determine the EMF and internal resistance of the battery pack.

3.4.19. The current in the conductor with resistance $R = 12\ \Omega$ decreases uniformly from the value $I_0 = 5\ A$ to the value $I = 0$ during the time $t = 10\ s$. Determine the amount of heat that is released in this conductor for a specified period of time.

3.4.20. The current in the conductor with resistance $R = 15\ \Omega$ increases uniformly from the value $I_0 = 0$ to some maximum value during the time $\tau = 5\ s$. During this time, an amount of heat equal to $Q = 10\ kJ$ was released in the conductor. Calculate the average current in the conductor for this period of time.

3.5. Answers to problems

3.4.1. $Q = 15\,C$.

3.4.2. $j = 6.1 \cdot 10^6\ A \cdot m^{-2}$.

3.4.3. $S = 34.2\ mm^2$.

3.4.4. $R = 2.58 \cdot 10^{-3}\ \Omega$.

3.4.5. $R = 18.8\ \Omega$.

3.4.6. $R = 250\ \Omega$; $\varepsilon_R = 20\ \%$.

3.4.7. $r_1 = 2.9\ \Omega$; $r_2 = 4.5\ \Omega$.

3.4.8. $I = 2\ A$.

3.4.9. $I = 7.5\ A$.

3.4.10. 1) $I = 3\ A,\ U = 0$; 2) $I = 0,\ U = 1.2\ V$.

3.4.11. $I = 0.5\ A$.

3.4.12. $I_1 = 3\ A$; $I_2 = 4\ A$; $I_3 = 1\ A$.

3.4.13. $I = 2\ A$.

3.4.14. $P_{max} = 15\ W$.

3.4.15. $\eta = 40\ \%$; $R = 297\ \Omega$.

3.4.16. $I = 20\ A$; $\eta = 17\ \%$.

3.4.17. $t_1 = 45\ min$; $t_2 = 10\ min$.

3.4.18. $E = 12\ V$; $r = 2\ \Omega$.

3.4.19. $Q = 10^3\ J$.

3.4.20. $< I > = 10\ A$.

CHAPTER 4. MAGNETIC FIELD

4.1. Basic formulas

The magnetic field intensity formed by the element dx of the conductor through which the current I, at a point that is at a distance of l, is determined according to the **Biot-Savart-Laplace law** (Jean-Baptiste Biot 1774 – 1862, Félix Savart 1791 – 1841, Pierre-Simon de Laplace 1749 – 1827)

$$dH = \frac{Idx \sin\alpha}{4\pi l^2}, \tag{4.1.1}$$

where α is the angle between $d\vec{x}$ and $\vec{l}$.

The vector is perpendicular to the plane that passes through $d\vec{x}$ and $\vec{l}$ and is directed according to the right screw rule. The right screw rule states that if the rotational motion of the screw occurs in the direction from $d\vec{x}$ to $\vec{l}$ along the shortest distance, then the translational motion of the screw coincides with the direction of the vector $d\vec{H}$.

The intensity of the magnetic field formed at a point A by a straight segment of a conductor, through which current I passes, is given by

$$H = \frac{I}{l}\left(cos\,\alpha_1 - cos\,\alpha_2\right), \tag{4.1.2}$$

where

α_1 and α_2 are the angles between the conductor and the directions to the point A from the ends of the segment;

l is the length of the conductor.

The intensity of the magnetic field formed by an infinitely long thin straight conductor with current at a point located at a distance of l from the conductor is

$$H = \frac{I}{2\pi l}. \tag{4.1.3}$$

The magnetic field intensity at the center of the circle arc with a radius R is

$$H = \frac{Id}{R^2}. \tag{4.1.4}$$

where d is the length of the arc.

The magnetic field intensity in the center of a circular conductor with current can be expressed by the relation

$$H = \frac{I}{2R}, \tag{4.1.5}$$

where R is the radius of the circular conductor.

The magnetic field intensity on the axis of a circular conductor with current is determined by the equation

$$H = \frac{IR^2}{2\left(R^2 + l^2\right)^{3/2}}, \tag{4.1.6}$$

where l is the distance from the center of the circular current to a point on the axis.

The magnetic field intensity at a point A on the axis of a direct solenoid is

$$H = 2\pi nI\left(cos\,\beta_1 - cos\,\beta_2\right), \tag{4.1.7}$$

where

β_1 is an acute (or right) angle between the axis of the solenoid and the direction from point A to some element of the extreme turn of the solenoid;

β_2 is an obtuse angle between the axis of the solenoid and the direction from point A to some element of the other extreme turn of the solenoid;

n is the number of turns per unit length of the solenoid;

I is the current passes through the solenoid.

The magnetic field intensity on the axis of a long straight solenoid (or toroid) at points near its middle is

$$H = nI. \tag{4.1.8}$$

The **magnetic moment of the circuit with current** I is given by

$$\vec{p}_m = \mu\mu_0 I\vec{S}, \tag{4.1.9}$$

where

μ is the relative magnetic permeability of the medium in which the current-carrying circuit is located;

μ_0 is the magnetic constant;

$\vec{S}$ is a vector that is numerically equal to the contour area S and is directed along the normal $\vec{n}$, constructed to the contour plane according to the right screw rule.

The **circulation of the magnetic field intensity vector** $\vec{H}$ along a closed loop that covers the current I, is

$$\oint \vec{H}_l d\vec{l} = I, \tag{4.1.10}$$

where
$\vec{H}_l$ is the component of the magnetic field intensity vector $\vec{H}$, directed tangentially to the contour that contains the element $d\vec{l}$;
I is the current that is covered by the circuit.

The functional relationship between magnetic induction, magnetic intensity and relative magnetic permeability has the form

$$\vec{B} = \mu\mu_0\vec{H}, \tag{4.1.11}$$

where $\vec{B}$ s the magnetic field induction vector.

The **flux of magnetic induction** through a flat surface S is determined by the formula

$$\Phi = \left(\vec{B}\vec{S}\right) = BS\cos\alpha, \tag{4.1.12}$$

where α is the angle between vectors $\vec{B}$ and $\vec{S}$.

The **Hopkinson formula** (John Hopkinson 1849 – 1898**)** for numerical calculations of the magnetic field flux has the form

$$\Phi = \frac{(IN)}{\left(\frac{l}{\mu\mu_0 S}\right)}, \tag{4.1.13}$$

where
l is the length of the magnetic circuit;
$\frac{l}{\mu\mu_0 S} = R_m$ is the magnetic resistance;
N is the number of closed circuits with current.

For the case when the magnetic circuit consists of several sections connected in series with circuit lengths l_1, l_2, …, areas S_1, S_2, … and relative magnetic permeabilities μ_1, μ_2, …, the **equivalent magnetic resistance** is equal to

$$R_m = \frac{l_1}{\mu_1\mu_0 S_1} + \frac{l_2}{\mu_2\mu_0 S_2} + ... \qquad (4.1.14)$$

In this case, we can write the following formula for the magnetic flux

$$\Phi = \frac{IN}{\left[\left(\frac{l_1}{\mu_1\mu_0 S_1}\right) + \left(\frac{l_2}{\mu_2\mu_0 S_2}\right) + ...\right]}. \qquad (4.1.15)$$

According to **Ampère's law**, the modulus of force that acts on a current-carrying conductor in a magnetic field is

$$F = IBdx\,sin\gamma, \qquad (4.1.16)$$

where
I is the current;
B is the induction of the magnetic field;
$d\vec{x}$ is an element of a current-carrying conductor;
γ is the angle between the vectors $d\vec{x}$ and $\vec{B}$.

The **force of interaction between conductors** that passes in two rectilinear parallel conductors is

$$F = \mu\mu_0 \frac{I_1 I_2}{2\pi d} dx, \qquad (4.1.17)$$

where
I_1 and I_2 are the currents in the first and second conductors, respectively;
d is the distance between conductors;
dx is the length of the first conductor (the length of the second conductor is assumed to be very large compared to the distance d).

The torque module of a pair of forces that acts on a coil (or coil) with a current is

$$M = Bp_m\,sin\alpha, \qquad (4.1.18)$$

where
B is the induction of the magnetic field in which the circuit with current is located;
p_m is the magnetic moment of the circuit with current;
α is the angle between the vectors $\vec{B}$ and $\vec{p}_m$.

The **lifting force** [4, p. 118] **of an electromagnet** is given by the equation

$$F = \frac{B^2}{2\mu\mu_0} S . \tag{4.1.19}$$

The **Lorentz force** modulus (Hedrik Antoon Lorentz 1853 – 1928) is determined by a functional relationship that is linear with respect to the charge

$$F_L = QB\upsilon \sin\alpha , \tag{4.1.20}$$

where
Q is a single charge moving in a magnetic field;
B is the induction of the magnetic field;
υ is the speed of the charge;
α is the angle between the vectors $\vec{B}$ and $\vec{\upsilon}$.

Since it is perpendicular to $\vec{\upsilon}$, $\vec{F}_L$ does not do any work on the particle with charge Q.

The **work of moving a conductor** (or a closed loop) with current in a magnetic field is equal to

$$A = I\Delta\Phi = I(\Phi_2 - \Phi_1), \tag{4.1.21}$$

where
I is the current passing through the conductor;
Φ_1 is the magnetic flux inside the loop at its initial position;
Φ_2 is the magnetic flux inside the circuit in its final position.

4.2. Problem-solving framework

The main characteristics of the magnetic field are two quantities: magnetic induction $\vec{B}$ and magnetic field intensity $\vec{H}$. These quantities are related by the following relationship

$$\vec{B} = \mu\mu_0\vec{H} . \tag{4.2.1}$$

The calculation of the magnetic field intensity of the current that passes through a conductor of arbitrary geometric shape is based on the following theoretical dependencies: 1) Biot-Savart-Laplace formula; 2) the theorem on the circulation of the magnetic field vector.

When using the Biot-Savart-Laplace formula, it is necessary to first select a certain current element with a length of dx on the conductor. The elementary intensity of the magnetic field H_1 (according to the first method) formed by such an element is calculated according to the formula (see the notation for formula (4.1.1))

$$dH_1 = \frac{Idx\,sin\,\alpha}{4\pi\,l_2}. \qquad (4.2.2)$$

We can find the total intensity of the magnetic field at some point by integrating formula (4.2.2). In this case, we must always remember that the magnetic field intensity is a vector quantity.

Often, when determining the intensity of a magnetic field, calculations are simplified if we use the concept of a magnetizing force, which is proportional to the algebraic sum of the currents covered by a given closed cirquit. This algebraic sum of currents is equal to the linear integral of the magnetic field intensity along the contour (L)

$$\oint_{(L)} \vec{H}d\vec{l} = \sum_{k=1}^{n} I_k\,. \qquad (4.2.3)$$

The second method of solving problems is convenient to use if the circuit that covers the currents can be chosen with the correct geometric shape. In this case, the magnetic field intensity at each point of such a circuit should be the same.

The electrodynamic force that acts on a conductor of length l in a uniform magnetic field with an induction of $\vec{B}$, is determined by integrating the elementary electrodynamic force (Ampère's law)

$$dF = IBdl\,sin\gamma \qquad (4.2.4)$$

along the entire length of the conductor.

When calculating the force of interaction between two conductors with currents, Ampère's law is also used. In this case, the value B refers to the induction of magnetic field that one conductor generated in the place where the other conductor is located.

4.3. Problem-solving examples

Problem 4.3.1

Problem description. The current $I = 2\ A$ flows in a straight wire of length $l = 10\ cm$. Determine the magnetic field intensity formed by this current at a point that is perpendicular to the middle of the conductor at a distance of $d = 6\ cm$ from it.

Known quantities: $I = 2\ A$, $l = 10\ cm$, $d = 6\ cm$.

Quantities to be calculated: H .

Problem solution. We will use the Biot-Savart-Laplace law to calculate the magnetic field intensity

$$H = \int \frac{I \sin\varphi}{4\pi r^2} dl . \qquad (1)$$

where
I is the current;
dl is the current element module $d\vec{l}$;
r is the modulus of the radius vector $\vec{r}$, drawn from the current element to point A, where the magnetic field intensity is determined;
φ is the angle between the vectors $d\vec{l}$ and $\vec{r}$.

For the system specified in the condition of the problem, we can write the following geometric relations

$$dl = \frac{r d\varphi}{\sin\varphi}, \qquad (2)$$

and

$$r = \frac{d}{\sin\varphi}, \qquad (3)$$

where d is the distance from the middle of the straight wire segment, through which the current passes, to point A.

Now we can rewrite equation (1)

$$H = \frac{I}{4\pi d} \int_{\varphi_1}^{\varphi_2} sin\varphi \, d\varphi . \quad (4)$$

After integration we get

$$H = \frac{I}{4\pi d}\left(cos\varphi_1 - cos\varphi_2\right). \quad (5)$$

In addition, the following relations are valid for the angle φ_2

$$cos\varphi_2 = cos(\pi - \varphi_1) = -cos\varphi_1 \quad (6)$$

and on the angle φ_1

$$cos\varphi_1 = \frac{l}{\sqrt{4d^2 + l^2}} , \quad (7)$$

where l s the length of the wire carrying the current I .

As a result, the intensity of the magnetic field formed by the current I at point A, is given by

$$H = \frac{Il}{2\pi d\sqrt{4d^2 + l^2}} . \quad (8)$$

Substituting known values, we get

$$H = 3\ A \cdot m^{-1}. \quad (9)$$

Answer. The intensity of the magnetic field formed by the current I at the point A, is $H = 3\ A \cdot m^{-1}$.

Problem 4.3.2

Problem description. Currents $I_1 = 10\ A$ and $I_2 = 10\ A$ are carried in two long straight wires, the distance between which is $d = 5\ cm$. Determine the magnetic field intensity at a point that is located in the middle between the wires for the following cases when the wires are parallel and the directions of the currents coincide with each other; when the wires are parallel and the directions of the currents are opposite; and finally, when the wires are arranged in mutually perpendicular directions.

Known quantities: $I_1 = 10\ A,\ I_2 = 10\ A,\ d = 5\ cm.$

Quantities to be calculated: 1) $H_1,\ H_2$; 2) $H_1,\ H_2$; 3) $H_1,\ H_2$.

Problem solution. The resulting magnetic field intensity is equal to the vector sum of the field intensities generated by each current separately

$$\vec{H} = \vec{H}_1 + \vec{H}_2. \quad (1)$$

where

$\vec{H}_1$ is the intensity of the magnetic field generated by the current I_1;

$\vec{H}_2$ is the intensity of the magnetic field generated by the current I_2.

In the case when the vectors $\vec{H}_1$ and $\vec{H}_2$ are directed along one straight line, the geometric sum can be replaced by an algebraic sum

$$H = H_1 + H_2. \quad (2)$$

To solve the problem for all three cases, it is necessary to determine the modules and directions of the vectors $\vec{H}_1$ and $\vec{H}_2$. The direction of the magnetic field intensity vector formed by the current flowing through an infinitely long straight wire is determined by the right screw rule. The module of the same vector can be calculated in accordance with the formula

$$H = \frac{I}{2\pi r}, \quad (3)$$

where

I is the current;

r is the distance from the wire to the point at which the modulus of the vector $\vec{H}$ is determined.

In this problem, in all three cases, the modules of vectors $\vec{H}_1$ and $\vec{H}_2$ will be the same, since the points at which the magnetic field strength is determined are at the same distance from the wires, and, moreover, the current strengths are also the same.

Let us determine the magnetic field strength in each case.

1) Due to the same direction of currents and the peculiarities of applying the right screw rule, for the resulting magnetic field intensity we get

$$H = \frac{I}{\pi d} - \frac{I}{\pi d} = 0. \quad (4)$$

2) In this case, the vectors $\vec{H}_1$ and $\vec{H}_2$ at the midpoint are directed in the same direction, so the absolute value of the vector of the resulting magnetic field strength is equal to

$$H = \frac{I}{\pi d} + \frac{I}{\pi d} = \frac{2I}{\pi d}. \quad (5)$$

Numerically

$$H = 130\ A \cdot m^{-1}. \quad (6)$$

3) The intensities of the magnetic fields formed by currents flowing through mutually perpendicular wires, at a point that is in the middle between the wires, are also mutually perpendicular. Therefore, the modulus of the resulting magnetic field intensity is equal to

$$H = \sqrt{H_1^2 + H_2^2}\ . \quad (7)$$

Substituting known values, we get

$$H = 90\ A \cdot m^{-1}. \quad (8)$$

Answer. The module of the resulting magnetic field intensity at a point located in the middle between the wires is equal to: 1) $H_1 = H_2 = 0$; 2) $H_1 = H_2 = 130\ A \cdot m^{-1}$; 3) $H_1 = H_2 = 90\ A \cdot m^{-1}$.

Problem 4.3.3

Problem description. A ring wire with a radius of $R = 11\,cm$, carries an electric current of $I = 14\ A$. Determine the magnetic field intensity in the center of the ring; and also at a point that is perpendicular to the plane of the ring drawn from its center, at a distance of $l = 10\ cm$ from its center.

Known quantities: $R = 11\,cm$, $I = 14\ A$, $l = 10\,cm$.

Quantities to be calculated: H_1, H_2.

Problem solution. To simplify the solution of the problem, we will divide the circular contour with a radius of R into elements dx. Next, we will denote the distance from point A, where the magnetic field intensity is determined, to these elements with the symbol r. The angle α etween directions $\vec{r}$ and $d\vec{x}$ is $\pi/2$.

The intensity of the magnetic field, which is formed by the contour element dx at point A, is determined according to the Biot-Savart-Laplace formula

$$H = \frac{Idx}{4\pi r^2}, \qquad (1)$$

where I is the current passing through a circular conductor.

The resulting magnetic field intensity $\vec{H}$ is the vector sum of elementary intensities $d\vec{H}$. The vector $\vec{H}$ is directed along the axis of the circular current. The modulus of the vector $\vec{H}$ is

$$H = \int \frac{I}{4\pi r^2} sin\,\beta\, dx, \qquad (2)$$

where β is the angle between the direction from the point A to the center of the circular conductor and the vector $\vec{r}$.

Functional relationships $\beta = f(R,r)$ and $r = f(R,l)$, where l is the distance from a point A to the center of a circular conductor, can be defined by the following equations

$$sin\,\beta = \frac{R}{r}, \qquad (3)$$

and

$$r = \sqrt{R^2 + l^2}. \qquad (4)$$

In this case, we can rewrite formula (2) as follows

$$H = \int \frac{IRdx}{4\pi\sqrt{\left(R^2+l^2\right)^3}} = \frac{IR}{4\pi\sqrt{\left(R^2+l^2\right)^3}} \int dx =$$

$$= \frac{IR}{4\pi\sqrt{\left(R^2+l^2\right)^3}} \cdot (2\pi R) = \frac{IR^2}{2\left(R^2+l^2\right)^{3/2}}. \quad (5)$$

Numerically

$$H_2 = H = 26 \ A \cdot m^{-1}. \quad (6)$$

Now we will find the magnetic field intensity for the second case. The value l for the center of the circular conductor is zero: $l = 0$. Therefore, formula (5) in this case, we can write as follows

$$H = \frac{IR^2}{2R^3} = \frac{I}{2R}. \quad (7)$$

Substituting known values, we get

$$H_1 = H = 64 \ A \cdot m^{-1}. \quad (8)$$

It should be noted that for distances l, greater than the radius R of the circular contour, the value R in the denominator of formula (5) can be neglected

$$H\big|_{l>>R} \approx \frac{IR^2}{2l^3}. \quad (9)$$

For such distances l we get the dependence $H = f\left(\frac{1}{l^3}\right)$.

Answer. The magnetic field intensity in the center of a circular current-carrying wire is $H_1 = 64 \ A \cdot m^{-1}$. The magnetic field intensity on the axis passing through the center of the circular wire is $H_2 = 26 \ A \cdot m^{-1}$.

Problem 4.3.4

Problem description. The current $I = 0.5\ A$ passes through a direct solenoid with a linear density of turns equal to $n = 15\ cm^{-1}$. The length of the solenoid is $l = 10 cm$, and the diameter of its base is $d = 4\ cm$. Determine the magnetic field strength at the center of the solenoid; and also in the center of one of its bases.

Known quantities: $n = 15\ cm^{-1}$, $I = 0.5\ A$, $l = 10 cm$, $d = 4\ cm$.

Quantities to be calculated: H_1, H_2.

Problem solution. At a distance of x from the center of the solenoid, we select a small element of length dx. The number of turns that are placed on this element (n is the linear density of the turns of the solenoid) is ndx. Denoting the current in each turn with the symbol I, we can consider the element dx as a circular current $Indx$.

The elementary intensity of the magnetic field in the center of the solenoid, formed by this element, is equal to

$$dH = \frac{IR^2 n}{2\left(R^2 + x^2\right)^{3/2}} dx, \tag{1}$$

where R is the radius of the loop.

The following geometric relationship can be written for a given solenoid

$$x = Rtg\beta, \tag{2}$$

where β is the angle between the axis of the solenoid and the radius vector drawn from the center of the solenoid to the element dx.

Let's rewrite equation (2)

$$dx = -R\frac{d\beta}{sin^2 \beta}. \tag{3}$$

Besides

$$R^2 + x^2 = \frac{R^2}{sin^2 \beta}. \tag{4}$$

Substituting the expressions dx and $R^2 + x^2$ into equation (1), we get

$$dH_1 = \frac{In \sin\beta}{2} d\beta . \tag{5}$$

Since the intensity in the center of the solenoid for all elements dx is directed along the axis of the solenoid, then to determine the resulting value of the magnetic field intensity, it is necessary to integrate equation (5) over the variable β. В результате мы можем записать такое уравнение

$$H_1 = \frac{In}{2}\int_{\beta_1}^{\beta_2} \sin\beta \, d\beta = \frac{In}{2}\left(\cos\beta_1 - \cos\beta_2\right), \tag{6}$$

where β_1 and β_2 are the angles for the ends of the solenoid.

For the quantities β_1 and β_2 we can write the following geometric relation

$$\cos\beta_1 = -\cos\beta_2 = \frac{l}{\sqrt{d^2 + l^2}}, \tag{7}$$

where

l is the length of the solenoid;

d is the diameter of the base of the solenoid.

Finally, for intensity H_1 we get

$$H_1 \frac{Inl}{\sqrt{d^2 + l^2}} . \tag{8}$$

Numerically

$$H_1 = 700 \; A \cdot m^{-1}. \tag{9}$$

The magnetic field intensity at the center of one of the bases of the solenoid can be determined using formula (6), in which one of the corners is considered as a right angle (for example, $\beta_2 = \pi / 2$)

$$H_2 = \frac{In}{2}\cos\beta_1' = \frac{Inl}{2\sqrt{R^2 + l^2}} . \tag{10}$$

Substituting known values, we get

$$H_2 = 375\ A \cdot m^{-1}. \tag{11}$$

Answer. The magnetic field intensity at the center of the solenoid is $H_1 = 700\ A \cdot m^{-1}$. The magnetic field intensity at the center of one of the bases of the solenoid is $H_2 = 375\ A \cdot m^{-1}$.

Problem 4.3.5

Problem description. Determine the magnetic field intensity at a point located on the axis of a flat spiral at a distance of h from the area of the spiral. The spiral is located between circles with radii R and r. The total number of turns of the spiral is N.

Known quantities: h, R, r, N.

Quantities to be calculated: H.

Problem solution. First of all, we select at a distance of x from the center of the spiral an element of length dx. This element has a number of turns equal to

$$N' = \frac{N}{R - r}, \tag{1}$$

where
N is the total number of turns of the spiral;
R is the radius of the outer circle of the spiral;
r is the radius of the inner circle of the spiral.

The intensity of the magnetic field formed by these turns at the point A, specified in the problem condition is

$$dH = \frac{Ix^2}{2\left(h^2 + x^2\right)^{3/2}} \cdot \frac{N}{R - r} dx, \tag{2}$$

where
I is the current passing in a spiral;
h is the distance from the point A to the plane of the spiral.

The resulting magnetic field intensity can be determined by integrating equation (2)

$$H = \frac{IN}{2(R-r)} \int_r^R \frac{x^2}{\left(h^2 + x^2\right)^{3/2}} dx. \qquad (3)$$

As a result of integration, we get

$$H = \frac{IN}{2(R-r)} \left[\frac{R}{\sqrt{h^2 + R^2}} + ln \frac{R + \sqrt{h^2 + R^2}}{r + \sqrt{h^2 + r^2}} - \frac{r}{\sqrt{h^2 + r^2}} \right]. \qquad (4)$$

The magnetic field intensity at the center of the spiral (for $h = 0$) is

$$H = \frac{IN}{2(R-r)} ln\left(\frac{R}{r}\right). \qquad (5)$$

Answer. The magnetic field intensity at point A, specified in the condition of the problem, is $H = \frac{IN}{2(R-r)} \left[\frac{R}{\sqrt{h^2 + R^2}} + ln \frac{R + \sqrt{h^2 + R^2}}{r + \sqrt{h^2 + r^2}} - \frac{r}{\sqrt{h^2 + r^2}} \right].$

Problem 4.3.6

Problem description. A wooden ball with a radius of $R = 10\ cm$ is wound with a thin wire so that all the turns are parallel to each other. The coils fit snugly together and cover half the ball in one layer. The wire carries a current of $I = 1\ A$. The total number of turns is $N = 30$. Determine the magnetic field intensity at the center of the sphere.

Known quantities: $R = 10\ cm$, $I = 1\ A$, $N = 30$.

Quantities to be calculated: H .

Problem solution. To solve the problem, we select at a distance of x from the center of the ball an element of length with a radius dx. The number of turns located on this element is equal to

$$N' = \frac{N}{R} dx, \qquad (1)$$

where
N is the total number of turns;

R is the radius of the sphere.

The elementary intensity of the magnetic field at the center of the ball is

$$dH = \frac{2\pi I r^2 N dx}{4\pi R^4}, \tag{2}$$

where

$r = \sqrt{R^2 - x^2}$;

I is the current passing through the coils.

Let's rewrite equation (2)

$$dH = \frac{IN\left(R^2 - x^2\right)}{2R^4}. \tag{3}$$

The total magnetic field intensity at the center of the ball can be found by integrating equation (3) with respect to the variable x

$$H = \frac{IN}{2R^4}\int_0^R \left(R^2 - x^2\right)dx = \frac{IN}{3R}. \tag{4}$$

Substituting known values, we get

$$H = 100\ A \cdot m^{-1}. \tag{5}$$

Answer. The magnetic field intensity at the center of the ball is $H = 100\ A \cdot m^{-1}$.

Problem 4.3.7

Problem description. A ring with a diameter of $d = 10\ cm$ of lead wire with a cross-sectional area $S = 0.7\ mm^2$ carries a current $I = 7\ A$. As a result, the temperature in the wire rises almost to the melting point. At this temperature, the tensile strength of lead is $P_0 = 2 \cdot 10^6\ N \cdot m^{-2}$. The ring is placed in a magnetic field whose induction is $B = 0.2\ T$. The plane of the ring is perpendicular to the direction of the magnetic field. Determine the pressure acting on the cross section of the wire.

Known quantities: $d = 10\ cm$, $S = 0.7\ mm^2$, $I = 7\ A$, $\rho_0 = 2 \cdot 10^6\ N \cdot m^{-2}$, $B = 0.2\ T$.

<u>Quantities to be calculated</u>: P.

<u>Problem solution</u>. According to Ampère's law, the force with which the magnetic field acts on a conductor element with current is proportional to the magnetic field induction

$$dF = BIdl\,sin\alpha, \qquad (1)$$

where
I is the current passing through the conductor;
B is the induction of the magnetic field;
dl is a current element;
α is the angle between the vectors $d\vec{l}$ and $\vec{B}$.

Since the magnetic field is perpendicular to the plane of the ring, the force will be directed along the radius and in this case $\alpha = \pi/2$.

We will draw the axis Y in such a way that it divides the ring into two equal halves. Then the projection of the elementary force on the axis Y is equal to

$$dF_Y = BIR\,sin\varphi\,d\varphi, \qquad (2)$$

where
R is the radius of the ring;
φ s the angle between the radius of the ring and the end of the segment dl.

The modulus of the vector $d\vec{l}$ is included in the following geometric relation

$$dl = Rd\varphi. \qquad (3)$$

Now we can integrate equation (2)

$$F_Y = IBR\int_{-\pi/2}^{\pi/2} sin\varphi\,d\varphi = 0. \qquad (4)$$

Therefore, the projection of the force on one half of the ring is zero.

Similarly, we can find the projection of the force on the axis X, perpendicular to the axis Y

$$F_Y = IBR\int_{-\pi/2}^{\pi/2} cos\,\varphi\,d\varphi = 2IBR. \qquad (5)$$

Considering that the projection of the force F_X acts on both cross sections of the ring, we get

$$P = \frac{F_X}{2S} = \frac{2IBR}{2S} = \frac{IBd}{2S}, \quad (6)$$

where
S is the cross-sectional area of the wire;
d is the diameter of the ring.

Substituting known values, we get

$$P = 10^6 \ N \cdot m^{-2}. \quad (7)$$

The ring break corresponds to the fulfillment of the inequality

$$P > P_0, \quad (8)$$

where P_0 is the tensile strength of lead.

An analysis of inequality (9) shows that the ring will not break under the action of this force.

Answer. The pressure acting on the cross section of the wire is $P = 10^6 \ N \cdot m^{-2}$.

Problem 4.3.8

Problem description. Determine the force with which the magnetic field of a straight infinite wire with a current I, acts on a ring circuit with a current I_1. Current I is located in the plane of the circular contour. Ring radius is R. The distance from the center of the ring to the wire carrying the current I, is d.

Known quantities: I, I_1, R, d.

Quantities to be calculated: F.

Problem solution. All forces that act on the elements of the ring are directed from the center along the corresponding radii. o solve the problem, we will choose point M on the ring. The distance from point M to the wire carrying current I is x. In this case, the intensity of magnetic field created by the current I at point M, is

$$H = \frac{I}{2\pi x}. \qquad (1)$$

The force that acts on the current element dl in the ring is

$$dF = \frac{\mu\mu_0 I_1 I dl}{2\pi x}, \qquad (2)$$

where
μ is the relative magnetic permeability of the medium containing the ring and the straight wire;
μ_0 is the magnetic constant;
I_1 is the current. that flows in the ring.

Now we will decompose the force into two components: parallel and perpendicular to a straight infinite wire. If element M is connected to a symmetrical element M', then all components parallel to current I, cancel each other out. Only perpendicular components remain, the resultant of which is equal to

$$F = \int \frac{2\mu\mu_0 I_1 I \cos\alpha dl}{2\pi x}, \qquad (3)$$

where α is the angle between the perpendicular drawn from the center of the circle to the straight wire and the line connecting the center of the circle to point M.

The quantities dl, α, x and dx are related by the following geometric relations

$$dl = \frac{dx}{\sin\alpha} \qquad (4)$$

and

$$ctg\alpha = \frac{d - x}{\sqrt{R^2 - (d - x)^2}}, \qquad (5)$$

where
d is the distance from the center of the ring to the wire carrying the current I;
R is the radius of the ring.

We substitute the resulting equations (4) and (5) into equation (3)

$$F = \frac{\mu\mu_0 I_1 I}{\pi} \int\limits_{d-R}^{d+R} \frac{d-x}{x\sqrt{R^2-(d-x)^2}}\,dx. \tag{6}$$

After integration, we get an equation for the force acting on a ring conductor with current

$$F = \frac{\mu\mu_0 I_1 I}{\pi}\left(\frac{d}{\sqrt{d^2-R^2}}-1\right). \tag{7}$$

Answer. The force acting on a ring conductor with current is equal to $F = \frac{\mu\mu_0 I_1 I}{\pi}\left(\frac{d}{\sqrt{d^2-R^2}}-1\right)$.

Problem 4.3.9

Problem description. A rectangular frame with current is in a magnetic field parallel to the lines of magnetic induction. A torque $M = 10^{-2}\ N \cdot m$ is acting on the frame. Calculate the work of the field forces when the frame is rotated through an angle of $\alpha = 30°$.

Known quantities: $M = 10^{-2}\ N \cdot m$, $\alpha = 30°$.

Quantities to be calculated: A.

Problem solution. The work of the forces of the magnetic field during the movement of the frame with current is

$$A = I(\Phi_2 - \Phi_1), \tag{1}$$

where
I is the current passing through the frame;
Φ_1 is the flux of magnetic induction crossing the frame before turning;
Φ_2 is the final flux of magnetic induction crossing the frame after the turn.

The flux of magnetic induction crossing the frame is

$$\Phi = BS\cos\alpha, \tag{2}$$

where

B is the induction of magnetic field;
S is the induction of magnetic field;
α is the angle between vectors $\vec{B}$ and $\vec{S}$.

For the case when the frame is placed parallel to the magnetic field lines, the normal to the plane of the frame forms an angle of $\alpha_1 = \pi / 2$. Then

$$\Phi_1 = BS\, cos\, \alpha_1 = 0. \tag{3}$$

For the case when the frame rotates by an angle equal to $30°$, he angle between the normal and the vector $\vec{B}$ will be equal to $\alpha_2 = 60°$. Therefore, the flux of magnetic induction after the turn is equal to

$$\Phi_2 = BS\, cos\, \alpha_2 = \frac{BS}{2}. \tag{4}$$

The difference between the fluxes of magnetic induction is determined by the relation

$$\Phi_2 - \Phi_1 = \frac{BS}{2}. \tag{5}$$

The torque of the forces that act on the frame with current in its initial position is proportional to the induction of the magnetic field on the current

$$M = BIS. \tag{6}$$

Therefore, the current passing through the frame is

$$I = \frac{M}{BS}. \tag{7}$$

Therefore, for the work of the magnetic field forces, we can write the following equation

$$A = I\left(\Phi_2 - \Phi_1\right) = \frac{MBS}{2BS} = \frac{M}{2}. \tag{8}$$

Substituting the given data, we find

$$A = 5 \cdot 10^{-3}\ J. \tag{9}$$

Answer. The work of the field forces when the frame is rotated is $A = 5 \cdot 10^{-3}\ J$.

Problem 4.3.10

Problem description. An alternating current is used to operate the electromagnet. The maximum magnetic field induction is $B_m = 1\,T$ and varies according to the law $B = B_m \sin \omega t$. The area of the electromagnet is $S = 2\ cm^2$. Consider that during the time during which the induction of the magnetic field is too small to hold the load, this load, however, does not have time to fall. Calculate the lifting force of the electromagnet.

Known quantities: $B_m = 1\,T$, $B = B_m \sin \omega t$, $S = 2\ cm^2$.

Quantities to be calculated: F.

Problem solution. The lifting force of an electromagnet can be determined according to the formula

$$F = \frac{B^2}{2\mu_0 \mu} S, \qquad (1)$$

where
B is the induction of the magnetic field in an electromagnet;
μ_0 is the electric constant;
μ is the relative magnetic permeability;
S is the area of the electromagnet.

Since the magnetic induction changes according to a sinusoidal law, in order to solve the problem, it is necessary to calculate the effective value of the induction

$$B_e = \frac{B_m}{2}, \qquad (2)$$

where B_m is the maximum value of the magnetic field induction.

Substituting formula (2) into formula (1), we obtain

$$F = \frac{B_m^2}{4\mu_0 \mu} S. \qquad (3)$$

Numerically,

$$F = 40\ N. \tag{4}$$

Answer. The lifting force of the electromagnet is $F = 40\ N$.

Problem 4.3.11

Problem description. An electron, having a very low initial speed, passed through a uniform electric field with a potential difference of $U = 1000\,V$. After that, the electron enters a uniform magnetic field with induction $B = 10^{-2}\ T$. The direction of the magnetic field is perpendicular to the electric field intensity vector. Determine the trajectory of the electron.

Known quantities: $U = 1000\,V$, $B = 10^{-2}\ T$.

Quantities to be calculated: R.

Problem solution. We will assume that an electron, having a speed of υ, moves in an electric field from the start point A в to the end point C. In time dt the length of this path will be equal to $dl = AC$, and the shape of the trajectory is a straight line parallel to the speed υ. The movement of an electron in section dl is equivalent to the presence of current I, directed from point C to point A. This current is

$$I = \frac{e}{dt}, \tag{1}$$

where e is the elementary charge.

The force of action of a uniform magnetic field on the current element Idl is

$$F = IBdl\,sin\alpha, \tag{2}$$

where
B is the induction of the magnetic field;
α is the angle between the vectors $d\vec{l}$ and $\vec{B}$.

According to the conditions of problems $\alpha = \pi/2$ and $sin\alpha = 1$, therefore, substituting equation (1) into equation (2), we get

$$dF = eB\frac{dl}{dt} = e\upsilon B, \tag{3}$$

where υ is the speed of an electron at the moment it enters the region where the magnetic field acts.

The Lorentz force, which is determined by equation (3), is perpendicular to the direction of electron motion. Therefore, this force only changes the direction of the velocity and causes the normal acceleration of the electron to appear

$$a = \frac{\upsilon^2}{R}, \tag{4}$$

where R is the radius of the corresponding circle.

Therefore, the centripetal force that acts on the electron is

$$dF = \frac{m\upsilon^2}{R} = e\upsilon B, \tag{5}$$

where m is the electron mass.

We found out that the shape of the electron trajectory is a circle and the radius of this circle can be determined from equation (5)

$$R = \frac{m\upsilon}{eB}. \tag{6}$$

We only need to determine the speed of the electron. We use the following reasoning for this. If the electron passed through the potential difference U in the electric field, then the electric forces did the work and changed the kinetic energy of the electron by the value

$$eU = \frac{m\upsilon^2}{2} - \frac{m\upsilon_0^2}{2}, \tag{7}$$

where υ_0 s the initial speed of the electron.

According to the condition of the problem, the initial speed of the electron is approximately equal to zero $\upsilon_0 \approx 0$. Consequently

$$eU = \frac{m\upsilon^2}{2}. \tag{8}$$

The electron speed determined from equation (9) is equal to

$$\upsilon = \sqrt{\frac{2eU}{m}}\,. \tag{9}$$

As a result, by substituting equation (9) into equation (6), we obtain a formula for determining the radius of a circle

$$R = \frac{1}{B}\sqrt{\frac{2mU}{e}}\,. \tag{10}$$

Substituting known values, we get

$$R = 0.01\,m\,. \tag{11}$$

Answer. The electron motion trajectory is a circle with a radius $R = 0.01\,m$.

Problem 4.3.12

Problem description. A proton with a velocity of $\upsilon = 20\ km \cdot s^{-1}$, flies into a uniform magnetic field at an angle of $\alpha = 30°$ to the direction of the field intensity vector. The modulus of the magnetic field intensity is equal to $H = 2.4 \cdot 10^3\ A \cdot m^{-1}$. Determine the radius and pitch of the helix along which the proton will move in this case.

Known quantities: $\upsilon = 20\ km \cdot s^{-1}$, $\alpha = 30°$, $H = 2.4 \cdot 10^3\ A \cdot m^{-1}$.

Quantities to be calculated: R, h.

Problem solution. A charged particle that flies into a magnetic field is affected by the Lorentz force, which is perpendicular to the direction of the field and the particle's velocity

$$F = e\upsilon B\, sin\,\alpha = \mu_0 \mu\, He\upsilon_n\,, \tag{1}$$

where
e is the elementary charge;
υ is the speed of the proton;
B is the induction of the magnetic field;
α is the angle between the vectors $\vec{\upsilon}$ and $\vec{B}$;
μ_0 is the magnetic constant;

μ is the relative magnetic permeability of the medium in which the proton moves (according to the condition of the problem $\mu \approx 1$);
H is the magnetic field intensity;
υ_n is the normal component of the proton velocity ($\upsilon_n = \upsilon \sin\alpha$).

The Lorentz force is always perpendicular to the direction of the velocity. Therefore, the velocity modulus of a proton does not change under the action of the Lorentz force. It should also be taken into account that, since the speed of the proton does not change, the Lorentz force also remains constant.

A force constant in absolute value, the direction of which is perpendicular to the velocity of a charged particle, is the reason for the motion of this particle in a circle. Consequently, a proton, flying into a magnetic field, will move in a circle in a plane perpendicular to the vector of the magnetic field. The speed of the proton in this case will be equal to the normal component of its initial speed. At the same time, the proton will move along the field with a speed υ_τ. The value υ_τ is equal to the projection of the initial velocity of the proton on the direction of the magnetic field.

As a result of simultaneous movement in a circle and in a straight line, the proton will move along a helix. Next, we will determine the radius and pitch of this helix. The Lorentz force, being the cause of the proton's motion in a circle, can be represented as a centripetal force. Consequently

$$\mu_0 \mu He\upsilon \sin\alpha = \frac{m\upsilon^2}{R}, \tag{2}$$

where
m is the mass of the proton;
R is the radius of the circle.

Equation (2) allows us to determine the radius of the circle

$$R = \frac{m\upsilon}{\mu_0 \mu eH \sin\alpha}. \tag{3}$$

Numerically,

$$R = 0.14\ m. \tag{4}$$

The pitch of the helix is equal to the path taken by the proton along the direction of the field at a speed υ_τ in the time it takes for the proton to make one turn

$$h = \upsilon_\tau T, \tag{5}$$

where T is the rotation period of the proton.

The rotation period of a proton can be defined as the ratio of the circumference to the normal component of the velocity

$$T = \frac{2\pi R}{\upsilon_n}. \tag{6}$$

Now we will rewrite equation (5) taking into account equation (6)

$$h = \frac{2\pi \upsilon_\tau R}{\upsilon_n} \tag{7}$$

or

$$h = \frac{2\pi R \cos\alpha}{\sin\alpha} = 2\pi R \cot\alpha . \tag{8}$$

Substituting known values, we get

$$h = 1.5\ m. \tag{9}$$

Answer. The radius of the proton helix is $R = 0.14\ m$. The pitch of the proton helix is $h = 1.5\ m$.

4.4. Level 1 problems

4.4.1. Determine the induction of the magnetic field in the center of a thin ring through which the current $I = 10\ A$ passes. The radius of the ring is $r = 5\ cm$.

4.4.2. The magnetic field intensity at the center of a circular coil with a radius of $r = 8\ cm$ is $H = 30\ A \cdot m^{-1}$. Determine the magnetic field intensity on the axis of the coil at a point located at a distance of $d = 6\ cm$ from the center of the coil.

4.4.3. A coil with a length of $l = 20\ cm$ contains $N = 100$ turns. A current of $I = 5\ A$ passes through the coil winding. The coil diameter is $d = 20\ cm$. Determine the induction of the magnetic field at a point that is located on the axis of the coil at a distance $a = 10\ cm$ from its end.

4.4.4. A long straight solenoid made of wire with a diameter of $d = 0.5\ mm$ is wound so that the turns fit snugly together. The current passing through the solenoid is $I = 4\ A$. Calculate the magnetic field intensity inside the solenoid. The thickness of the insulation in solving the problem can be neglected..

4.4.5. The solenoid winding is made of a thin wire with turns tightly adjacent to each other. The length of the coil is $l = 1\ m$, and its diameter is $d = 2\ cm$. An electric current flows through the winding. Calculate the dimensions of the area on the center line within which the magnetic field induction can be calculated using the infinite solenoid formula with an error not exceeding $0.1\,\%$.

4.4.6. Two long parallel wires are $r = 5\ cm$ apart. The wires carry equal currents $I = 10\ A$ in opposite directions. Determine the magnetic field strength at a point that is $r_1 = 2\ cm$ from one wire and $r_2 = 3\ cm$ from the other wire.

4.4.7. Two infinitely long straight parallel wires carry currents $I_1 = 50\ A$ and $I_2 = 100\ A$ in opposite directions. The distance between the wires is $d = 20\ cm$. Determine the induction of the magnetic field at a point removed at a distance of $r_1 = 25\ cm$ from the first wire and at a distance of from the second wire.

4.4.8. An electron in an unexcited hydrogen atom moves around the nucleus in a circle with a radius of $r = 53\ pm$. Calculate the equivalent circular current and magnetic field intensity at the center of the circle.

4.4.9. A straight wire carrying a current $I = 1\ kA$, is located in a uniform magnetic field perpendicular to the lines of induction. Calculate the force with which the magnetic field acts on a piece of wire with a length of $l = 1\ m$, if the magnetic field strength is $B = 1\ T$.

4.4.10. A straight wire with a length of $l = 10\ cm$, through which a current of $I = 20\ A$ passes, is in a uniform magnetic field with an induction of $B = 0.01\ T$. Calculate the angle between the directions of the vector $\vec{B}$ and the current if a force equal to $F = 10\ mN$ acts on the wire .

4.4.11. A wire in the form of a thin half-ring with a radius of $R = 10\ cm$ is in a uniform magnetic field with an induction $B = 50\ mT$. An electric current $I = 10\ A$ passes through the wire. Determine the force acting on the wire if the plane of the semiring is perpendicular to the lines of magnetic field induction, and the power wires are outside the field.

4.4.12. Two parallel straight wires of length $l = 2.5\ m$ each carry the same currents $I = 1\ kA$. The distance between the wires is $d = 20\ cm$. Calculate the strength of the currents interaction.

4.4.13. Three parallel straight wires at the same distance $a = 10\ cm$ from each other carry the same currents $I = 100\ A$. The directions of the currents are the same in two wires. Calculate the force that acts on a segment of length $l = 1\ m$ of each wire.

4.4.14. The magnetic moment of the coil is $p_m = 0.2\ J \cdot T^{-1}$. The coil diameter is $d = 10\ cm$. Calculate the current in the coil.

4.4.15. A direct current flows through the ring wire. The magnetic field induction on the axis of the ring at a distance of $d = 1\ m$ from its plane is $B = 10\ nT$. Determine the magnetic moment of a ring with current. When solving the problem, assume that the radius of the ring is much less than the distance d.

4.4.16. An electron in an unexcited hydrogen atom moves around the nucleus in a circle with a radius of $r = 53\ pm$. Calculate the magnetic moment of the equivalent circular current and the mechanical moment acting on the circular current if the atom is placed in a magnetic field whose lines of induction are parallel to the plane of the electron's orbit. The magnetic field induction is $B = 0.1\ T$.

4.4.17. On a disk with a radius of $R = 10\ cm$ there is an electric charge $Q = 0.2\ \mu C$. The disc rotates uniformly at a frequency of $n = 20\ s^{-1}$ about an axis perpendicular to the plane of the disc and passing through its center. Calculate the magnetic moment of the circular current created by the disk, as well as the ratio of the magnetic moment to the angular momentum, if the mass of the disk is $m = 100\ g$.

4.4.18. A galvanometer frame containing $N = 200$ turns of a thin wire is suspended on an elastic thread. The area of the frame is $S = 1\ cm^2$. The normal to the plane of the frame is perpendicular to the lines of magnetic induction. The magnetic field induction is $B = 5\ mT$. When a current of $I = 2\ \mu A$ was passed through the galvanometer, the frame turned through an angle of $\alpha = 30\ °$. Determine the torsion constant of the thread.

4.4.19. An electron moves in a uniform magnetic field perpendicular to the lines of induction. The magnetic field induction is $B = 0.1\ T$. The radius of curvature of the electron trajectory is $R = 0.5\ cm$. Determine the force acting on the electron from the magnetic field.

4.4.20. An electron that flew into a cloud chamber left a trace in the form of an arc of a circle with a radius of $R = 10\ cm$. The chamber is in a uniform magnetic field. The magnetic field induction is $B = 10\ T$. Calculate the kinetic energy of an electron.

4.5. Answers to problems

4.4.1. $B = 1.26 \cdot 10^{-4}\ T$.

4.4.2. $H = 15.4\ A \cdot m^{-1}$.

4.4.3. $B = 6.05 \cdot 10^{-4}\ T$.

4.4.4. $H = 8 \cdot 10^{3}\ A \cdot m^{-1}$.

4.4.5. $\Delta l = 6.84 \cdot 10^{-1}\ m$.

4.4.6. $H = 132\ A \cdot m^{-1}$.

4.4.7. $B = 2.12 \cdot 10^{-5}\ T$.

4.4.8. $I = 1.1 \cdot 10^{-3}\ A;\ H = 1 \cdot 10^{5}\ A \cdot m^{-1}$.

4.4.9. $F = 1 \cdot 10^{3}\ N \cdot m^{-1}$.

4.4.10. $\alpha = 30\ ^\circ$.

4.4.11. $F = 0.1\ N$.

4.4.12. $F = 2.5\ N$.

4.4.13. $F = 3.46 \cdot 10^{-2}\ N$.

4.4.14. $I = 25.5\ A$.

4.4.15. $p_m = 5 \cdot 10^{-2}\ A \cdot m^2$.

4.4.16. $p_m = 9.4 \cdot 10^{-23}\ A \cdot m^2$; $M = 9.4 \cdot 10^{-25}\ N \cdot m$.

4.4.17. $p_m = 6.28 \cdot 10^{-8}\ A \cdot m^2$; $p_m / L = 1 \cdot 10^{-6}\ C \cdot kg^{-1}$.

4.4.18. $C = 3.32 \cdot 10^{-10}\ N \cdot m / rad$.

4.4.19. $F = 1.4 \cdot 10^{-12}\ N$.

4.4.20. $E_k = 4.8 \cdot 10^{-11}\ J$.

CHAPTER 5. ELECTROMAGNETIC INDUCTION. MAGNETIC FIELD ENERGY

5.1. Basic formulas

An **electromotive force** occurs in a conductor provided that the magnetic field around this conductor changes. The magnitude of the **electromotive force of induction** in a closed circuit is determined by the rate of change of the magnetic flux that crosses the circuit (**Faraday's law**)

$$E_i = -N\frac{d\Phi}{dt} = -\frac{d\Psi}{dt}, \tag{5.1.1}$$

where
N is the number of turns in a closed circuit;
Φ is the magnetic flux of one turn;
Ψ is flux linkage ($\Psi = N\Phi$).

The “–” sign in formula (5.1.1) indicates that the direction of the electromotive force of induction is considered positive when the magnetic flux in the closed circuit decreases.

The electromotive force of induction in a conductor that moves in a uniform magnetic field is

$$E_i = Bl\upsilon \sin\alpha, \tag{5.1.2}$$

where
l is the conductor length;
B is the magnetic field induction;
υ is the conductor speed;
α is the angle between the vectors $\vec{\upsilon}$ and $\vec{B}$.

The electromotive force of induction, which occurs in a frame rotating in a uniform magnetic field, is

$$E_i = BNS\omega \sin(\omega t), \tag{5.1.3}$$

where
S is the area of the frame;
ω is the angular velocity of the frame;
N is the number of turns of the frame;

t is the time during which the frame was in rotational motion.

The amount of electric charge that passes in the closed circuit when the magnetic flux changes is determined by the ratio

$$\Delta Q = -\frac{\Delta \Phi}{R}, \tag{5.1.4}$$

where

$\Delta \Phi$ is the change in magnetic flux;

R is the circuit resistance.

A change in current in a wire loop is the cause of a change in the magnetic flux and the appearance of electromagnetic induction (the **phenomenon of self-induction**). The phenomenon of self-induction was discovered by the American physicist Joseph Henry (Joseph Henry 1797 – 1878). The electromotive force of self-induction is proportional to the change in current

$$E_s = -L\frac{dI}{dt}, \tag{5.1.5}$$

where

L is the inductance, which depends on the size and shape of the conductor;

I s the current that passes through the conductor;

dI/dt is the rate of current change in the conductor.

The **inductance of the solenoid** (toroid) can be determined according to the formula

$$L = \frac{\mu_0 \mu N^2 S}{l}, \tag{5.1.6}$$

where

μ_0 is the magnetic constant;

μ is the relative magnetic permeability of the core of the solenoid (toroid);

N is the number of turns of the solenoid (toroid);

S is the solenoid cross-sectional area;

l is the solenoid length (toroid).

The relative magnetic permeability of the solenoid (toroid) core depends on the magnetic field intensity. In all cases of calculating the inductance of a solenoid (toroid) with a core to determine the relative magnetic permeability, first of all, it is necessary to use a graph of the dependence of the magnetic field on the magnetic field intensity. Then we should make calculations according to the formula

$$\mu = \frac{B}{\mu_0 H}, \tag{5.1.7}$$

where H is the magnetic field intensity.

The dependence of the current on time after connecting the inductance to the electrical circuit has the form

$$I = \frac{E}{R}\left[1 - exp\left(-\frac{R}{L}t\right)\right], \tag{5.1.8}$$

where

E is the electromotive force of the current source;

t is the time that has elapsed since the closing of the electrical circuit.

The dependence of the current on time after the opening of the electrical circuit has the form

$$I = I_0\, exp\left(-\frac{R}{L}t\right), \tag{5.1.9}$$

where

I_0 is the current at the initial moment of time (the moment of opening the electrical circuit);

t is the time that has elapsed since the opening of the electrical circuit.

Consider a system of two wire loops. A change in the current in one loop is the cause of the occurrence of an **electromotive force of mutual induction** in the second loop

$$E_{21} = -M\frac{dI_{21}}{dt} \tag{5.1.10}$$

or

$$E_{12} = -M\frac{dI_{12}}{dt}, \tag{5.1.11}$$

where M is the mutual induction coefficient.

The **energy of the magnetic field** that occurs in the circuit (solenoid) is determined by the equation

$$W = \frac{LI^2}{2}, \tag{5.1.12}$$

where
I is the current in the circuit;
L is the inductance of the circuit.

The **volumetric energy density of magnetic field** is

$$\omega = \frac{\mu_0 \mu H^2}{2} = \frac{B^2}{2\mu_0 \mu} = \frac{BH}{2}. \tag{5.1.13}$$

5.2. Problem-solving framework

To calculate the electromotive force of induction according to Faraday's law, it is necessary to calculate only the change in the magnetic flux through the area bounded by a closed loop. In this case, the reason for the change in the magnetic flux does not matter. In particular, as the reasons for such a change, one can indicate: a change in the shape of the contour; displacement of the contour in a non-uniform magnetic field; change over time in the magnitude of the induction of the magnetic field. Solving problems for calculating the electromotive force of induction should be accompanied by determining the dependence of the magnetic induction flux on time. Only then can we proceed to the calculation of the electromotive force of induction by determining the derivative of the flux of magnetic induction with respect to time. Similarly, we can calculate the electromotive force of self-induction. In this case, it should be borne in mind that the change in the flux of magnetic induction through a fixed circuit is associated with the current that passes through the same circuit.

If the law of current change with time and the inductance of the circuit are known, then the electromotive force of self-induction can be determined according to the formula

$$E = L\frac{dI}{dt}. \tag{5.2.1}$$

In order to calculate the inductance of an arbitrary closed circuit, it is necessary to determine the induction flux that permeates the area bounded by this circuit. In this case, the induction flux is determined only by the current passing in the circuit. Then the inductance is equal to the ratio of the induction flux to the current strength in the circuit

$$L = \frac{\Phi}{I}. \tag{5.2.2}$$

The calculation of the mutual inductance of two circuits also comes down to determining the flux of magnetic induction. To do this, we must consider the ratio of the flux that permeates the first circuit to the current in the second circuit

$$M = \frac{\Phi_{12}}{I_2}. \tag{5.2.3}$$

5.3. Problem-solving examples

Problem 5.3.1

Problem description. A metal rod OA of length $l = 0.4\ m$ rotates with an angular velocity of $\omega = 10\ s^{-1}$ around a point O in a plane perpendicular to the magnetic field lines. The magnetic field induction is $B = 10^{-3}\ T$. Determine the electromotive force of induction between points O and A.

Known quantities: $l = 0.4\ m$, $\omega = 10\ s^{-1}$, $B = 10^{-3}\ T$.

Quantities to be calculated: U .

Problem solution. We will consider an element of length dx, which is at a distance of x from the center O. The flux of magnetic induction through an elementary area dS is

$$d\Phi = BdS\cos\alpha = Bxdxd\varphi, \tag{1}$$

where
B is the induction of magnetic field;

α is the angle between the vectors $\vec{B}$ and $d\vec{S}$;
φ is the elementary angle through which the rod is turned.

The electromotive force generated in element dx, is

$$dE = -\frac{d\Phi}{dt} = -B\omega x dx, \tag{2}$$

where
ω is the angular velocity of the rod;
dt is the elementary time of rotation of the rod.

The total electromotive force that occurs in the rod is

$$E = -\int_0^l B\omega x dx = -B\omega \frac{l^2}{2}, \tag{3}$$

where l is the length of the rod.

If the electrical circuit is open, then this electromotive force is the cause of the potential difference between the points O and A

$$\varphi_A - \varphi_O = U = \frac{B\omega l^2}{2}. \tag{4}$$

If the electrical circuit is closed (for example, the end A of the rod slides along the conductive section of the electrical circuit, which is connected to the point O through resistance R), then a current appears in the circuit

$$I = \frac{U}{R}. \tag{5}$$

We substitute numerical values in the formula (4)

$$U = 8 \cdot 10^{-4}\ V. \tag{6}$$

Answer. The electromotive force of induction between points O and A is $U = 8 \cdot 10^{-4}\ V$.

Problem 5.3.2

Problem description. The rod AB moves parallel to itself, sliding along a circular conductor at a constant speed of υ. The conductor is in a uniform magnetic field. The magnetic field lines are perpendicular to the conductor. Determine the electromotive force of induction in the rod.

Known quantities: $B \perp AB$, υ.

Quantities to be calculated: E.

Problem solution. To solve the problem, we consider an infinitely narrow strip of conductor dx, which is located at a distance of x from a point on a round conductor in the direction of its radius towards the center. The area of an infinitely narrow strip is

$$dS = AB \cdot dx, \tag{1}$$

where AB is the length of the longer side of the infinitely narrow strip.

Suppose that the rod starts its movement from point C on the round conductor towards point E, which is located on the radius of the round conductor, closer to its center. The path passed by the rod is

$$x = \upsilon t, \tag{2}$$

where
υ is the rod speed;
t is the time of the rod movement on the segment CE.

For this problem, we can write the following geometric relation

$$AB = 2\sqrt{R^2 - (R - x)^2}, \tag{3}$$

where R is the radius of the round conductor.

Let us rewrite equation (2) for elementary increments of the path and time

$$dx = \upsilon dt. \tag{4}$$

We can now write the following equation for the area of an infinitely narrow strip

$$dS = 2\sqrt{2RX - x^2}\,\upsilon\, dt = 2\upsilon\sqrt{\upsilon t(2R - \upsilon t)}\, dt. \tag{5}$$

The elementary magnetic flux in this case is

$$d\Phi = BdS = 2B\upsilon\sqrt{\upsilon t(2R-\upsilon t)}\,dt, \tag{6}$$

where B is the induction of a uniform magnetic field.

According to Faraday's law, the electromotive force of induction in the rod AB at the moment of time t is

$$E = \frac{d\Phi}{dt} = 2B\upsilon\sqrt{\upsilon t(2R-\upsilon t)}\ . \tag{7}$$

Answer. Electromotive force of induction in the rod AB at the moment of time t is $E = \frac{d\Phi}{dt} = 2B\upsilon\sqrt{\upsilon t(2R-\upsilon t)}$.

Problem 5.3.3

Problem description. The frame, which covers area $S = 1400\ cm^2$, consists of several turns of the conductor. The frame resistance is $R = 4.7\ \Omega$. The number of turns of the frame is $N = 100$. The frame is placed in a vertical plane and connected to a ballistic galvanometer, the sensitivity of which is $C = 2\cdot 10^{-6}\ C/div$. The frame quickly rotates around the diameter so that its plane becomes parallel to the direction of the Earth's magnetic field. The initial angle between the frame diameter and the direction of the Earth's magnetic field lines is $\alpha = 30°$. The resistance of the galvanometer is $R = 9.4\ \Omega$. The intensity of the Earth's magnetic field is $H = 16\ A\cdot m^{-1}$. Determine the amount of deviation on the scale of the galvanometer.

Known quantities: $S = 1400\ cm^2$, $R = 4.7\ \Omega$, $N = 100$, $C = 2\cdot 10^{-6}\ C/div$, $\alpha = 30°$, $R = 9.4\ \Omega$, $H = 16\ A\cdot m^{-1}$.

Quantities to be calculated: γ .

Problem solution. For the initial position, the magnetic field flux through the plane enclosed by the frame is

$$\varPhi_1 = BSN \sin\alpha, \tag{1}$$

where
B is the induction of the earth's magnetic field;
S is the area of the frame;
N is the number of turns of the frame;
α is the angle between the normal to the box plane and the vector $\vec{B}$.

The flux of magnetic induction for the end position of the frame is

$$\varPhi_2 = 0. \tag{2}$$

Therefore, the resulting change in the flux of magnetic induction is

$$\Delta\varPhi = NBS \sin\alpha. \tag{3}$$

The magnitude of the electric charge that has passed in the electric circuit as a result of the occurrence of an induction current can be found from the relation

$$Q = \frac{NBS \sin\alpha}{R + R_0} = \frac{\mu_0 \mu HSN \sin\alpha}{R + R_0}, \tag{4}$$

where
μ_0 is the electric constant;
μ is the relative magnetic permeability of the environment (for the conditions of the problem, we can assume that $\mu \approx 1$);
H Earth's magnetic field intensity;
R is the total resistance of the frame turns;
R_0 is the resistance of the ballistic galvanometer.

The number of scale divisions corresponding to a change in the readings of a ballistic galvanometer is equal to the ratio of the electric charge Q to the sensitivity of the galvanometer

$$\gamma = \frac{\mu_0 \mu HSN \sin\alpha}{(R + R_0)C}, \tag{5}$$

where C is the sensitivity of a ballistic galvanometer.

Numerically,

$$\gamma = 10\ div. \quad (6)$$

Answer. The deviation of the ballistic galvanometer is $\gamma = 10\ div$.

Problem 5.3.4

Problem description. A frame with an area of $S = 1000\ cm^2$ and an ohmic resistance of $R = 0.5\ \Omega$ was first placed parallel to the lines of induction of the Earth's magnetic field, and then turned so that its plane became perpendicular to the lines of magnetic induction. Calculate the amount of charge that is induced in the frame. The induction of the Earth's magnetic field is $B = 5 \cdot 10^{-5}\ T$.

Known quantities: $S = 1000\ cm^2$, $R = 0.5\ \Omega$, $B = 5 \cdot 10^{-5}\ T$.

Quantities to be calculated: Q.

Problem solution. The magnetic field induction flux crossing the frame is equal to for the initial position

$$\Phi_1 = 0. \quad (1)$$

The magnetic field induction flux crossing the frame in the final position, when the plane of the frame is perpendicular to the lines of magnetic induction, is

$$\Phi_2 = BS, \quad (2)$$

where
B is the induction of the earth's magnetic field;
S is the area of the frame.

Therefore, the change in the magnetic field induction flux crossing the frame is determined by the relation

$$\Delta\Phi = \Phi_2 - \Phi_1 = BS. \quad (3)$$

The amount of charge induced in the frame is

$$Q=\frac{\Delta\Phi}{R}=\frac{\Phi_2-\Phi_1}{R}=\frac{BS}{R}, \tag{4}$$

where R is the ohmic resistance of the frame.

Substituting known values, we get

$$Q=10^{-5}\ C. \tag{5}$$

Answer. The charge that is induced in the frame is $Q=10^{-5}\ C$.

Problem 5.3.5

Problem description. The length and cross-sectional area of the solenoid are, respectively $l=50\ cm$ and $S=10\ cm^2$. The total number of turns of the solenoid is $N=3000$. The relative magnetic permeability of the substance concentrated inside the solenoid is equal to $\mu=1$. Calculate the coefficient of self-induction of the solenoid.

Known quantities: $l=50\ cm$, $S=10\ cm^2$, $N=3000$, $\mu=1$.

Quantities to be calculated: L.

Problem solution. The magnetic field intensity inside the solenoid is

$$H=\frac{N}{l}I, \tag{1}$$

where
N is the total number of turns of the solenoid;
l is the length of the solenoid;
I is the current flowing through the coils of the solenoid.

If the winding of the coil is bifilar, i.e., in the second layer the turns are wound in opposite sense, backwards along the solenoid (with the same density), then the magnetic field is zeroinside as well as outside the solenoid.

The magnetic field induction flux [5, p. 956] through the cross section of one coil is determined by the relation

$$\Phi_0 = BS = \mu_0 \mu HS, \qquad (2)$$

where
B is the induction of the magnetic field;
S is the cross-sectional area of the solenoid;
μ_0 is the magnetic constant;
μ is the relative magnetic permeability of the substance inside the solenoid;
H is the intensity of the magnetic field.

The total flux of magnetic induction through all turns is

$$\Phi = N\Phi_0 = \mu_0 \mu HSN. \qquad (3)$$

Let us rewrite equation (3) taking into account equation (1)

$$\Phi = \mu_0 \mu SI \frac{N^2}{l}. \qquad (4)$$

According to the formula that determines the relationship between the flux of magnetic induction and electric current, we can write

$$\Phi = LI, \qquad (5)$$

where L is the inductance of the solenoid.

Now we can rewrite equation (5) as follows

$$L = \frac{\Phi}{I} = \frac{\mu_0 \mu SN^2}{l}. \qquad (6)$$

Numerically

$$L = 0.023\ H. \qquad (7)$$

Answer. The coefficient of self-induction of the solenoid is $L = 0.023\ H$.

Problem 5.3.6

Problem description. Determine the inductance of a toroid with an iron core at a current of $I_1 = 0.3\ A$. The core centerline radius is $R = 30\ cm$. The number of winding turns is $N = 10000$. The cross-sectional area of the core is $S = 5\ cm^2$. Calculate the inductance for the case when the current is $I_2 = 0.1\ A$. The inhomogeneity of the magnetic field within the cross section of the core can be neglected.

Known quantities: $I_1 = 0.3\ A$, $R = 30\ cm$, $N = 10000$, $S = 5\ cm^2$, $I_2 = 0.1\ A$.

Quantities to be calculated: L_1, L_2.

Problem solution. The magnetic field intensity in the core of the toroid is

$$H = \frac{IN}{2\pi R}, \tag{1}$$

where
I is the current flowing through the turns of the toroid;
N is the number of turns of the toroid;
R is the radius of the centerline of the toroid.

The magnetic flux in the core is

$$\Phi = BS, \tag{2}$$

where
B is the induction of the magnetic field in the core;
S is the cross-sectional area of the core.

The magnetic field induction is determined from the dependency graph $B = f(H)$ for the magnet from which the core is made. The inductance of the toroid is equal to the ratio of the total magnetic flux for all turns of the toroid to the current in these turns

$$L = \frac{N\Phi}{I} = \frac{BSN}{I}. \tag{3}$$

We substitute numerical data for the current I_1

$$L_1 = 23.3\ H. \tag{4}$$

We substitute numerical data for the current I_2

$$L_2 = 54.5\ H. \tag{5}$$

Answer. The inductance of the toroid in the first case is $L_1 = 23.3\ H$. The inductance of the toroid in the second case is $L_2 = 54.5\ H$.

Problem 5.3.7

Problem description. Determine the coefficient of self-induction of a coaxial cable, which consists of two coaxial cylindrical conductors, between which there is a substance with a relative magnetic permeability μ. The radii of the inner and outer cylinders are equal R_1 and R_2, respectively. The cable length is l.

Known quantities: μ, R_1, R_2, l.

Quantities to be calculated: L.

Problem solution. The primary goal of the problem is to determine the flux of magnetic induction through one half of the axial section. In this case, it should be taken into account that the intensity of the magnetic field generated by the current that passes in the hollow conductor (cylinder) in the inner region of the cylinder is zero. The magnetic field intensity between the cylindrical surfaces of the conductor is determined only by the current that passes in the inner cylinder, since the magnetic field intensity of the outer cylinder is also zero. The magnetic field intensities generated by currents in a cylindrical conductor are equal for areas outside the cylinder and along the axis of the cylinder. Therefore, the magnetic field intensity between the cylinders is

$$H = \frac{I}{2\pi x}, \tag{1}$$

where
I is the current flowing through the cylindrical conductor;
x is the distance measured from the axis of the cylinder.

The magnetic field strength outside both cylinders is equal to the sum of strengths equal in magnitude and opposite in sign, formed by oppositely directed currents $|I_1| = |I_2| = I$. Therefore, the magnetic field intensity for the entire area outside both cylinders is zero.

Now we divide the area of the axial section into elementary bands with area

$$dS = l\,dx, \tag{2}$$

where
l is length of the band (equal to the length of the cable);
dx is the bandwidth.

The elementary magnetic flux through the area is equal dS is

$$d\Phi = BdS = \mu_0 \mu \frac{I \cdot l}{2\pi x} dx, \tag{3}$$

where
B is the induction of the magnetic field;
μ_0 is the magnetic constant;
μ is the relative magnetic permeability.

The total magnetic flux through half the cross-sectional area of a coaxial cable is

$$\Phi = \mu_0 \mu \frac{I \cdot l}{2\pi} \int_{R_1}^{R_2} \frac{dx}{x} = \mu_0 \mu \frac{I \cdot l}{2\pi} ln \frac{R_2}{R_1}, \tag{4}$$

where
R_1 is the radius of the inner cylinder of the coaxial cable;
R_2 is the radius of the outer cylinder of the coaxial cable.

Finally, the coefficient of self-inductance of the coaxial cable is

$$L = \frac{\Phi}{I} = \mu_0 \mu \frac{l}{2\pi} ln \frac{R_2}{R_1}. \tag{5}$$

Answer. The coefficient of self-induction of a coaxial cable is $L = \mu_0 \mu \frac{l}{2\pi} ln \frac{R_2}{R_1}$.

Problem 5.3.8

Problem description. Calculate the mutual induction coefficient of two circular currents. Circular currents are located in parallel planes with a common axis. One circular current is much smaller than the other. The radii of the circular current loops are $R = 0.5\ m$ and $r = 0.1\ m$, respectively. The distance between the planes of circular currents is $d = 1.5\ m$.

Known quantities: $R = 0.5\ m$, $r = 0.1\ m$, $d = 1.5\ m$.

Quantities to be calculated: M .

Problem solution. The intensity of the magnetic field on the axis of the circular current is determined by the relation

$$H = \frac{IR^2}{2\left(R^2 + d^2\right)^{3/2}}, \tag{1}$$

where
I is the current;
R is the radius of the conductor loop through which the current flow;
d is the distance from a fixed point to the plane of the circular current, measured along an axis passing through its center.

The flux of magnetic field induction through the plane of small circular current is

$$\Phi = BS = \frac{\mu_0 \mu I\, R^2 \pi r^2}{2\left(R^2 + d^2\right)^{3/2}}, \tag{2}$$

where
B is the induction of the magnetic field;
S is the area covered by the small circular current;
r is the radius of the conductor contour for small circular current;
μ_0 is the magnetic constant;
μ is the relative magnetic permeability of the substance in which the circular currents are located (according to the conditions of the problem $\mu \approx 1$).

The coefficient of mutual induction is equal to the ratio of the flux of magnetic induction to the current

$$M = \frac{\Phi}{I}. \qquad (3)$$

Hence

$$M = \frac{\mu_0 \mu \pi R^2 r^2}{2\left(R^2 + d^2\right)^{3/2}}. \qquad (4)$$

Substituting known values, we get

$$M = 1.2 \cdot 10^{-9} \ H. \qquad (5)$$

Answer. The mutual induction coefficient of two circular currents is $M = 1.2 \cdot 10^{-9} \ H$.

Problem 5.3.9

Problem description. The solenoid is connected to the battery. The electromotive force of the battery is $E = 8\,V$. The ohmic resistance of the solenoid is $R = 2\ \Omega$. The current in this electrical circuit reaches a value of $I = 1A$ after a time $\tau = 0.01\,s$. Calculate the coefficient of self-induction of the solenoid. The internal resistance of the battery can be neglected.

Known quantities: $E = 8\,V$, $R = 2\ \Omega$, $I = 1A$, $\tau = 0.01\,s$.

Quantities to be calculated: L.

Problem solution. After the solenoid is included in the electric circuit, the current in it after some time t will be determined by the relation

$$I = I_0\left[1 - exp\left(-\frac{R}{L}t\right)\right], \qquad (1)$$

where
I_0 is the current that will be established in the electrical circuit after the attenuation of induction phenomena;
R is the resistance of the solenoid;
L is the coefficient of the solenoid self-induction.

According to Ohm's law

$$I_0 = \frac{E}{R}, \tag{2}$$

where E is the electromotive force of the battery.

Now we can rewrite equation (1) with respect to the self-induction coefficient of the solenoid

$$L = \frac{R\tau}{ln\left(\frac{E}{E - IR}\right)}, \tag{3}$$

where τ is the time interval after which the current in the electric circuit reaches the value I.

Numerically

$$L = 0.07\ H. \tag{4}$$

Answer. The self-induction coefficient of the solenoid is $L = 0.07\ H$.

Problem 5.3.10

Problem description. To measure the mutual induction coefficient of two solenoids, an electrical circuit is used, which includes two resistances R_1 and R_2, a capacitor with an electrical capacity C, a galvanometer, and a battery with an electromotive force E. The resistance values R_1 and R_2 and the capacitance of the capacitor are related to each other in such a way that the current passing through the galvanometer when connecting and disconnecting the electrical circuit from the battery was equal to zero. Determine the coefficient of mutual induction of the solenoids.

Known quantities: R_1, R_2, C, E.

<u>Quantities to be calculated</u>: M .

<u>Problem solution</u>. When an electrical circuit is connected to the battery, an electric current appears. When the electrical circuit is subsequently disconnected from the battery, the current will change, as a result of which an electromotive force will appear in the circuit with the galvanometer

$$E = -L\frac{dI_1}{dt}, \tag{1}$$

where
L is the inductance of each of the solenoids;
dI_1 / dt is the change in current strength when an electrical circuit is opened.

Now we can write Kirchhoff's second law for a circuit with a galvanometer

$$L\frac{dI_1}{dt} = IR_0 + I_2R_2, \tag{2}$$

where
R_0 is the internal resistance of the galvanometer;
I is the current passing through the galvanometer;
I_2 is the amount of discharge current that passes through the resistance R_2.

We integrate equation (2)

$$\int LdI_1 = \int IR_0dt + \int I_2R_2dt. \tag{3}$$

According to the condition of the problem, the current does not pass through the galvanometer, therefore

$$\int IR_0dt = 0. \tag{4}$$

For current I_2 we can write the following relation

$$\int I_2dt = Q, \tag{5}$$

where Q is the electric charge on the capacitor plates.

The consequence of equations (3) - (4) is the relation

$$LI_1 = R_2 Q. \tag{6}$$

According to Ohm's law, the voltage at the ends of resistance R_1 is

$$U = I_1 R_1. \tag{7}$$

The relationship between the electric charge of a capacitor and the potential difference across the resistance R_1 can be represented as an equation

$$Q = CU. \tag{8}$$

In this case, the mutual induction coefficient of the solenoids is

$$M = L = CR_1 R_2. \tag{9}$$

Answer. The coefficient of mutual induction of solenoids is $M = CR_1 R_2$.

Problem 5.3.11

Problem description. The electrical circuit consists of a solenoid connected in series with an inductance of $L = 10\ H$ and a group of two resistances $r = 24.24\ \Omega$ and $R = 2400\ \Omega$. The resistors, in turn, are connected in parallel. The resistance group and the solenoid are connected to a battery with an electromotive force of $E = 24\,V$. Calculate the voltage at the ends of the parallel connection of the resistances for the case when the resistance r is connected to the electrical circuit, and for the case when the resistance r is disconnected from the electrical circuit.

Known quantities: $L = 10\ H$, $r = 24.24\ \Omega$, $R = 2400\ \Omega$, $E = 24\,V$.

Quantities to be calculated: U_1, U_2.

Problem solution. Consider the case when the resistance r is connected to an electric circuit and an electromotive force of self-induction arises in it

$$E_i = -L\frac{dI}{dt}, \tag{1}$$

where
L is the inductance of the solenoid;
dI/dt is the change in current in the circuit over time.

The current in an electrical circuit can be determined using the following relation

$$E - L\frac{dI}{dt} = \frac{Rr}{R+r}I, \tag{2}$$

where
E is the electromotive force of the battery;
R is the second resistance from a group of resistors connected in parallel.

Let us rewrite equation (2) as follows

$$\frac{dt}{L(R+r)} = \frac{dI}{(R+r)E - rRI}. \tag{3}$$

Now we can integrate both sides of equation (3)

$$ln[(R+r)E - rRI] = -\frac{rR}{L(R+r)}t + lnC, \tag{4}$$

where C is the constant of integration.

An arbitrary constant lnC can be determined from the conditions

$$I = I_0 \text{ for } t = 0. \tag{5}$$

Then

$$lnC = ln[(R+r)E - rRI_0]. \tag{6}$$

At the initial moment of time, the current in the electric circuit is equal to

$$I_0 = \frac{E}{R}. \tag{7}$$

Now equation (4) can be written as

$$ln\frac{(R+r)E - r\,RI}{(R+r)E - r\,E} = -\frac{r\,R}{L(R+r)}t\,. \tag{8}$$

From here we determine the current

$$I = \frac{E(R+r)}{Rr}\left\{1 - \frac{R}{R+r}exp\left[-\frac{r\,R}{L(R+r)}t\right]\right\}. \tag{9}$$

The voltage between the ends of a group of resistors connected in parallel is determined by the relation

$$U_1 = \frac{r\,RI}{R+r} = E\left\{1 - \frac{R}{R+r}exp\left[-\frac{r\,R}{L(R+r)}t\right]\right\}. \tag{10}$$

Substituting known values, we get

$$U_1 = 0.24\ V\,. \tag{11}$$

Consider the case of disconnection of the resistance r from the electrical circuit. In this case, the processes in the electrical circuit are determined by the relation

$$E + L\frac{dI}{dt} = RI\,. \tag{12}$$

The current at the initial moment of time is given by

$$I_0 = \frac{E(R+r)}{r\,R}. \tag{13}$$

Then the solution of equation (12) will have the form

$$I = \frac{E}{R}\left[1 + \frac{R}{r}exp\left(-\frac{R}{L}t\right)\right]. \tag{14}$$

The voltage between the ends of a group of resistors connected in parallel in this case is equal to the voltage between the ends of the resistance R

$$U_2 = IR = E\left[1 + \frac{R}{r} exp\left(-\frac{R}{L}t\right)\right]. \tag{15}$$

Let us now insert the given data

$$U_2 = 2400\,V. \tag{16}$$

Answer. The voltages when connecting and disconnecting resistance r are $U_1 = 0.24\ V$ and $U_2 = 2400\,V$, respectively.

Problem 5.3.12

Problem description. Current $I = 20\,A$ passes through a wire with a radius of $R_1 = 1\ mm$, placed along the axis of a sufficiently thin metal tube. Then this current passes to the bottom of the tube, to the center of which a wire is soldered, and returns back along the surface of the tube. The radius and length of the tube are equal, respectively $R_2 = 5\ cm$ and $l = 20\ cm$. Determine the energy of the magnetic field of the conductor.

Known quantities: $I = 20\,A$, $R_1 = 1\ mm$, $R_2 = 5\ cm$, $l = 20\ cm$.

Quantities to be calculated: W .

Problem solution. The energy of the magnetic field that occurs when the magnetic flux changes can be represented by the equation

$$dW = Id\Phi, \tag{1}$$

where
I is the current;
$d\Phi$ is an elementary change in the magnetic flux.

The magnetic field intensity inside the tube at a distance of x from its axis is

$$H = \frac{I}{2\pi x}. \tag{2}$$

The calculation of the magnetic field intensity by formula (2) can be performed without taking into account the edge effects. In addition, when solving the problem, we assume that there is no magnetic field outside the tube.

The elementary magnetic flux through a strip of a radial partition with a thickness dx can be determined using the following relation

$$d\Phi = \mu_0 \mu H dS = \mu_0 \mu \frac{I}{2\pi x} l\, dx, \tag{3}$$

where

μ_0 is the magnetic constant;

μ is the relative magnetic permeability;

l is the length of the strip;

dS is the elementary area of the strip.

Now we can integrate equation (1)

$$W = \int_{R_1}^{R_2} \mu_0 \mu \frac{I^2}{2\pi x} l\, dx = \mu_0 \mu \frac{I^2 l}{2\pi} ln \frac{R_2}{R_1}, \tag{4}$$

where

R_1 is the radius of the wire;

R_2 is the radius of the tube.

Numerically

$$W = 6 \cdot 10^{-3}\ J. \tag{5}$$

Answer. The energy of the magnetic field of the conductor is $W = 6 \cdot 10^{-3}\ J$.

5.4. Level 1 problems

5.4.1. A straight wire of length $l = 10\ cm$ is placed in a uniform magnetic field of induction $B = 1\ T$. The ends of the straight wire are closed by a flexible wire outside the field. The resistance of the entire circuit is $R = 0.4\ \Omega$. Calculate the power required to move the wire perpendicular to the magnetic field lines at a speed of $\upsilon = 20\ m \cdot s^{-1}$.

5.4.2. A rod of length $l = 10\ cm$ rotates in a uniform magnetic field with an induction of $B = 0.4\ T$ in a plane perpendicular to the lines of induction of the field. The axis of rotation passes through one of the ends of the rod. Determine the potential difference at the ends of the rod at a rotational speed of $n = 16\ s^{-1}$.

5.4.3. In a uniform magnetic field with induction $B = 0.35\ T$ the frame rotates uniformly. Frame rotation frequency is $n = 480\ min^{-1}$. The frame contains $N = 1500$ turns with an area of $S = 50\ cm^2$. The axis of rotation is in the plane of the frame and is perpendicular to the lines of induction. Determine the maximum electromotive force of induction that occurs in the frame.

5.4.4. A short solenoid containing $N = 1000$ turns rotates uniformly in a uniform magnetic field with an induction of $B = 0.4\ T$. The angular velocity of rotation is $\omega = 5\ rad \cdot s^{-1}$. Rotation occurs about an axis coinciding with the diameter of the solenoid, which, in turn, is perpendicular to the lines of magnetic field induction. Determine the instantaneous value of the electromotive force of induction for those moments of time when the angle between the plane of the coil and the lines of induction of the magnetic field is $\alpha = 60\ ^\circ$. The cross-sectional area of the solenoid is $S = 100\ cm^2$.

5.4.5. A straight magnet was inserted into a wire ring attached to a ballistic galvanometer. The amount of electric charge that has passed through the circuit is $Q = 10\ \mu C$. The resistance of the galvanometer circuit is $R = 30\ \Omega$. Calculate the magnetic flux that crosses the ring.

5.4.6. At a distance of $a = 1\ m$ from a long straight wire with a current of $I = 1\ kA$ here is a ring with a radius of $r = 1\ cm$. The ring is located so that the magnetic flux through it is maximum. Determine the amount of charge that will flow through the ring when the current in the conductor is turned off. Ring resistance is $R = 10\ \Omega$. The magnetic field within the ring can be considered uniform.

5.4.7. The inductance of the solenoid is $L = 0.03\ mH$. A current equal to $I = 0.6\ A$, flows through the solenoid. When the electrical circuit is opened, the current changes almost to zero in a time of $\Delta t = 120\ \mu s$. Determine the average value of the electromotive force of induction that occurs in the circuit.

5.4.8. The inductance of the solenoid is $L = 2\ mH$. The current with a frequency of $\nu = 50\ Hz$, flowing through the solenoid, changes according to a sinusoidal law. Determine the average electromotive force of induction that occurs over the time interval during which the current in the solenoid changes from the minimum to the maximum value. The amplitude value of the current strength is $I_0 = 10\ A$.

5.4.9. The inductance of a solenoid with a length of $l = 1\ m$, wound in one layer on a non-magnetic frame, is $L = 1.6\ mH$. The cross-sectional area of the solenoid is $S = 20 cm^2$. Determine the number of turns per centimeter of solenoid length.

5.4.10. The inductance of the solenoid is $L = 4\ mH$. The number of turns of the solenoid is $N = 600$. Determine the magnetic flux if the current flowing through the winding is $I = 12\ A$.

5.4.11. The number of turns of the iron core solenoid winding is $N = 500$. The length of the solenoid core is $l = 50\ cm$. The current flowing through the winding increases from a value of $I_1 = 0.1\ A$ to a value of $I_2 = 1\ A$. Calculate the relative change in solenoid inductance.

5.4.12. The electrical circuit consists of a solenoid with an inductance of $L = 1\ H$ and a resistance of $R = 10\Omega$. The current source can be turned off without breaking the circuit. Determine the time after which the current will decrease a thousand times compared to its original value.

5.4.13. A solenoid with an inductance of $L = 0.5\ H$ and a resistance of $R_1 = 8\ \Omega$ 66 is connected to a current source with an internal resistance of $R_2 = 2\ \Omega$. Calculate the time during which the current in the solenoid, increasing, reaches a value that differs from the maximum by 1%.

5.4.14. The average rate of change of the magnetic flux in the betatron is $\Delta\Phi / \Delta t = 50\, Wb \cdot s^{-1}$. The betatron is rated at $T = 60\ MeV$ energy. Determine the number of turns of the electron in orbit during the time of accelerated motion, as well as the path passed by the electron, provided that the radius of its orbit is $r = 20\ cm$.

5.4.15. An electron in a betatron moves in an orbit with a radius of $r = 0.4\ m$ and receives a kinetic energy of $T = 20\ eV$ in one turn. Calculate the rate of change of the magnetic field induction, considering this rate to be constant over a fixed period of time.

5.4.16. The number of turns of the solenoid is $N = 1000$. The current in the solenoid winding is $I = 1\ A$. The magnetic flux through the cross section of the solenoid is $\Phi = 0.1\, mWb$. Calculate the energy of the magnetic field.

5.4.17. The current that flows through the winding of the toroid is $I = 0.6\ A$. The diameter of the turns of wire is $d = 0.4\ mm$. The coils fit snugly together. In this case, the thickness of the wire insulation can be neglected. The cross-sectional area of the toroid core is $S = 4\ cm^2$. The midline diameter of the toroid is $D = 30\ cm$. Determine the energy of the magnetic field in the steel core of the toroid.

5.4.18. Calculate the energy density of the magnetic field in the iron core of a closed solenoid if the intensity of the magnetizing field is $H = 1.2\ kA \cdot m^{-1}$.

5.4.19. At a certain current, the energy density of the magnetic field of a solenoid without a core is $\omega = 0.2\ J \cdot m^{-3}$. Calculate the relative increase in field energy density at the same current if the solenoid has an iron core.

5.4.20. The number of turns for each centimeter of the length of a toroid with a non-magnetic core is $n = 10\ cm^{-1}$. Determine the energy density of the magnetic field for the case when current $I = 16\ A$ flows through the winding.

5.5. Answers to problems

5.4.1. $P = 10\ W$.

5.4.2. $U = 201\ mV$.

5.4.3. $E_{max} = 1.32 \cdot 10^{2}\ V$.

5.4.4. $E_i = 1\ V$.

5.4.5. $\Phi = 3 \cdot 10^{-4}\ Wb$.

5.4.6. $Q = 6.28 \cdot 10^{-5}\ C$.

5.4.7. $< E >= 1.5 \cdot 10^{-1}\ V$.

5.4.8. $< E >= 4\ V$.

5.4.9. $n = 8\ cm^{-1}$.

5.4.10. $\Phi = 8 \cdot 10^{-5}\ Wb$.

5.4.11. $L_2 / L_1 = 1/58$.

5.4.12. $\tau = 6.9 \cdot 10^{-1}\ s$.

5.4.13. $t = 2.3 \cdot 10^{-1}\ s$.

5.4.14. $N = 1.2 \cdot 10^{6}$; $L = 1.51 \cdot 10^{6}\ m$.

5.4.15. $\Delta B / \Delta t = 4 \cdot 10^{1}\ T \cdot s^{-1}$.

5.4.16. $W = 5 \cdot 10^{1}\ J$.

5.4.17. $W = 3.24 \cdot 10^{-1}\ J$.

5.4.18. $\omega = 8 \cdot 10^{2}\ J \cdot m^{-3}$.

5.4.19. $\omega_2 / \omega_1 = 1.6 \cdot 10^{3}$.

5.4.20. $\omega = 1.61 \cdot 10^{2}\ J \cdot m^{-3}$.

CHAPTER 6. ELECTROMAGNETIC OSCILLATIONS AND WAVES

6.1. Basic formulas

Ohm's law for a section of an alternating current circuit has the form

$$I_e = \frac{U_e}{Z}, \tag{6.1.1}$$

where
I_e is the effective value of the current;
U_e is the effective value of the voltage;
Z is the total (effective) resistance.

The **effective values of current and voltage** are determined by the following equations

$$I_e = \frac{I_0}{\sqrt{2}}, \tag{6.1.2}$$

$$U_e = \frac{U_0}{\sqrt{2}}, \tag{6.1.3}$$

where
I_0 is the amplitude value of the sinusoidal current;
U_0 is the amplitude value of the sinusoidal voltage.

Effective resistance can be defined as a function of active (ohmic), capacitive and inductive resistances

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}, \tag{6.1.4}$$

where
ω is the circular frequency of change in current and voltage values;
R is an active (ohmic) resistance;
L is the inductance;

C is the electric capacity;
$\omega L - 1/(\omega C)$ is the reactance;
ωL is the inductive reactance;
$1/(\omega C)$ is the capacitance.

The term effective resistance or impedance was introduced by the English engineer, mathematician and physicist Heaviside (Oliver Heaviside 1850 – 1925).

The **phase shift between current and voltage** is determined according to the formula

$$tg\varphi = \frac{\left(\omega L - \dfrac{1}{\omega C}\right)}{R}. \qquad (6.1.5)$$

The total resistance for series-connected active and inductive resistances is

$$Z = \sqrt{R^2 + (\omega L)^2}. \qquad (6.1.6)$$

The total resistance for parallel connected active and inductive resistances is

$$Z = \frac{\omega LR}{\sqrt{R^2 + (\omega L)^2}}. \qquad (6.1.7)$$

The phase shift for series-connected active and inductive resistances is

$$tg\varphi = \frac{\omega L}{R}. \qquad (6.1.8)$$

The phase shift for parallel-connected active and inductive resistances is determined by the relation

$$tg\varphi = \frac{R}{\omega L}. \qquad (6.1.9)$$

The impedance for series-connected active and capacitive resistances can be represented by the equation

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}\,. \tag{6.1.10}$$

The impedance for parallel connected resistance and capacitance can be determined using the following relation

$$Z = \frac{R}{\sqrt{1 + (\omega RC)^2}}\,. \tag{6.1.11}$$

The phase shift for series-connected active and capacitive resistances is determined by the relation

$$tg\varphi = \frac{1}{\omega RC}\,. \tag{6.1.12}$$

The phase shift for active and capacitive resistances connected in parallel can be represented by the equation

$$tg\varphi = -\omega RC\,. \tag{6.1.13}$$

A **solenoid**, which is characterized by active resistance and inductance in an alternating current circuit, can be considered as a series connection of active and inductive resistances. The solenoid, which is characterized by active resistance and electrical capacitance in an alternating current circuit, can be considered as a parallel connection of active and capacitive resistances.

The **active power of alternating current**, which is released in the electrical circuit in one period, is given by

$$P = I_e U_e \cos\varphi\,. \tag{6.1.14}$$

For the case when the electrical circuit contains only active resistance ($\varphi = 0$), the active power is

$$P = I_e U_e = I_e^2 R = \frac{U_e^2}{R}\,. \tag{6.1.15}$$

For the case where the electrical circuit contains only capacitive or only inductive resistance ($cos\varphi = 0$), the active power can be determined using the following relationship

$$P = 0. \tag{6.1.16}$$

The period of electromagnetic oscillations in a circuit, which consists of electrical capacitance, inductance and active resistance connected in series, is determined according to the formula

$$T = \frac{2\pi}{\sqrt{\left(\frac{1}{LC}\right) - \left(\frac{R}{2L}\right)^2}}. \tag{6.1.17}$$

For the case when the resistance of the electric circuit is sufficiently small and obeys a strong inequality

$$\left(\frac{R}{2L}\right)^2 << \frac{1}{LC}, \tag{6.1.18}$$

the period of undamped electromagnetic oscillations is determined by the **Thomson formula**

$$T = 2\pi\sqrt{LC}. \tag{6.1.19}$$

Note that an LC circuit can considered a special case where the resistance R in RLC circuit goes to zero [6, p. 58].

Formula (6.1.19) was obtained by the British physicist, mechanic and engineer Thomson (William Thomson 1824 – 1907).

Electromagnetic oscillations are damped for the case when the resistance of the electrical circuit is not equal to zero. In this case, the potential difference on the capacitor plates changes with time t according to the law

$$U = U_0\, exp(-\delta t) cos(\omega t), \tag{6.1.20}$$

where δ is the attenuation coefficient.

Formula (6.1.20) is given for the case when the time is counted from the moment when the potential difference on the capacitor plates has a maximum value. The functional relationship between the attenuation coefficient, active resistance and inductance has the form

$$\delta = \frac{R}{2L}. \tag{6.1.21}$$

Electromagnetic oscillations in an electrical circuit, the resistance of which is not equal to zero, are characterized by a **logarithmic decrement**

$$\kappa = \delta T. \tag{6.1.22}$$

For the case when $\delta = 0$, electromagnetic oscillations will be undamped, and the dependence of voltage on time has the form

$$U = U_0 \cos(\omega t). \tag{6.1.23}$$

Q factor of the oscillatory circuit is

$$Q = \frac{R_x}{R}, \tag{6.1.24}$$

where R_x is the wave impedance of the oscillating circuit.

The functional relationship between wave impedance, inductance and capacitance can be represented using the following equation

$$R_x = \sqrt{\frac{L}{C}}. \tag{6.1.25}$$

6.2. Problem-solving framework

Most of the typical problems can be solved analytically using formulas (6.1.1) - (6.1.25). However, the solution of problems concerning alternating currents is greatly facilitated by using the graphical method, which is also called the method of vector diagrams of voltages and currents.

A **vector voltage diagram** (voltage triangle) for an alternating current circuit with a solenoid connected in series and an electrical capacity is compiled as follows. On a certain scale, along an arbitrary axis X the voltage vector on the active resistance

$\vec{U}_R$ is plotted. The voltage across the inductive reactance leads the current in phase by $\pi/2$. Therefore, the vector $\vec{U}_L$ is perpendicular to $\vec{U}_R$ in the direction of increasing angles. Vector $\vec{U}_C$ on the capacitance lags the current in phase by $\pi/2$. Therefore, vectors $\vec{U}_L$ and $\vec{U}_C$ are directed in opposite directions. The resulting voltage $\vec{U}$ is a simple geometric sum. From the voltage triangle (see Fig. 6.1) we can easily determine the phase shift between current and voltage (angle φ).

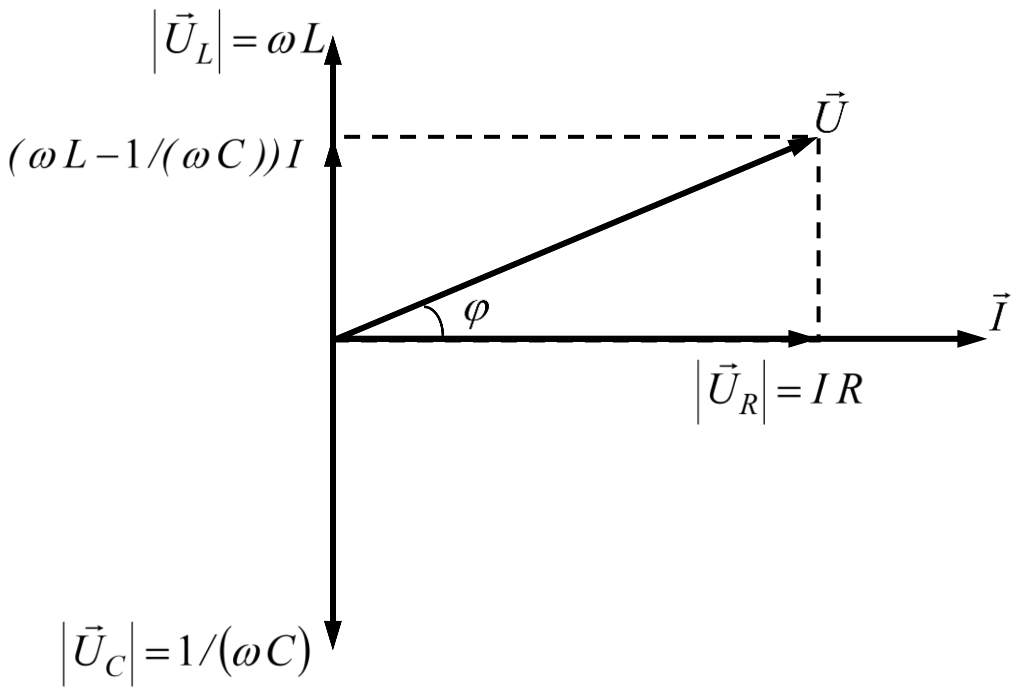

Figure 6.1. Vector voltage diagram.

To graphically determine the impedance of alternating current for the case of known values of active, inductive and capacitive resistances, we can build a triangle of resistance. The legs of this triangle are R and $\left(\omega L - \dfrac{1}{\omega C}\right)$, and the hypotenuse is equal to the total resistance $Z = \sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$. The resistance triangle can also be useful for determining the phase shift between current and voltage

$$tg\varphi = \frac{\left(\omega L - \dfrac{1}{\omega C}\right)}{R}. \qquad (6.2.1)$$

6.3. Problem-solving examples

Problem 6.3.1

Problem description. Determine the impedance of the solenoid and the phase shift between current and voltage in the AC circuit, if the values of active and inductive resistances are equal, respectively $R = 1.5\ \Omega$ and $X_L = 2\ \Omega$.

Known quantities: $R = 1.5\ \Omega$, $X_L = 2\ \Omega$.

Quantities to be calculated: Z, φ.

Problem solution. We will solve the problem graphically by constructing a triangle of resistance. First of all, we will choose the scale $1\ cm = 0.5\ \Omega$. On this scale, the leg that corresponds to the active resistance is equal to $R = 1.5/0.5 = 3\ cm$. The length of the leg, which corresponds to the inductive reactance, is $X_L = 2/0.5 = 4\ cm$. From an arbitrary point O (см. рис. 6.2) we will construct a segment $OA = 3\ cm$, which corresponds to the active resistance $R = 1.5\ \Omega$. Through point A we draw a straight line perpendicular to segment OA and construct segment $FB = 4\ cm$ on it, which corresponds to the inductive reactance $X_L = 2\ \Omega$. We connect points O and B with a straight line, which corresponds to the total resistance Z. Measuring segment OB allows us to determine its length ($5\ cm$). Therefore, the total resistance of this connection is $Z = 2.5\ \Omega$.

An analysis of the geometric features of the triangle of resistance makes it possible to determine the magnitude of the phase shift between current and voltage: $\varphi = 53°$.

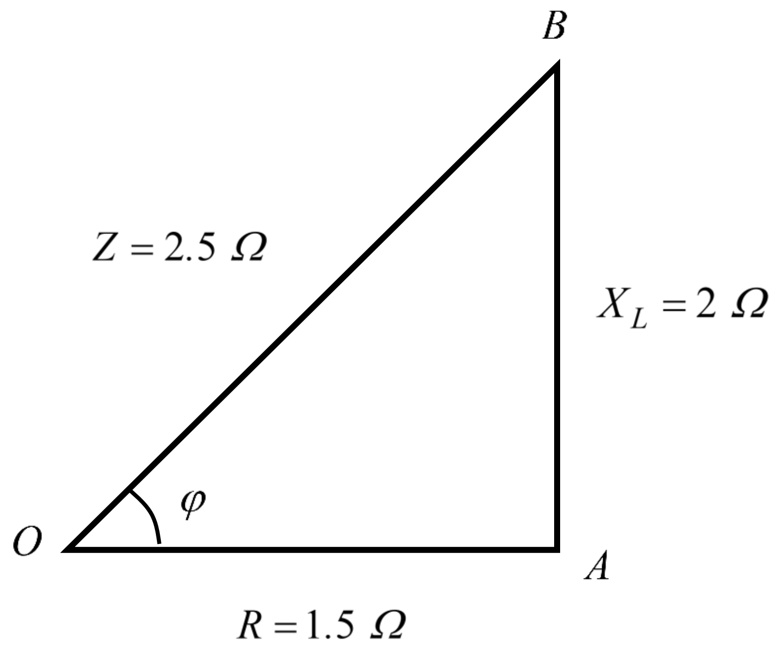

Figure 6.2. Resistance triangle.

Answer. The impedance of the connection is $Z = 2.5\,\Omega$. The phase shift between current and voltage is $\varphi = 53°$.

Problem 6.3.2

Problem description. In an AC circuit, the voltage at the solenoid terminals is $U = 120\ V$. The phase shift between current and voltage is $\varphi = 37\ °$. Determine active and inductive voltage.

Known quantities: $U = 120\ V$, $\varphi = 37\ °$.

Quantities to be calculated: U_R, U_L.

Problem solution. To solve the problem, we will build a stress triangle with a scale of $1\ cm - 20V$. On this scale, the hypotenuse of the stress triangle is $120/20 = 6\ cm$.

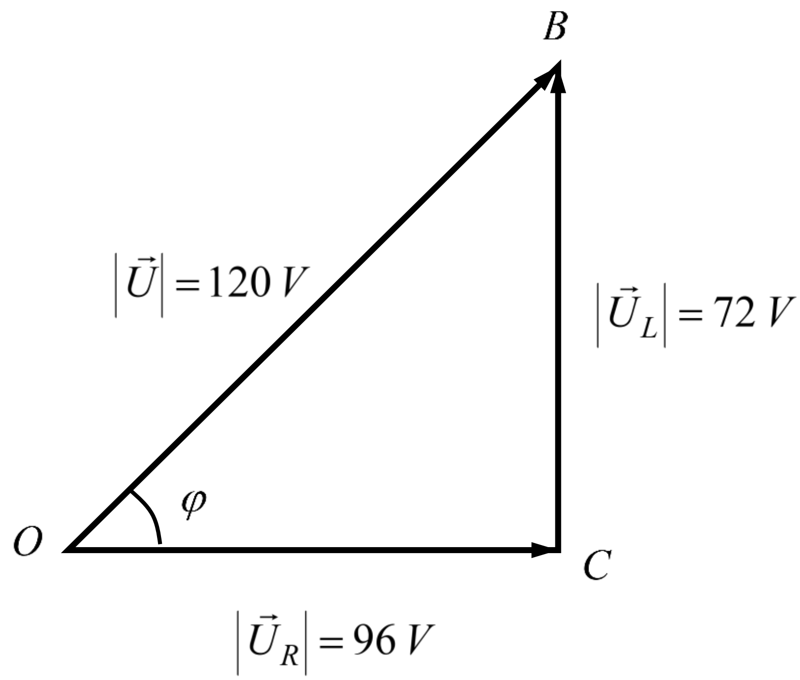

Figure 6.3. Voltage triangle.

From an arbitrary point O (see Fig. 6.3) we draw a line OC. Further, at an angle of $\varphi = 37\ ^\circ$ to this line, we will postpone the segment $OB = 6\ cm$. This segment corresponds to the voltage vector $\vec{U}$ at the solenoid terminals. From point B let's drop perpendicular CB to line OC, which corresponds to the vector of inductive voltage $\vec{U}_L$. In turn, segment OC corresponds to the active voltage vector $\vec{U}_R$.

An analysis of the geometric features of the stress triangle allows us to determine the length of the segments OC and BC. Therefore, the active voltage is $U_R = 20 \times 4.8 = 96\ V$. The inductive voltage is $U_L = 20 \times 3.6 = 72\ V$.

Answer. The active voltage is $U_R = 96\ V$. The inductive voltage is $U_L = 72\ V$.

Problem 6.3.3

Problem description. Determine the impedance and value $cos\,\varphi$ for an alternating current circuit, which consists of active resistance $R = 8\ \Omega$, inductive resistance $X_L = 20\ \Omega$ and capacitance $X_C = 26\ \Omega$.

<u>Known quantities</u>: $R = 8\ \Omega$, $X_L = 20\ \Omega$, $X_C = 26\ \Omega$.

<u>Quantities to be calculated</u>: Z, $\cos\varphi$.

<u>Problem solution</u>. The reactance in the AC circuit is

$$X = X_L - X_C, \tag{1}$$

where
X_L is an inductive reactance;
X_C is the capacitance.

Substituting known values, we get

$$X = -6\ \Omega. \tag{2}$$

The value of the reactance is negative, therefore, the load in the AC circuit has a capacitive character.

Next, we will choose a scale $1\ cm - 2\ \Omega$ and build a resistance triangle (see Fig. 6.4) similar to the case of an electrical circuit with active resistance and electric capacitance. An analysis of the geometric features of the resistance triangle allows us to determine the values of the impedance and the cosine of the phase shift between current and voltage.

The total resistance of the electrical circuit in this case is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + X^2}. \tag{3}$$

Numerically

$$Z = 10\ \Omega. \tag{4}$$

The cosine of the phase shift between current and voltage is

$$\cos\varphi = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + X^2}}. \tag{5}$$

Substituting known values, we get

$$cos\,\varphi = 0.8. \qquad (6)$$

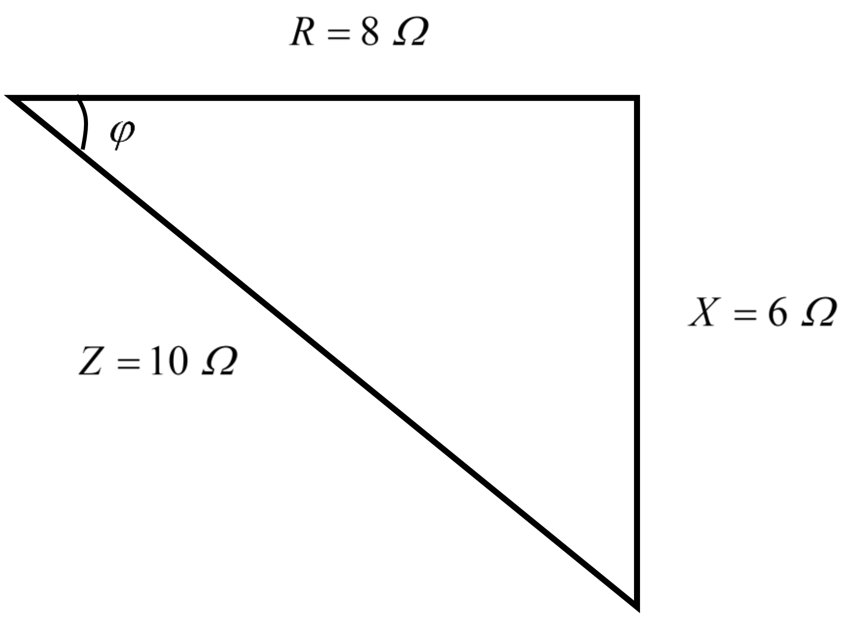

Figure 6.4. Resistance triangle for an electrical circuit with active, capacitive and inductive resistances.

Answer. The total resistance of the electrical circuit is $Z = 10\ \Omega$. The cosine of the phase shift between current and voltage is $cos\,\varphi = 0.8$.

Problem 6.3.4

Problem description. The electrical circuit (see Fig. 6.5) is characterized by the following parameters: the power recorded by the wattmeter is $P = 940\,W$; the voltage recorded by the voltmeter is $U = 220\,V$; the current recorded by the ammeter is $I = 5\ A$. Determine the resistance values R_1, X_1 and build a vector diagram. Calculate the voltage in section KM and the phase shift between current and voltage.

Known quantities: $P = 940\,W$, $U = 220\,V$, $I = 5\ A$.

Quantities to be calculated: R_1, X_1, U_{R_1}, U_{X_1}, vector diagram, φ_2.

Problem solution. The active power in the section with a resistance R_2 is

$$P_2 = I^2 R_2^2, \qquad (1)$$

where I is the current flowing through the section.

Substituting known values, we have

$$P_2 = 550\,W. \qquad (2)$$

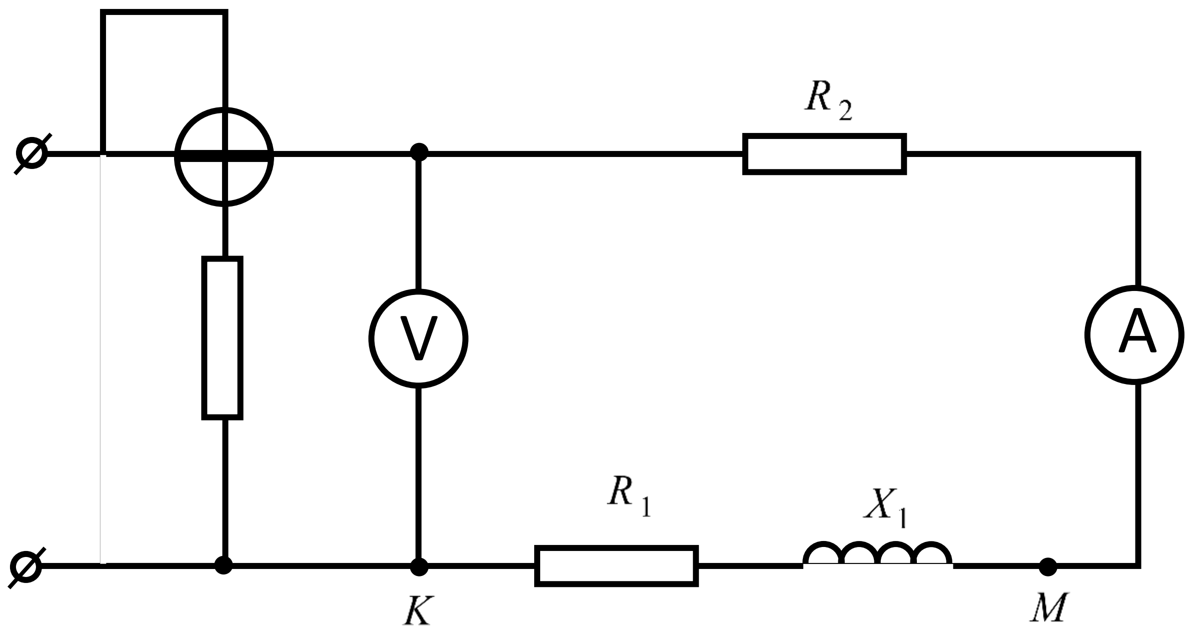

Figure 6.5. Electrical circuit for problem 6.3.4.

Active power in section KM (see Fig. 6.5) is

$$P_1 = P - P_2, \qquad (3)$$

where P is the wattmeter reading.

Numerically

$$P_1 = 390\,W. \qquad (4)$$

Analysis of equation (1) allows us to determine the resistance value R_1

$$R_1 = \frac{P_1}{I^2}. \tag{5}$$

Substituting known values, we get

$$R_1 = 15.6\ \Omega. \tag{6}$$

The phase shift cosine φ between current and voltage can be determined using the equation

$$cos\,\varphi = \frac{P}{IU}, \tag{7}$$

where U is the voltage value, which is determined by the readings of the voltmeter.

Numerically

$$cos\,\varphi = 0.855. \tag{8}$$

The reactance for a given phase shift is

$$X_1 = Z\,sin\,\varphi = \frac{U}{I}\,sin\,\varphi, \tag{9}$$

where Z is the total resistance.

Substituting known values, we have

$$X_1 = 22.8\ \Omega. \tag{10}$$

The voltage across the active resistance R_1 is

$$U_{R_1} = IR_1. \tag{11}$$

Numerically

$$U_{R_1} = 78\,V. \tag{12}$$

The voltage across the active resistance R_2 is

$$U_{R_2} = IR_2. \tag{13}$$

Substituting known values, we get

$$U_{R_2} = 110\,V. \tag{14}$$

The voltage across the reactance X_1 is

$$U_{X_1} = IX_1. \tag{15}$$

Numerically

$$U_{X_1} = 114\,V. \tag{16}$$

Using the calculated values of stresses, we can build a vector diagram (see Fig. 6.6).

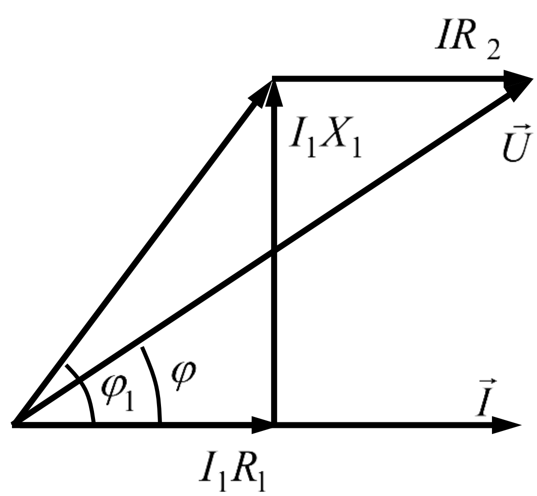

Figure 6.6. Vector diagram for problem 6.3.4.

Analysis of the vector diagram allows us to calculate the phase shift between current and voltage

$$tg\varphi_2 = 1.46; \qquad \varphi_2 = 55°30'. \tag{17}$$

Answer. The active resistance is $R_1 = 15.6\ \Omega$. The reactance is $X_1 = 22.8\ \Omega$. The voltage across the active resistance (section KM) is $U_{R_1} = 78\,V$. The voltage across the reactance (section KM) is $U_{X_1} = 114\,V$. The vector diagram is shown in Fig. 6.6. The phase shift between current and voltage in section KM is $\varphi_2 = 55°30'$.

Problem 6.3.5

Problem description. The light bulb and solenoid are connected in parallel and connected to a $U = 120\,V$ AC supply. The active resistance of the light bulb is $R = 70\ \Omega$. The active and reactive resistances of the solenoid are $R_L = 12\ \Omega$ and $X_L = 16\ \Omega$. Determine the current in the supply wires and the phase shift between current and voltage.

Known quantities: $U = 120\,V$, $R = 70\ \Omega$, $R_L = 12\ \Omega$, $X_L = 16\ \Omega$.

Quantities to be calculated: I_1, I_2, φ.

Problem solution. The electrical circuit, which includes a light bulb and a solenoid connected in parallel, is shown in Fig. 6.7.

The current in the light bulb is

$$I_1 = \frac{U}{R}, \tag{1}$$

where

U is the voltage in the AC circuit;

R is the resistance of the bulb.

Substituting known values, we get

$$I_1 = 1.71\ A. \tag{2}$$

The phase shift between the current in the bulb and the voltage is zero ($\varphi = 0$).

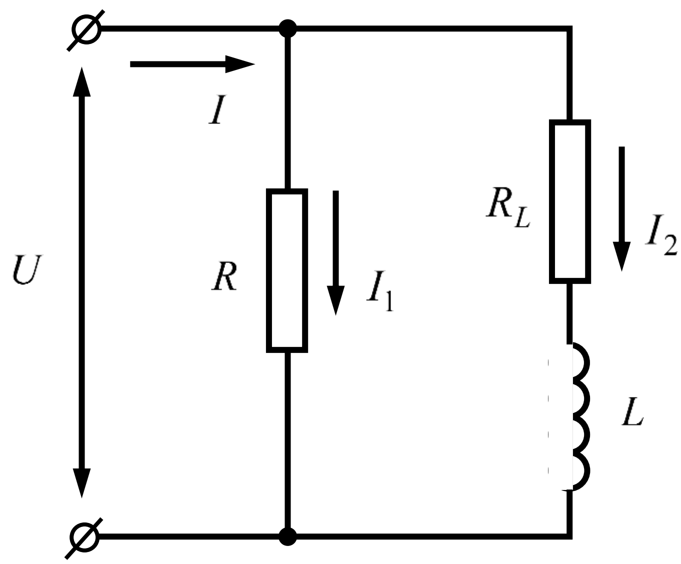

Figure 6.7. Electrical circuit for problem 6.3.5.

Solenoid impedance is

$$Z = \sqrt{R_L^2 + X_L^2}\,, \tag{3}$$

where
R_L is the active resistance of the solenoid;
X_L is the reactance of the solenoid.
Numerically

$$Z = 20\ \Omega. \tag{4}$$

The current in the solenoid is

$$I_2 = \frac{U}{Z}. \tag{5}$$

Substituting known values, we get

$$I_2 = 6\ A. \tag{6}$$

The phase shift between the current in the solenoid and the voltage is

$$\cos\varphi_2 = \frac{R_L}{Z}. \tag{7}$$

Substituting known values, we have

$$\varphi_2 = 53^\circ. \tag{8}$$

To build a vector diagram (see Fig. 6.8), we will choose the scale: $1\ cm - 1\ A$. In the horizontal direction, we will plot the voltage vector $\vec{U}$. Current $\vec{I}_1$ is in phase with the voltage, so we will build the vector $\vec{I}_1$ in the direction of the vector $\vec{U}$. Between the current $\vec{I}_2$ and the voltage $\vec{U}$ there is a phase difference φ_2, so we will direct the vector $\vec{I}_2$ at an angle $\varphi_2 = 53^\circ$ towards the lag.

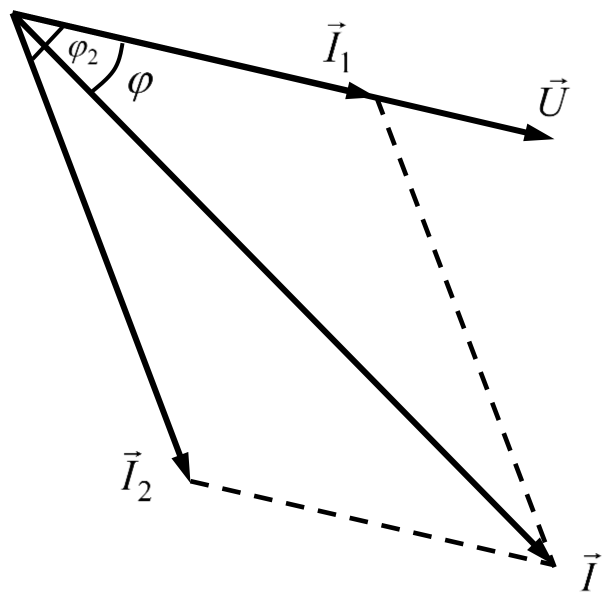

Figure 6.8. Vector diagram for problem 6.3.5.

The current vector $\vec{I}$ in the supply wires is equal to the vector sum of the vectors $\vec{I}_1$ and $\vec{I}_2$. Analysis of the geometric features of the vector diagram shown in Fig. 6.8., allows us to get the next numerical value for the module of the vector $\vec{I}$

$$I = 7.2\ A. \tag{9}$$

The phase shift between current and voltage in a given electrical circuit is equal to the angle φ on the vector diagram

$$\varphi = 42\ °. \tag{10}$$

<u>Answer</u>. The currents in the supply wires are: $I_1 = 1.71\ A$ and $I_2 = 6\ A$. The phase shift between current and voltage in an electrical circuit is $\varphi = 42\ °$.

Problem 6.3.6

<u>Problem description</u>. The AC circuit contains a solenoid with an inductance of $L = 10\ mH$ and a capacitor with an electrical capacity of $C = 400\ \mu F$. The resistance values $R_1 = R_2 = R$ change simultaneously by the same amount. Current resonance occurs at the following frequencies: $f_1 = 50\ Hz$, $f_2 = 100\ Hz$ and $f_3 = 200\ Hz$. The voltage in the electrical circuit is $U = 120\ V$. Determine the amount of resistance that corresponds to this resonance. Calculate the values of currents for branched and unbranched parts of the electrical circuit.

<u>Known quantities</u>: $L = 10\ mH$, $C = 400\ \mu F$, $R_1 = R_2 = R$, $f_1 = 50\ Hz$, $f_2 = 100\ Hz$, $f_3 = 200\ Hz$, $U = 120\ V$.

<u>Quantities to be calculated</u>: R_r, I_1, I_2, I_3.

<u>Problem solution</u>. The current resonance condition in the electrical circuit shown in Fig. 6.9 for the case $R_1 = R_2 = R$ can be written as follows

$$\frac{\omega L}{R^2 + (\omega L)^2} = \frac{\omega C}{(\omega RC)^2 + 1}, \tag{1}$$

or

$$\frac{X_L}{R^2 + X_L^2} = \frac{X_C}{R^2 + X_C^2}, \tag{2}$$

where
ω is the circular frequency of alternating current;
L is the inductance of the solenoid;
R is active resistance;
C is the capacitance of the capacitor;
X_L is an inductive reactance;
X_C is the capacitance.

Analysis of equations (1) and (2) shows that the current resonance corresponds to the following condition for the resistance value

$$R = R_r = \sqrt{\frac{L}{C}}. \tag{3}$$

where R_r is the resonant impedance.

Therefore, the value of the resonant impedance does not depend on the frequency.

The current passing through resistance R_1, is

$$I_1 = \frac{U}{\sqrt{R^2 + (\omega L)^2}} = \frac{U}{\sqrt{R^2 + (2\pi f L)^2}}, \tag{4}$$

where
U is the voltage in the circuit;
f is the line frequency of the AC.

The current passing through resistance R_2, is

$$I_2 = \frac{U}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} = \frac{U}{\sqrt{R^2 + (2\pi f C)^2}}. \tag{5}$$

The current in the supply wires is

$$I = \sqrt{I_1^2 + I_2^2}. \tag{6}$$

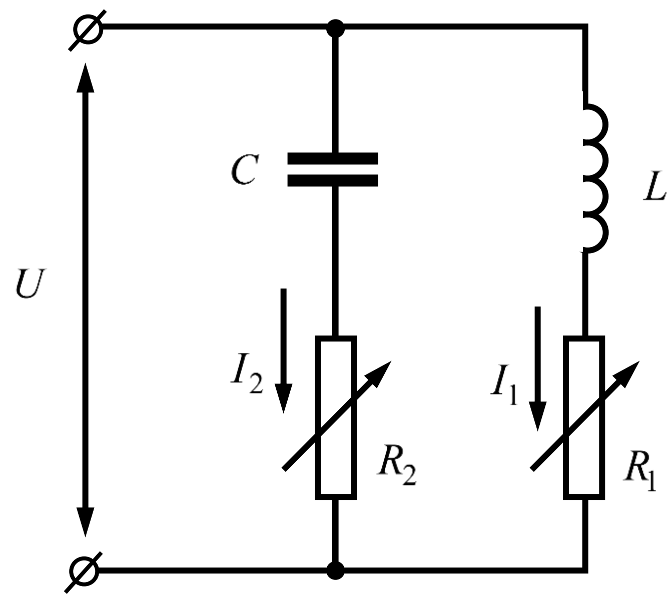

Figure 6.9. Electrical circuit for problem 6.3.6

We substitute numerical values for $f_1 = 50\ Hz$

$$R_r = 5\ \Omega, \quad I_1 = 20.3\ A, \quad I_2 = 12.7\ A, \quad I = 24\ A. \tag{7}$$

We substitute numerical values for $f_2 = 100\ Hz$

$$R_r = 5\ \Omega, \quad I_1 = 15\ A, \quad I_2 = 18.75\ A, \quad I = 24\ A. \tag{8}$$

We substitute numerical values for $f_3 = 50\ Hz$

$$R_r = 5\ \Omega, \quad I_1 = 8.9\ A, \quad I_2 = 22.4\ A, \quad I = 24\ A. \tag{9}$$

Answer. The resonant impedance is $R_r = 5\ \Omega$. The currents for branched and unbranched parts of the electrical circuit are equal for $f_1 = 50\ Hz$: $I_1 = 20.3\ A$, $I_2 = 12.7\ A$, $I = 24\ A$; for $f_2 = 100\ Hz$: $I_1 = 15\ A$, $I_2 = 18.75\ A$, $I = 24\ A$; for $f_3 = 50\ Hz$: $I_1 = 8.9\ A$, $I_2 = 22.4\ A$, $I = 24\ A$.

Problem 6.3.7

Problem description. The AC circuit is energized at $U = 120\,V$. The current in the electrical circuit changes with a frequency of $f = 50\,Hz$. The electrical circuit contains a solenoid with an inductance of $L = 0.3\,H$ and a resistance of $R = 10\,\Omega$. Determine the amplitude of the current and the phase shift between current and voltage in the electrical circuit.

Known quantities: $U = 120\,V$, $f = 50\,Hz$, $L = 0.3\,H$, $R = 10\,\Omega$.

Quantities to be calculated: I, φ.

Problem solution. The amplitude of the current can be determined using the following relationship

$$I = \sqrt{2} I_e = \frac{U_e \sqrt{2}}{\sqrt{R^2 + (\omega L)^2}}, \tag{1}$$

where

I_e is the effective value of the current;

U_e is the effective value of the voltage;

R is the active resistance;

ω is the circular frequency of the change in current in an electrical circuit;

L is the inductance of the solenoid.

Substituting known values, we get

$$I = 1.68\ A. \tag{2}$$

The phase shift between current and voltage can be determined by the formula

$$\varphi = arctan\left(\frac{\omega L}{R}\right). \tag{3}$$

Numerically

$$\varphi = 43\,°. \tag{4}$$

Answer. The current amplitude is $I = 1.68\ A$. The phase shift between current and voltage is $\varphi = 43\,°$.

Problem 6.3.8

Problem description. A three-phase consumer, which has an active resistance $R = 8\ A$ and an inductive resistance $X_L = 6\ \Omega$, in each phase, is connected to a three-phase current network with a linear voltage $U_{LN} = 380\,V$. Determine the voltage in each phase of the consumer and the currents in each phase and linear wires, if the phases are connected according to schemes «wye» and «delta».

Known quantities: $R = 8\ A$, $X_L = 6\ \Omega$, $U_{LN} = 380\,V$.

Quantities to be calculated: U_{PH}, I_{PH}, I_{LN}.

Problem solution. Due to the fact that the load in the electrical circuit is uniform, it is enough to calculate the voltage in one phase.

Consider the "wye" schema. The voltage in each phase is

$$U_{PH} = \frac{U_{LN}}{\sqrt{3}}, \tag{1}$$

where U_{LN} is the line voltage.

Numerically

$$U_{PH} = 220\,V. \tag{2}$$

The impedance of each phase is determined by the ratio

$$Z_{PH} = \sqrt{R_{PH}^2 + X_{L,PH}^2}, \tag{3}$$

where

$R_{PH} = R$ is the active resistance of each phase;

$X_{L,PH} = X_L$ is the inductive reactance of each phase.

Substituting known values, we get

$$Z_{PH} = 10\ \Omega. \tag{4}$$

The current in each phase and in the line wire is

$$I_{PH} = I_{LN} = \frac{U_{PH}}{Z_{PH}}, \tag{5}$$

where

I_{PH} is the phase current;

I_{LN} is the current in the line.

Substituting known values, we find

$$I_{PH} = I_{LN} = 22\ A. \tag{6}$$

Consider the "delta" scheme. The voltage in each phase is

$$U_{PH} = U_{LN} = 380\,V. \tag{7}$$

The impedance of each phase is

$$Z_{PH} = 10\ \Omega. \tag{8}$$

The current in each phase is

$$I_{PH} = \frac{U_{PH}}{Z_{PH}} = 38\ A. \tag{9}$$

The current in each line wire is

$$I_{LN} = \sqrt{3} I_{PH}. \tag{10}$$

Numerically

$$I_{LN} = 66\ A. \tag{11}$$

Answer. "Wye" scheme. The voltage in each phase is $U_{PH} = 220\,V$. The current in each phase and in the line wire is $I_{PH} = I_{LN} = 22\ A$. "Delta" scheme. The voltage in each phase is $U_{PH} = 380\ V$. The current in each phase is $I_{PH} = 38\ A$. The current in each line wire is $I_{LN} = 66\ A$.

Problem 6.3.9

Problem description. The oscillating circuit consists of capacitors connected in series with electric capacitance $C = 10\ \mu F$, non-inductive resistance $R = 20\ \Omega$ two solenoids with inductances $L_1 = 0.2\ mH$ and $L_2 = 0.4\ mH$ and a very small active resistance. Determine the period of free oscillations in this oscillatory circuit for the cases of series and parallel connection of solenoids.

Known quantities: $C = 10\ \mu F$, $R = 20\ \Omega$, $L_1 = 0.2\ mH$, $L_2 = 0.4\ mH$.

Quantities to be calculated: T_1, T_2.

Problem solution. The period of free oscillations in an electromagnetic oscillatory circuit can be determined using the following relationship

$$T = \frac{2\pi}{\sqrt{\left(\frac{1}{LC}\right) - \left(\frac{R}{2L}\right)^2}}, \qquad (1)$$

where
C is the capacitance of the capacitor;
R is active resistance;
L is the total inductance of the two solenoids.

Consider two types of connection of solenoids: 1) serial connection, 2) parallel connection.

For the case of series connection (see Fig. 6.10), the same current I flows in both solenoids.

The electromotive force in the first solenoid is

$$E_1 = -L_1 \frac{dI}{dt}, \qquad (2)$$

where L_1 is the inductance of the first solenoid.

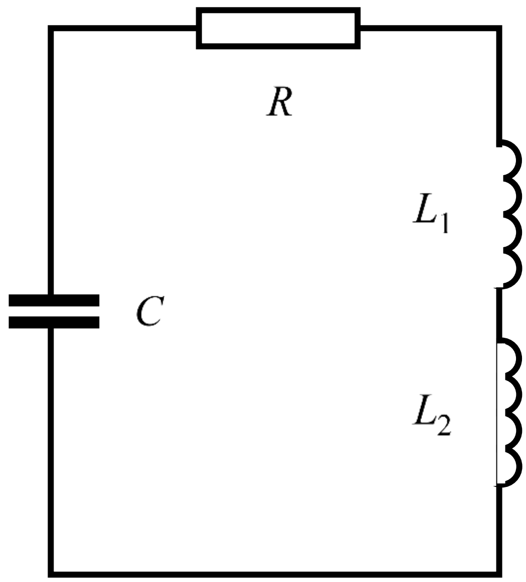

Figure 6.10. Electric circuit # 1 to problem 6.3.9.

The electromotive force in the second solenoid is

$$E_2 = -L_2 \frac{dI}{dt}, \tag{3}$$

where L_2 is the inductance of the second solenoid.

The resulting electromotive force in the oscillatory circuit is given by

$$E = E_1 + E_2 = -(L_1 + L_2)\frac{dI}{dt} = -L\frac{dI}{dt}, \tag{4}$$

where L is the inductance of the entire resonant circuit.

By analogy with formula (1), for the period T_1 of free oscillations in an oscillatory circuit, for the case of a series connection of two solenoids, we can write the equation

$$T_1 = \frac{2\pi}{\sqrt{\left[\frac{1}{(L_1 + L_2)C}\right] - \left[\frac{R^2}{4(L_1 + L_2)^2}\right]}}. \qquad (5)$$

Substituting known values, we have

$$T_1 = 1.68 \cdot 10^{-3}\ s. \qquad (6)$$

Consider a parallel connection of two solenoids (see Fig. 6.11). We will assume that the voltage between points A and B changes over time t according to the law

$$U = U_0\, sin(\omega t), \qquad (7)$$

where
U_0 is the amplitude value of the voltage;
ω is the circular frequency of voltage change.

In this case, the amplitude of the current I_1 in the first solenoid is

$$I_{01} = \frac{U_0}{\omega L_1}. \qquad (8)$$

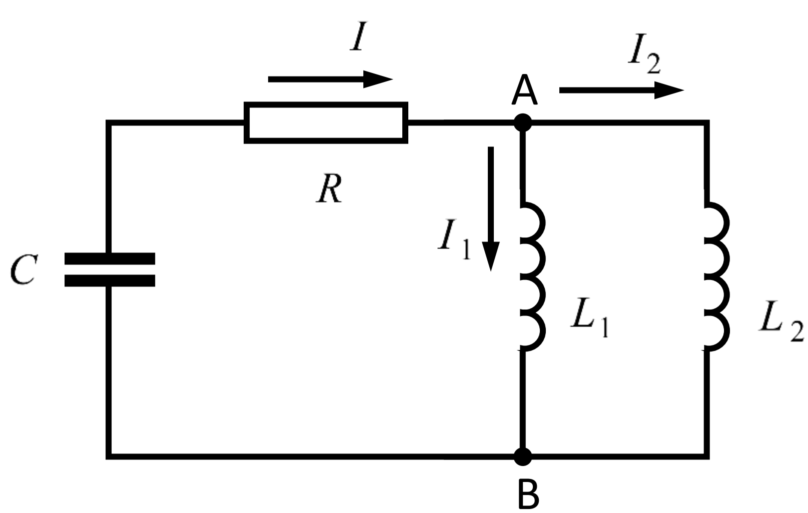

Figure. 6.11. Electric circuit # 2 to problem 6.3.9.

The current amplitude in the second solenoid is

$$I_{02} = \frac{U_0}{\omega L_2}. \qquad (9)$$

The phase shift between current I_1 and voltage U is

$$tg\varphi_1 = \frac{\omega L_1}{R_1}, \qquad (10)$$

where R_1 is the active resistance of the first solenoid.

The phase shift between current I_2 and voltage U is

$$tg\varphi_2 = \frac{\omega L_2}{R_2}, \qquad (11)$$

where R_2 is the active resistance of the first solenoid.

According to the condition of the problem, the active resistances of both solenoids are very small, therefore

$$\varphi_1 \approx \varphi_2 = \frac{\pi}{2}. \qquad (12)$$

The dependence of the current I_1 on time has the form

$$I_1 = \frac{U_0}{\omega L_1} sin\left(\omega t - \frac{\pi}{2}\right) = -\frac{U_0}{\omega L_1} cos(\omega t). \qquad (13)$$

The dependence of the current I_2 on time has the form

$$I_2 = \frac{U_0}{\omega L_2} sin\left(\omega t - \frac{\pi}{2}\right) = -\frac{U_0}{\omega L_2} cos(\omega t). \qquad (13)$$

The current in the unbranched part of the oscillatory circuit is equal to the sum of the currents I_1 and I_2

$$I = I_1 + I_2 = -\frac{U_0}{\omega}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)cos(\omega t). \tag{14}$$

An analysis of the last equation shows that the same time dependence can be obtained if one solenoid with equivalent inductance is included in the oscillatory circuit

$$L = \frac{1}{\left(\frac{1}{L_1} + \frac{1}{L_2}\right)} = \frac{L_1 L_2}{L_1 + L_2}. \tag{15}$$

Now we can substitute the expression for the equivalent inductance into equation (1) and obtain a formula for the period T_2 of free oscillations in an oscillatory circuit with two solenoids connected in parallel

$$T_2 = \frac{2\pi}{\sqrt{\left[\frac{(L_1 + L_2)}{L_1 L_2 C}\right] - \left[\frac{(L_1 + L_2)^2 R^2}{4 L_1^2 L_2^2}\right]}}. \tag{16}$$

Substituting known values, we get

$$T_2 = 1.45 \cdot 10^{-3}\ s. \tag{17}$$

Answer. The period of free oscillations in the oscillatory circuit when two solenoids are connected in series is $T_1 = 1.68 \cdot 10^{-3}\ s$. The period of free oscillations in the oscillatory circuit with a parallel connection of two solenoids is $T_2 = 1.45 \cdot 10^{-3}\ s$.

Problem 6.3.10

Problem description. In an oscillatory circuit, the inductance of which is $L = 0.01\ H$, the charge of the capacitor decreases by 10 times over a period of $T = 10^{-5}\ s$. Determine the resistance of the oscillatory circuit.

Known quantities: $L = 0.01\ H$, $Q / Q_1 = 10$, $T = 10^{-5}\ s$.

<u>Quantities to be calculated</u>: R.

<u>Problem solution</u>. The charge of the capacitor at any time t is determined by the relation

$$Q = Aexp\left(-\frac{R}{2L}t\right)cos(\omega t - \varphi), \tag{1}$$

there
A is a constant;
R is the active resistance of the oscillating circuit;
ω is the circular frequency of the oscillating circuit;
L is the inductance;
φ is the phase shift.

The charge of the capacitor at time $t + T$ is

$$Q = Aexp\left[-\frac{R}{2L}(t+T)\right]cos[\omega(t+T) - \varphi], \tag{2}$$

where
T is the period of the oscillatory circuit.

Divide equation (2) by equation (1)

$$\frac{Q}{Q_1} = exp\left(\frac{R}{2L}T\right) \cdot \frac{cos(\omega t - \varphi)}{cos[\omega(t+T) - \varphi]}. \tag{3}$$

The relationship between the circular frequency and the period of the oscillatory circuit has the form

$$\omega = \frac{2\pi}{T}. \tag{4}$$

Now we can rewrite equation (3), given that the period of the cosine is 2π

$$\frac{Q}{Q_1} = exp\left(\frac{RT}{2L}\right). \tag{5}$$

Therefore, the resistance of the oscillatory circuit is

$$R = \frac{2L}{T} ln\left(\frac{Q}{Q_1}\right). \qquad (6)$$

Numerically

$$R = 4.605 \cdot 10^3 \ \Omega. \qquad (7)$$

Answer. The resistance of the oscillatory circuit is $R = 4.605 \cdot 10^3 \ \Omega$.

Problem 6.3.11

Problem description. The oscillatory circuit consists of a capacitor with an electrical capacity of $C = (1/3) \cdot 10^{-8} \ F$ and a solenoid with an inductance of $L = 1.2 \cdot 10^{-5} \ H$. Solenoid resistance is $R = 6 \ \Omega$. Determine the period of free oscillations, the attenuation coefficient and the logarithmic decrement.

Known quantities: $C = (1/3) \cdot 10^{-8} \ F$, $L = 1.2 \cdot 10^{-5} \ H$, $R = 6 \ \Omega$.

Quantities to be calculated: T, β, λ.

Problem solution. The oscillation period of the electromagnetic circuit can be determined using the following relationship

$$T = \frac{2\pi}{\sqrt{\left(\frac{1}{LC}\right) - \left(\frac{R}{2L}\right)^2}}, \qquad (1)$$

where
L is the inductance of the solenoid;
C is the capacitance of the capacitor;
R is the ohmic resistance.

Substituting known values, we get

$$T = 1.48 \cdot 10^{-6}\ s. \quad (2)$$

The attenuation coefficient is determined by the relation

$$\beta = \frac{R}{2L}. \quad (3)$$

Substituting known values, we find

$$\beta = 2.5 \cdot 10^{6}\ s^{-1}. \quad (4)$$

The logarithmic decrement characterizes the decrease in the oscillation amplitude in one period

$$\lambda = ln\left(\frac{A_n}{A_{n+1}}\right) = ln\left\{\frac{A_0\, exp(-\beta t)}{A_0\, exp[-\beta(t+T)]}\right\}, \quad (5)$$

where
A_n and A_{n+1} are amplitudes that differ by one period;
A_0 is the initial amplitude;
t is the current time.
Numerically

$$\lambda = 3.7. \quad (6)$$

Answer. The period of free oscillations is $T = 1.48 \cdot 10^{-6}\ s$. The attenuation coefficient is $\beta = 2.5 \cdot 10^{6}\ s^{-1}$. The logarithmic decrement is $\lambda = 3.7$.

Problem 6.3.12

Problem description. The oscillatory circuit of the receiver is tuned to a wavelength $\lambda = 500\ m$. The inductance and active resistance of the oscillatory circuit are equal, respectively $L = 2 \cdot 10^{-6}\ H$ and $R = 10\ \Omega$. Determine the natural frequency of oscillation of the circuit.

Known quantities: $\lambda = 500\ m$, $L = 2 \cdot 10^{-6}\ H$, $R = 10\ \Omega$.

Quantities to be calculated: ν_0.

Problem solution. The resonant period of the circuit oscillations is determined by the Thomson formula

$$T = 2\pi\sqrt{LC}\,, \tag{1}$$

where
L is the inductance of the oscillating circuit;
C is the capacitance of the oscillating circuit.

The natural oscillation frequency of the circuit is

$$\nu_0 = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC}\right) - \left(\frac{R}{2L}\right)^2}\,, \tag{2}$$

where R is the ohmic resistance of the oscillating circuit.

The relationship between the length of the electromagnetic wave and the oscillation period of the circuit has the form

$$\lambda = cT\,, \tag{3}$$

where
λ is the wavelength of the electromagnetic wave;
C is the speed of light in a vacuum.

Let us rewrite equation (3) taking into account the Thomson formula

$$\lambda = 2\pi c\sqrt{LC}\,. \tag{4}$$

Now we can write a mathematical expression for the natural frequency of oscillation of the circuit, taking into account equations (2) and (4)

$$\nu_0 = \frac{1}{2\pi}\sqrt{\left(\frac{4\pi^2 c^2}{\lambda^2}\right) - \left(\frac{R^2}{4L^2}\right)}\,. \tag{5}$$

Substituting known values, we get

$$\nu_0 = 4.5 \cdot 10^5 \ Hz. \quad (6)$$

Answer. The frequency of natural oscillations of the circuit is $\nu_0 = 4.5 \cdot 10^5 \ Hz$.

6.4. Level 1 problems

6.4.1. A coil with an inductance of $L = 1 \ mH$ and an air capacitor, consisting of two round plates with a diameter of $D = 20 \ cm$ each, are connected in parallel. The distance between the plates is $D = 1 \ cm$. Determine the period of electromagnetic oscillations.

6.4.2. Capacitor with capacitance $C = 500 \ pF$ is connected in parallel with a solenoid with length $l = 40 \ cm$ and cross-sectional area $S = 5 \ cm^2$. The number of turns of the coil is $N = 1000$. The core is made of non-magnetic material. Determine the period of electromagnetic oscillations of such a circuit.

6.4.3. The oscillatory circuit consists of a solenoid with an inductance of $L = 20 \ \mu H$ and a capacitor with an electrical capacity of $C = 80 \ nF$. The value of the electrical capacity may deviate from the specified value by 2%. Calculate the change in the wavelength at which the given oscillatory circuit resonates.

6.4.4. The oscillatory circuit consists of a solenoid with an inductance of $L = 1.6 \ mH$ and a capacitor with an electric capacitance of $C = 0.04 \ \mu F$. The maximum voltage across the capacitor plates is $U = 200 \ V$. Determine the maximum current in the circuit. The resistance of the electromagnetic circuit can be neglected.

6.4.5. The oscillatory circuit consists of a capacitor with an electrical capacity of $C = 8 \ pF$ and a solenoid with an inductance of $L = 0.5 \ mH$. The maximum current in the oscillatory circuit is $I_m = 40 \ mA$. Determine the maximum voltage across the capacitor plates.

6.4.6. The length and cross-sectional area of the solenoid are equal, respectively $l = 50 \ cm$ and $S_1 = 3 cm^2$. The number of turns of the solenoid is $N = 1000$. The solenoid has no core and is connected in parallel with a capacitor. The area of each capacitor plate is $S_2 = 75 \ cm^2$. The distance between the plates is $d = 5 \ cm$. The dielectric located between the plates of the capacitor is the air medium. Determine the period of oscillation of the circuit.

6.4.7. The oscillatory circuit consists of a capacitor with electric capacitance $C = 1\ \mu F$ and a solenoid with inductance $L = 1\ mH$ connected in parallel. The resistance of the electromagnetic circuit is negligible. Determine the frequency of electromagnetic oscillations of this circui.

6.4.8. The inductance of the oscillating circuit is $L = 0.5\ mH$. The oscillatory circuit resonates at a wavelength of $\lambda = 300\ m$. Determine the capacitance of the capacitor.

6.4.9. The oscillatory circuit consists of a solenoid with an inductance of $L = 4\ \mu H$ and a capacitor with an electric capacitance of $C = 1.11\ nF$. Determine the wavelength at which the oscillatory circuit will resonate.

6.4.10. To demonstrate Hertz's experiments with the refraction of electromagnetic waves, a large prism made of paraffin is sometimes used. The dielectric constant and relative magnetic permeability of paraffin are equal, respectively $\varepsilon = 2$ and $\mu = 1$. Determine the refractive index of paraffin.

6.4.11. Two parallel wires immersed in glycerine are inductively connected to an electromagnetic oscillation generator with a frequency of $\nu = 420\ MHz$. The distance between the antinodes of standing waves on the wires is $l = 7\ cm$. The relative magnetic permeability of glycerol is approximately equal to unity. Determine the relative permittivity of glycerol.

6.4.12. The oscillatory circuit consists of a solenoid and a capacitor. The inductance of the solenoid is $L = 3 \cdot 10^{-5}\ H$. The area of each plate of a flat capacitor is $S = 100\ cm^2$. The distance between the plates of the capacitor is $d = 0.1\ mm$. The oscillatory circuit resonates at a wavelength of $\lambda = 750\ m$. Determine the relative permittivity of the medium that fills the space between the capacitor plates.

6.4.13. The oscillatory circuit consists of a capacitor with an electrical capacity of $C = 0.2\ \mu F$ and a solenoid with an inductance of $L = 5.07 \cdot 10^{-3}\ H$. The potential difference across the plates of the capacitor decreases three times in time $\tau = 10^{-3}\ s$. Determine the logarithmic decrement of the oscillatory circuit.

6.4.14. The oscillatory circuit consists of a capacitor and a long coil wound of copper wire with a cross-sectional area of $S = 0.1\,mm^2$. The length of the solenoid is $l = 40\,cm$. The error that occurs when calculating the oscillation period using the approximate formula $T = 2\pi\sqrt{LC}$, is $\varepsilon = 1\,\%$. Determine the capacitance of the capacitor.

6.4.15. Two capacitors with capacitances $C_1 = 0.2\,\mu F$ and $C_2 = 0.1\,\mu F$ are connected in series in an alternating current circuit with a voltage of $U = 220\,V$ and a frequency of $\nu = 50\,Hz$. Determine the current in the electrical circuit and the voltage on the first and second capacitors.

6.4.16. The capacitor and the light bulb are connected in series and are included in an alternating current circuit with a voltage of $U = 440\,V$ and a frequency of $\nu = 50\,Hz$. The current flowing through the bulb is $I = 0.5\,A$. The potential difference across the light bulb is $U = 110\,V$. Determine the capacitance of the capacitor.

6.4.17. A solenoid with active resistance $R = 10\,\Omega$ is connected to an alternating current circuit with a voltage of $U = 127\,V$ and a frequency of $\nu = 50\,Hz$. The solenoid absorbs power equal to $P = 400\,W$. The phase shift between voltage and current is $\varphi = 60\,°$. Determine the inductance of the solenoid.

6.4.18. Active resistance, solenoid and capacitor are included in an alternating current circuit with a voltage of $U = 220\,V$ and a frequency of $\nu = 50\,Hz$. The capacitance of the capacitor is $C = 35.4\,\mu F$. The inductance of the solenoid is $L = 0.7\,H$. The active resistance is $R = 100\,\Omega$. Calculate the current in the electrical circuit, as well as the voltage on the capacitor plates, solenoid and active resistance.

6.4.19. The active resistance and the solenoid are connected in parallel and connected to an alternating current circuit with a voltage of $U = 127\,V$ and a frequency of $\nu = 50\,Hz$. The power absorbed in the electrical circuit is $P = 404\,W$. The phase shift between voltage and current is $\varphi = 60\,°$. Calculate the values of active resistance and inductance.

6.4.20. The capacitor, solenoid and active resistance are connected in series in an alternating current circuit with a voltage of $U = 220\,V$. The voltage across the capacitor plates is twice as much as the voltage across the resistance. The voltage across the solenoid is three times greater than the voltage across the resistance. Determine the voltage across the active resistance.

6.5. Answers to problems

6.4.1. $T = 3.32 \cdot 10^{-8}\ s$.

6.4.2. $T = 5.57 \cdot 10^{-6}\ s$.

6.4.3. $\lambda = \left(2.38 \cdot 10^{3} \pm 2.38 \cdot 10^{1}\right) m$.

6.4.4. $I_m = 1\ A$.

6.4.5. $U_m = 3.17 \cdot 10^{2}\ V$.

6.4.6. $T = 6.28 \cdot 10^{-7}\ s$.

6.4.7. $\nu = 5.05 \cdot 10^{3}\ Hz$.

6.4.8. $C = 5.1 \cdot 10^{-11}\ F$.

6.4.9. $\lambda = 1.26 \cdot 10^{2}\ m$.

6.4.10. $n = 1.4$.

6.4.11. $\varepsilon = 2.6 \cdot 10^{1}$.

6.4.12. $\varepsilon = 6$.

6.4.13. $\kappa = 0.22$.

6.4.14. $C = 7 \cdot 10^{-7}\ F$.

6.4.15. $I = 4.6 \cdot 10^{-3}\ A$; $U_1 = 7.34 \cdot 10^{1}\ V$; $U_2 = 1.466 \cdot 10^{2}\ V$.

6.4.16. $C = 3.74 \cdot 10^{-6}\ F$.

6.4.17. $L = 5.5 \cdot 10^{-2}\ H$.

6.4.18. $I = 1.34\ A$; $U_C = 1.21 \cdot 10^{2}\ V$; $U_R = 1.34 \cdot 10^{2}\ V$; $U_L = 2.95 \cdot 10^{2}\ V$.

6.4.19. $R = 4 \cdot 10^{1}\ \Omega$; $L = 7.4 \cdot 10^{-2}\ H$.

6.4.20. $U_R = 1.56 \cdot 10^{2}\ V$.

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APPENDICES

Table A1. Greek alphabet

Name	Capital	Lower-case	Name	Capital	Lower-case
Alpha	A	α	Nu	N	ν
Beta	B	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	O	o
Delta	Δ	δ	Pi	Π	π
Epsilon	E	ε	Rho	P	ρ
Zeta	Z	ζ	Sigma	Σ	σ
Eta	H	η	Tau	T	τ
Theta	Θ	θ	Upsilon	ϒ	υ
Iota	I	ι	Phi	Φ	ϕ
Kappa	K	κ	Chi	X	χ
Lambda	Λ	λ	Psi	Ψ	ψ
Mu	M	μ	Omega	Ω	ω

Table A2. SI prefixes

Prefix		**Representation**	**Prefix**		**Representation**
Name	***Symbol***	***Base 10***	***Name***	***Symbol***	***Base 10***
yotta	Y	10^{24}	deci	d	10^{-1}
zeta	Z	10^{21}	centi	c	10^{-2}
exa	E	10^{18}	milli	m	10^{-3}
peta	P	10^{15}	micro	μ or u	10^{-6}
tera	T	10^{12}	nano	n	10^{-9}
giga	G	10^{9}	pico	p	10^{-12}

Name	*Symbol*	*Base 10*	*Name*	*Symbol*	*Base 10*
mega	M	10^{6}	femto	f	10^{-15}
kilo	k	10^{3}	atto	a	10^{-18}
hecto	h	10^{2}	zepto	z	10^{-21}
deca	da	10^{1}	yocto	y	10^{-24}

Table A3. SI base units

Unit name	**Unit symbol**	**Quantity name**	**Definition**
metre	m	length	The distance travelled by light in vacuum in 1/299792458 second.
kilogram	kg	mass	The kilogram is defined by taking the fixed numerical value of the Plank constant h to be 6.62607015×10^{-34} when expressed in the unit J×s, which is equal to $kg\times m^{2}\times s^{-1}$, where the metre and the second are defined in terms of c and $\Delta\nu_{Cs}$.
second	s	time	The second is define by taking the fixed numerical value of the caesium frequency $\Delta\nu_{Cs}$, the unperturbed ground-state hyperfine transition frequency of the ^{133}C atom, to be 9192631770 when expressed in the unit Hz, which is equal to s^{-1}.

Unit name	Unit symbol	Quantity name	Definition
ampere	A	electric current	The ampere is defined by taking the fixed numerical value of the elementary charge e to be $1.602176634\times10^{-19}$ when expressed in unit C, which is equal to A×s, where the second is defined in terms of $\Delta\nu_{Cs}$.
kelvin	K	thermodynamic temperature	The kelvin is defined by taking the fixed numerical value of the Boltzmann constant k to be 1.380649×10^{-23} $J\times K^{-1}$ ($J=kg\times m^2\times s^{-2}$), given the definition of the kilogram, the metre, and the second.
mole	mol	amount of substance	The amount of substance of exactly 6.02214076×10^{23} elementary entities. This number is the fixed numerical value of the Avogadro constant, N_A, when expressed in the unit mol^{-1} and is called the Avogadro number.
candela	cd	luminous intensity	The luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 5.4×10^{14} Hz and that has a radiant intensity in that direction of 1/683 watt per steradian.

Table A4. SI derived units

Unit name	Unit symbol	Unit Equivalents	Quantity name
hertz	Hz	s^{-1}	frequency
radian	rad	One radian is the angle subtended at the center of a circle by an arc that is equal in length to the radius of the circle.	angle

Unit name	Unit symbol	Unit Equivalents	Quantity name
steradian	sr	The solid angle subtended at the center of a unit sphere by a unit area on its surface	solid angle
newton	N	$kg \times m \times s^{-2}$	force, weight
pascal	Pa	$N/m^2 = kg \times m^{-1} \times s^{-2}$	pressure, stress
joule	J	$N \times m = kg \times m^2 \times s^{-2}$	energy, work, heat
watt	W	$J/s = kg \times m^2 \times s^{-3}$	power, radiant flux
coulomb	C	$A \times s$	electric charge
volt	V	$J/C = kg \times m^2 \times s^{-3} \times A^{-1}$	voltage, electromotive force
farad	F	$C/V = A^2 \times s^4 \times kg^{-1} \times m^{-2}$	electrical capacitance
ohm	Ω or Ohm	$V/A = kg \times m^2 \times s^{-3} \times A^{-2}$	electrical resistance, impedance
siemens	S	$1/Ohm = A^2 \times s^3 \times kg^{-1} \times m^{-2}$	electrical conductance
weber	Wb	$V \times s = kg \times m^2 \times s^{-2} \times A^{-1}$	magnetic flux
tesla	T	$Wb/m^2 = kg \times s^{-2} \times A^{-1}$	magnetic field strength
henry	H	$Wb/A = kg \times m^2 \times s^{-2} \times A^{-2}$	electrical inductance
degree Celsius	°C	K	temperature relative to 273.15 K
lumen	lm	$cd \times sr = cd$	luminous flux
lux	lx	$lm/m^2 = cd \times m^{-2}$	illuminance
becquerel	Bq	s^{-1}	radioactivity
gray	Gy	$J/kg = m^2 \times s^{-2}$	absorbed dose
sievert	Sv	$J/kg = m^2 \times s^{-2}$	equivalent dose
katal	kat	$mol/s = mol \times s^{-1}$	catalytic activity

Table A5. Physical constants

Quantity	Symbol	Value
Avogadro constant	N_A	6.0221415(10)×10^{23} mol^{-1}
Boltzmann constant	k	1.3806505(24)×10^{-23} J / K
Electric constant	ε_0	8.854187817×10^{-12} F×m^{-1}
Faraday constant	F	96485.3383(83) C×mol^{-1}
Gravitational constant	G	6.6742(10)×10^{-11} N×m^2 / kg^2
Magnetic constant	μ_0	4π×10^{-7} T×m /A (exact)
Molar gas constant	R	8.314472(15) J/(mol×K)
Planck constant	h	6.6260693(11)×10^{-34} J×s
Rydberg constant	R_H	1.0973731568525(73)×10^7 m^{-1}
Stefan-Boltzmann constant	σ	5.670400(40)×10^{-8} W×m^{-2}×K^{-4}
Wien displacement law constant	b	2.8977685(51)×10^{-3} m×K
Atomic mass unit	u	1.66053886(28)×10^{-27} kg
Electron mass	m_e	9.1093826(16)×10^{-31} kg
Neutron mass	m_n	1.67492728(29)×10^{-27} kg
Proton mass	m_p	1.67262171(29)×10^{-27} kg
Elementary charge	e	1.60217653(14)×10^{-19} C
Speed of light in vacuum	c	2.99792458×10^8 m /s
Bohr magnetron	μ_B	9.27400949(80)×10^{-24} J/T
Bohr radius	a_0	5.291772108(18)×10^{-11} m
Compton wavelength	λ_C	2.426310238(16)×10^{-12} m

Table A6. Astronomical data

Body	**Mass, kg**	**Equatorial radius, m**	**Perihelion/ Aphelion, m**	**Sidereal period**	**Orbital speed, km/s**
Sun	1.998×10^{30}	6.955×10^{8}	2.5×10^{20}(*)	2.3×10^{8} y(*)	2.2×10^{2}(*)
Moon	7.342×10^{22}	1.738×10^{6}	$(3.63/4.05)\times10^{8}$	27.321661 d	1.002
Mercury	3.301×10^{23}	2.440×10^{6}	$(4.60/6.98)\times10^{10}$	87.9691 d	47.362
Venus	4.867×10^{24}	6.052×10^{6}	$(1.08/1.09)\times10^{11}$	224.698 d	35.02
Earth	5.973×10^{24}	6.378×10^{6}	$(1.47/1.52)\times10^{11}$	365.25636 d	29.783
Mars	6.417×10^{23}	3.396×10^{6}	$(2.07/2.49)\times10^{11}$	686.971 d	24.007
Jupiter	1.898×10^{27}	7.149×10^{7}	$(7.40/7.78)\times10^{11}$	11.862 y	13.07
Saturn	5.683×10^{26}	6.027×10^{7}	$(1.35/1.51)\times10^{12}$	29.4571 y	9.68
Uranus	8.683×10^{25}	2.556×10^{7}	$(2.75/3.00)\times10^{12}$	84.01 y	6.81
Neptune	1.024×10^{26}	2.476×10^{7}	$(4.45/4.55)\times10^{12}$	164.79 y	5.4349

(*) – Milky Way

Table A7. Periodic table of elements

Name	$_{AN}$**Symbol** (*AN – atomic number*)	**Standard atomic weight**	**Name**	$_{AN}$**Symbol** (*AN – atomic number*)	**Standard atomic weight**
1	***2***	***3***	***1***	***2***	***3***
Actinium	$_{89}$Ac	227	Californium	$_{98}$Cf	251
Aluminium	$_{13}$Al	26.9815384	Carbon	$_{6}$C	12.011
Americium	$_{95}$Am	243	Caesium	$_{55}$Cs	132.905452
Antimony	$_{51}$Sb	121.760	Cerium	$_{58}$Ce	140.116

1	*2*	*3*	*1*	*2*	*3*
Argon	$_{18}Ar$	39.948	Chlorine	$_{17}Cl$	35.45
Arsenic	$_{33}As$	74.921595	Chromium	$_{24}Cr$	51.9961
Astatine	$_{85}At$	210	Cobalt	$_{27}Co$	58.933194
Barium	$_{56}Ba$	137.327	Copernicium	$_{112}Cn$	285
Berkelium	$_{97}Bk$	247	Copper	$_{29}Cu$	63.546
Beryllium	$_{4}Be$	9.0121831	Curium	$_{96}Cm$	247
Bismuth	$_{83}Bi$	208.98040	Darmstadtium	$_{110}Ds$	281
Bohrium	$_{107}Bh$	270	Dubnium	$_{105}Db$	268
Boron	$_{5}B$	10.81	Dysprosium	$_{66}Dy$	162.500
Bromine	$_{35}Br$	79.904	Einsteinium	$_{99}Es$	252
Cadmium	$_{48}Cd$	112.414	Erbium	$_{68}Er$	167.259
Calcium	$_{20}Ca$	40.078	Europium	$_{63}Eu$	151.964
Fermium	$_{100}Fm$	257	Phosphorus	$_{15}P$	30.9737620
Flerovium	$_{114}Fl$	289	Platinum	$_{78}Pt$	195.084
Fluorine	$_{9}F$	18.9984032	Plutonium	$_{94}Pu$	244
Francium	$_{87}Fr$	223	Polonium	$_{84}Po$	209
Gadolinium	$_{64}Gd$	157.25	Potassium	$_{19}K$	39.0983
Gallium	$_{31}Ga$	69.723	Praseodymium	$_{59}Pr$	140.90766
Germanium	$_{32}Ge$	72.630	Promethium	$_{61}Pm$	145
Gold	$_{79}Au$	196.966570	Protactinium	$_{91}Pa$	231.03588
Hafnium	$_{72}Hf$	178.49	Radium	$_{88}Ra$	226
Hassium	$_{108}Hs$	270	Radon	$_{86}Rn$	222
Helium	$_{2}He$	4.002602	Rhenium	$_{75}Re$	186.207
Holmium	$_{67}Ho$	164.930328	Rhodium	$_{45}Rh$	102.90549
Hydrogen	$_{1}H$	1.008	Roentgenium	$_{111}Rg$	282
Indium	$_{49}In$	114.818	Rubidium	$_{37}Rb$	85.4678

1	2	3	1	2	3
Iodine	$_{53}I$	126.90447	Ruthenium	$_{44}Ru$	101.07
Iridium	$_{77}Ir$	192.217	Rutherfordium	$_{104}Rf$	267
Iron	$_{26}Fe$	55.845	Samarium	$_{62}Sm$	150.36
Krypton	$_{36}Kr$	83.798	Scandium	$_{21}Sc$	44.955908
Lanthanum	$_{57}La$	138.90547	Seaborgium	$_{106}Sg$	269
Lawrencium	$_{103}Lr$	266	Selenium	$_{34}Se$	78.971
Lead	$_{82}Pb$	207.2	Silicon	$_{14}Si$	28.085
Lithium	$_{3}Li$	6.94	Silver	$_{47}Ag$	107.8682
Livermorium	$_{116}Lv$	293	Sodium	$_{11}Na$	22.9897693
Lutetium	$_{71}Lu$	174.9668	Strontium	$_{38}Sr$	87.62
Magnesium	$_{12}Mg$	24.305	Sulfur	$_{16}S$	32.06
Manganese	$_{25}Mn$	54.938043	Tantalum	$_{73}Ta$	180.94788
Meitnerium	$_{109}Mt$	278	Technetium	$_{43}Tc$	98
Mendelevium	$_{101}Md$	258	Tellurium	$_{52}Te$	127.60
Mercury	$_{80}Hg$	200.592	Tennessine	$_{117}Ts$	294
Molybdenum	$_{42}Mo$	95.95	Terbium	$_{65}Tb$	158.925354
Moscovium	$_{115}Mc$	290	Thallium	$_{81}Tl$	204.38
Neodymium	$_{60}Nd$	144.242	Thorium	$_{90}Th$	232.0377
Neon	$_{10}Ne$	20.1797	Thulium	$_{69}Tm$	168.934218
Neptunium	$_{93}Np$	237	Tin	$_{50}Sn$	118.710
Nickel	$_{28}Ni$	58.6934	Titanium	$_{22}Ti$	47.867
Nihonium	$_{113}Nh$	286	Tungsten	$_{74}W$	183.84
Niobium	$_{41}Nb$	92.90637	Uranium	$_{92}U$	238.02891
Nitrogen	$_{7}N$	14.007	Vanadium	$_{23}V$	50.9415
Nobelium	$_{102}No$	259	Xenon	$_{54}Xe$	131.293
Oganesson	$_{118}Og$	294	Ytterbium	$_{70}Yb$	173.045

1	*2*	*3*	*1*	*2*	*3*
Osmium	$_{76}Os$	190.23	Yttrium	$_{39}Y$	88.90584
Oxygen	$_{8}O$	15.999	Zinc	$_{30}Zn$	65.38
Palladium	$_{46}Pd$	106.42	Zirconium	$_{40}Zr$	91.224

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