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GROUP AND PHASE VELOCITIES OF LAMB WAVES

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Abstract. *This study is devoted to the theoretical analysis of the kinetic features of Lamb wave propagation in laminar composites. In particular, the dispersion dependences for the group and phase velocities, as well as the characteristic wave curves, were analyzed. The theoretical analysis was based on the assumption of the existence of multiple Lamb wave modes. Numerical solutions for an infinite number of possible wave modes were obtained based on the three-dimensional (3-D) theory of elasticity. Transcendental equations, which are the governing equations of the exact dispersion relations, were obtained. For symmetric laminates, a numerical calculation technique is proposed based on the possibility of imposing boundary conditions on the middle plane and the upper surface to separate symmetric and antisymmetric wave modes. A new semi-exact method for calculating Lamb wave group velocities in composites is developed. The analysis of the angular dependence of Lamb wave propagation was performed for the characteristic wave curves of the phase and group velocities. A detailed calculation was performed for the dispersive and anisotropic behavior of Lamb waves in two different types of symmetric laminates. The kinetics of Lamb wave propagation suggested the presence of symmetric and antisymmetric wave modes with narrow-band signals. In addition, the Gabor wavelet transform was used to extract the group velocities from the Lamb wave propagation time. The numerical values of group velocities obtained on the basis of the developed technique are in good agreement with data from the three-dimensional elasticity theory.*

Key words: *Lamb wave, laminated composites, dispersion relations, symmetric wave modes, group velocity, phase velocity.*

Introduction.

Modern composite materials are quite often used in a wide variety of industries. There are several reasons for the popularity of composites. These include high specific rigidity, specific strength, and fatigue strength. Fiber-reinforced composite materials, which have high mechanical strength, have naturally anisotropic characteristics, since such materials are made from carbon fibers and a matrix. On the other hand, composite materials have a high probability of developing defects due to complex manufacturing processes, unexpected external influences, and deterioration of mechanical properties during operation [1, 2]. An effective technique for performing inspection of large structures, including wave propagation over several meters depending on the material and geometry of the structure in question, is Lamb wave testing [3, 4]. High-speed monitoring of large structures such as pipes and trusses is not possible without the use of the basic equation for Lamb wave packets. It should be noted that the experimental ultrasonic test tests a local part of the structure containing composite inserts and elements only in the immediate vicinity of the ultrasonic wave sources. Therefore, such a technique requires the placement of wave sources over the entire surface of a large structure to perform a structural test, and the results are usually displayed as a C-scan [5, 6]. This testing process can be laborious and time-consuming. The advantage of using Lamb waves is the ability to simultaneously monitor large surfaces of structures. However, such methods are more complex than conventional ultrasonic testing, since Lamb waves have dispersive characteristics, i.e. the wave speed varies depending on the frequency, modes and plate thickness [7, 8]. Experimentally measured group and phase velocities of symmetric and antisymmetric Lamb waves in composite materials with anisotropic characteristics do not coincide with theoretical values, in particular, group velocities, which are calculated using the Lamb wave dispersion equation [9]. The reason for the existence of such a difference is the presence of a non-zero angle between the direction of the group velocity and the direction of the phase velocity in anisotropic materials. The standard technique involves calculating the values of group velocities of symmetric modes from the dispersion curves of the Lamb wave in an anisotropic plate of a laminar composite without taking into account the direction and magnitude of propagation of the wave packet.

This study analyzes the difference between the experimentally measured Lamb wave group velocities and the theoretical group velocities in laminar composite materials. In isotropic materials, Lamb waves can propagate in a circular pattern in the same direction of the wave vector and the direction of the energy flow. However, the direction of the wave vector in a composite plate having anisotropic characteristics due to the fiber arrangement is different from the direction of the energy flow except for the main axis [10]. It is necessary that wave propagation in composite materials with anisotropic characteristics be considered not only in terms of direction but also in terms of amplitude. When taking into account the difference in amplitudes, it is necessary to take into account that the fundamental symmetric mode S0 is characterized by reduced dispersion values for small values of the frequency multiplied by the thickness value.

Experiments on the analysis of the S0 mode in unidirectional laminated composite plates revealed a difference in the directions between the group velocity and the phase velocity. In addition, it was found that the direction of the energy flow coincides with the direction of the fibers of the laminated composites, with the exception of the perpendicular direction of the fibers. Theoretical analysis showed that the symmetric mode S0 of the group velocity for low frequencies can be used as a phase velocity in the region of low dispersions. The numerical values of the group velocities obtained as a result of processing the diagrams for the reciprocal values of the phase velocities are in good agreement with the measured values of the group velocity in unidirectional composite plates. In this study, the S0 mode dispersion properties of phase velocity are analyzed using Lamb wave equations for unidirectional, bidirectional and quasi-isotropic composite plates. The characteristic parameters of phase velocity dispersion are used to calculate the theoretical group velocity in the volume of the laminar composite. The retardation surface, which is related to the inverse of the phase velocity, is used to calculate the magnitude and direction of the group velocity. A rescaled group velocity curve, i.e., a curve from which the magnitude and direction of the group velocity are re-modified from the retardation surface, is obtained. The difference between the measured group velocities and the calculated group velocity from the dispersion curves is due to the direction and magnitude of the wave vector and the direction of the energy flow.

Materials and results

The numerical calculation methodology for Lamb wave propagation in laminated composite structures assumes that the interfaces between layers are ideally coupled. For each layer, the displacement components in the corresponding z-axis equation must be modified into exponential forms to account for the inhomogeneity of the multilayer laminate

$$U = A\exp(i\xi x), \qquad V = B\exp(i\xi y), \qquad W = -iC\exp(i\xi z), \tag{1}$$

where

U is the displacement in the x direction (control coefficient *A*);

V is the displacement in the y direction (control coefficient *B*);

W is the displacement in the z direction (control coefficient -*iC*);

z is the coordinate perpendicular to the fixed layer of the composite;

ξ is the fixed variable.

The general solution in each lamina is

$$\{U,V,W\} = \exp\{i\,[(k_x x + k_y y) - \omega t]\}\cdot \sum_j A_j \{1, R_j, S_j\} \exp(i\xi_j z), \tag{2}$$

where

ω is the angular frequency;

k_x, k_y are the projections of the wave vector onto the Cartesian x and y axes;

R, S are the characteristic coefficients of the layer.

Symmetrical and asymmetrical Lamb wave modes in conventional laminates cannot be separated. It should be noted that symmetrical laminates are used in engineering practice when designing composite structures. A reliable method for separating the two types of wave modes is to generate boundary conditions on both the upper and middle planes of the surface. For the upper boundary of the laminate, the boundary conditions can be written as follows

$$\{\sigma_z, \tau_{yz}, \tau_{xz}\}\big|_{z=h/2} = 0, \tag{3}$$

where

σ is the normal mechanical stress;

τ is the tangential mechanical stress;

h is the layer thickness.

The symmetry conditions for the entire laminate allow only half of the entire sample to be analyzed. In a subsequent step, the following conditions are imposed on the stress and displacement components in the mid-plane for symmetric modes

$$\{u, \upsilon, \sigma_z\}\big|_{z=0} = 0, \tag{4}$$

The implicit functional form $G(\omega, \boldsymbol{k}) = 0$, or $G(\omega, k, \theta) = 0$ can be used to formulate the dispersion relation between ω and k. This dispersion relation can be explicitly solved in the form of real roots $\omega = W(\boldsymbol{k})$, or $\omega = W(k, \theta)$.

The number of possible solutions with different functions W tends to infinity. Such solutions correspond to different wave modes. The phase velocity vector for plane modes is defined as $c_p = (\omega/k)\cdot(\mathbf{k}/|\mathbf{k}|) = (\omega/k^2)\mathbf{k}$. Therefore, its magnitude is $c_p = \omega/k$. The set of all statistical samples k from the origin for c_p at a given frequency forms the so-called velocity curve. The radius vectors of the velocity curves in the direction of a given k represent the admissible dispersion of the phase velocity of the different wave modes.

A set of points in phase space or a slowness curve can be defined by fixing the slowness vector $\mathbf{s} = \mathbf{k}/\omega$. The characteristics of the set of phase points can be simply formed from the velocity curve by geometric inversion, i.e. by mapping through the inverse radius.

The directions of the slowness and phase velocity vectors coincide. Thus, the inverse phase velocities can be measured from the origin to the slowness curves. The distance traveled per unit time is defined as the phase velocity. On the other hand, time as slowness is numerically equal to the time required to travel a unit distance. For volume (non-dispersive) waves, it is convenient to use the slowness curve. The reason for this is the fact that this group velocity curve does not depend on x.

In isotropic materials, the phase velocity depends only on the magnitude of the wave vector k. The phase velocity of anisotropic materials depends on the wave vector **k**, its magnitude, and the direction in which the wave propagates. For experimentally measured wave packets, the phase velocities are measured by tracking the wave peaks.

The numerical value for the group velocity can be determined by tracking the wave packet envelopes, namely $c_g = \mathrm{grad}_k W = \partial W/\partial k$. Provided that the closed form of the implicit function G has been previously determined, the group velocity can also be calculated as

$$c_g = -\left(\frac{\partial G}{\partial k}\right)\cdot\left(\frac{\partial G}{\partial \omega}\right)^{-1}. \qquad (5)$$

The gradient W ($\mathrm{grad}_k\, W$) in the polar coordinate system has a radial component $\partial W/\partial k$ in the direction $\boldsymbol{k}$ and an angular component $\partial W/k\partial\theta$, perpendicular to $\boldsymbol{k}$. After the coordinate transformation, the group velocity in the Cartesian coordinate system is equal to

$$\begin{Bmatrix} c_{gx} \\ c_{gy} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \dfrac{\partial W}{\partial k} \\ \dfrac{\partial W}{k\partial\theta} \end{Bmatrix}, \qquad (6)$$

where indexes “x” and “y” represent the components in x- and y-axes, respectively.

The magnitude and direction of group velocity represents this system of equations

$$c_g = \sqrt{c_{gx}^2 + c_{gy}^2}, \qquad \theta_g = \tan^{-1}\frac{c_{gy}}{c_{gx}}. \tag{7}$$

The skew angle θ or steering angles defined as

$$\theta = \theta_g - \theta'. \tag{8}$$

It makes sense to introduce the concept of a wave curve (or wave front curve) as the geometric locus of the group (radial) velocity vectors along all choices c_g from the origin at a given frequency. The radius vector connecting the origin (or source point) with a point on the wave curve is numerically equal to the distance traveled by the elastic disturbance per unit time.

In other words, the geometric concept of a wave curve essentially comes down to the concept of a geometric locus of points (or wave front) recorded per unit time by a disturbance emitted by a point source acting through the origin at time $t = 0$. Wave curves are of great importance in detecting mechanical damage.

The dispersion relation written for each Lamb wave mode can be expressed as an explicit function of W (k; h). This function is associated with a conical surface in a three-dimensional domain. In addition, the deceleration curve W (k; θ) = ω_0 is geometrically a level surface of W (k; θ). Differentiating both sides of the equation with respect to θ, the following relation can be obtained

$$\frac{\partial W}{\partial k}\frac{dk}{d\theta} + \frac{\partial W}{\partial \theta} = 0, \tag{9}$$

As an example of the use of numerical and symbolic methods for recording mechanical damage in a composite material, we can consider the results of the Lamb wave propagation analysis in graphite/epoxy resin AS4/3502. Two laminates are used in the tests: L_1 ($[+45_6/45_6]_s$) and L_2 ($[+45/45/0/90]_s$). The numerical results consist of dispersion curves (phase and group velocities) and three characteristic wave curves in two different types of laminates. The characteristic dispersion curves are illustrated in Fig. 1, 2. The dimensionless frequency $f_d = \omega h/c_T$ and the dimensionless velocity υ_{d1} =

c_p/c_T and $\upsilon_{d2} = c_g/c_T$ are used to normalize the physical frequency and velocity, respectively. In addition, c_T is defined as $(G_{12}/\rho)^{0.5}$ is the velocity of the transverse (in-plane shear) wave in the plate.

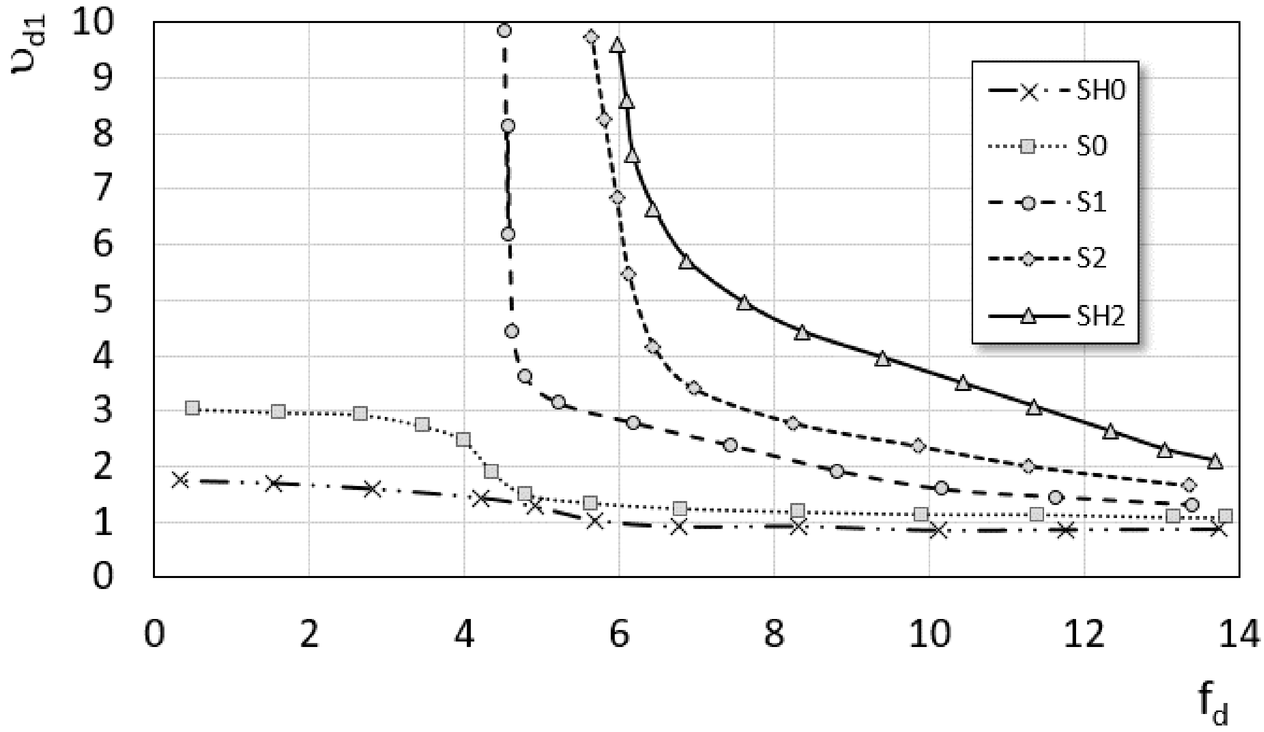

Figure 1. Dispersion curves $\upsilon_{d1} = \upsilon_{d1}$ (f_d) of Lamb waves along $\theta = 45^0$ (symmetric modes)

The group velocity of the SH0 and S0 modes has pronounced dispersion characteristics. However, even greater dispersion is observed for symmetric modes in the quasi-isotropic laminate [+45/45/0/90]s. In contrast, the dispersion of the A0 mode in both laminates is weaker beyond $\omega h/c_T = 1$. This feature is effectively used for structural monitoring of laminar composites.

Summary and conclusions.

The calculation method of the phase and group velocity kinetics of the laminated composite allowed us to obtain a general control equation that relates the mechanical normal and tangential stresses for each layer. The dispersion relations are written for the middle layer of the laminate sample for the case of propagation of both symmetric and antisymmetric modes of the Lamb wave packet. The observed dispersion dependences of the phase velocity were obtained as a result of numerical analysis of the velocity curve. The analysis showed that for non-isotropic composites, the characteristic value of the phase velocity depends both on the amplitude value of the wave vector and on the direction of propagation of the symmetric modes of the Lamb waves.

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