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# THE METHOD OF DISCRETE RIEMANN'S PROBLEM IN SOLVING SOME PROBLEMS OF MATHEMATICAL PHYSICS

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**The method of discrete Riemann's problems is developed to solve periodical non-stationary as well as dynamical problems of engineering. This is illustrated by means of a typical example which is propagation of harmonic heat waves in a periodical system of punches on a half plane. As it is the case in this method, this parabolic type mixed problem is converted to a singular integral equation with Hilbert kernel which is in turn reduced to infinite system of linear algebraic equations. The solution is found in the space $l_p(p>4)$. The error occurring due to the truncation has been estimated.**

The method of discrete Riemann's problems was originally proposed [1] for solving finite mixed steady problems. By gradual modifications and amplifications, it has become of wide applications to solutions of problems in several branches of mathematical physics [2]. Moreover, it has been developed further to solve initial – value problems with mixed boundary conditions [3], [4] this work, the propagation of harmonic heat waves in a periodical system of punches on a half plane is considered. The thermoelastic punch problems occurring in engineering mathematics, and a priori the corresponding problem of heat conductivity, have become of major interest in recent investigations.

The point of departure is the following heat equation.

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{1}{a_T}\frac{\partial T}{\partial t}, \quad (y>0) \tag{1}$$

with mixed boundary conditions

$$T(x,\,0,\,t)=e^{-iwt}f(x), \; if \;\; x\in\Delta_1 \tag{2}$$

$$\frac{\partial T(x, 0, t)}{\partial y} = 0, \; if \;\; x \in \Delta_2 \tag{3}$$

$$\lim_{y \to +\infty} |T(x, y, t)| < +\infty \tag{4}$$

where $\Delta_1 = [-a, a]$, $\Delta_2 = [-\pi, \pi] \setminus \Delta_1$, $a_T$ is the thermal diffusivity. The mixed conditions (2) and (3) are periodically continued over the whole x-axis with a period $2\pi$ as a reflection to the periodicity of the punches throughout the half-plane. For the sake of definiteness we assume $f(x)$ to be an even function. Consequently the problem (1)-(4) is of an even parity in the x-direction.

The mixed boundary conditions (2), (3) can be replaced by the two uniform and compatible ones:

$$T^*(x, 0) = f_-(x) + \phi_+(x), \; \frac{\partial T^*(x, 0)}{\partial y} = \phi_-(x) \tag{5}$$

where

$$\phi_+(x) = \begin{cases} 0 & if \;\; x \in \Delta_1 \\ unknown & if \;\; x \in \Delta_2 \end{cases}$$

$$\phi_-(x) = \begin{cases} unknown & if \;\; x \in \Delta_1 \\ 0 & if \;\; x \in \Delta_2 \end{cases}$$

$$\phi_-(x) = \begin{cases} f(x) & if \;\; x \in \Delta_1 \\ 0 & if \;\; x \in \Delta_2 \end{cases} \tag{6}$$

$$T(x, y, t) = e^{-iwt} T^*(x, y)$$

Applying finite Fourier transform whit respect to x to the boundary conditions (5) and the equation

$$\Delta T^* + \frac{iw}{a_T} T^* = 0$$

we arrive at following discrete Riemann's problem

$$\Phi_{n+} = -\lambda_n^{-1} \Phi_{n-} - F_{n-} \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{7}$$

where

$$\Phi_{n\pm} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_\pm(x) e^{-inx} dx$$

$$F_{n-} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_-(x) e^{-inx} dx$$

$$\lambda_n^2 = n^2 - \frac{iw}{a_T}$$

Multiplying (7) by $n$, we rewrite it in such a form

$$n\Phi_{n+} = -\operatorname{sgn}\left(n + \frac{1}{2}\right)\Phi_{n-} + \Gamma_n \Phi_{n-} - n\Phi_{n-} \qquad (n = \pm 1, \pm 2, \ldots) \quad (8)$$

where $\Gamma_n = \operatorname{sgn}\left(n + \frac{1}{2}\right) - n\lambda_n^{-1}$ and also $|\Gamma_n| = O\left(\frac{1}{n^2}\right)(n \to \infty)$. Additionally, the condition

$$-\sum_{n=-\infty}^{+\infty} \lambda_n^{-1} \Phi_{n-} = f(0) \quad (9)$$

determines the solution of problem (8) equivalent to that of (7).

Performing the inverse Fourier transform

$$-w\Phi_{n\pm} = \sum_{n=-\infty}^{+\infty} \Phi_{n\pm} e^{inx} = \phi_\pm(x) \quad (10)$$

using the formula

$$w^{-1} \operatorname{sgn}\left(n + \frac{1}{2}\right)\Phi_{n-} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\phi_-(t) e^{it}}{e^{it} - e^{ix}} dt \quad (11)$$

which can be found in [1], and taking into consideration that $\phi_+'(x) = 0$ as $x \in \Delta_1$ we reduce the discrete problem (8) to the singular integral equation

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\phi_-(\xi) d\xi}{1 - e^{i(x-\xi)}} = \frac{1}{2\pi} \int_{-a}^{a} \gamma(x - \xi) \phi_-(\xi) d\xi + if'(x), \ (x \in \Delta_1) \quad (12)$$

to find an even solution of equation (11) we represent the kernel of this equation in the form

$$\frac{e^{i\xi}}{e^{i\xi} - e^{ix}} = \frac{1}{2}\left(1 - i\cot\frac{\xi - x}{2}\right) \quad (13)$$

Substituting (13) into (12) and using the convolution formula [1]

$$\frac{1}{2\pi} \int_{-a}^{a} \gamma(x - \xi) \phi_-(\xi) d\xi = \sum_{n=-\infty}^{+\infty} \Gamma_n \Phi_{n-} e^{inx} \quad (14)$$

and taking into account that $\Gamma_0 = 1$, $\Phi_{n-} = \Phi_{-n-}$, $\Gamma_n = -\Gamma_{-n}$ $(n \neq 0)$, we obtain the singular integral equation

$$\frac{1}{2\pi}\int_{-a}^{a}\cot\frac{\xi - x}{2}\phi_{-}(\xi)d\xi = -2\sum_{n=1}^{+\infty}\Gamma_n\Phi_{n-}\sin nx - f'(x) \tag{15}$$

Inverting Hilbert-type integral on the left of equation (15) in the class of integrable functions [2] we reduce equation (15) to the form

$$\phi_{-}(x) = -\frac{1}{\pi X(x)}\sum_{n=1}^{+\infty}\Gamma_n V_n(x)\int_{-a}^{a}\phi_{-}(y)\cos ny dy = g(x) \tag{16}$$

where

$$g(x) = \frac{1}{\pi X(x)}\left[m(x) + a_0\cos\frac{x}{2}\right],\quad X(x) = \sqrt{2(\cos x - \cos a)}$$

$$m(x) = \frac{1}{2\pi}\int_{-a}^{a}\frac{X(\xi)f'(\xi)d\xi}{\sin\frac{\xi - x}{2}},\quad V_n(x) = \frac{1}{2\pi}\int_{-a}^{a}\frac{X(\xi)\sin n\xi d\xi}{\sin\frac{\xi - x}{2}},$$ $a_0$ is a constant.

Calculating integral $V_n(x)$ we express it in the form

$$V_n(x) = \sum_{m=0}^{n}\mu_{n-m}(\cos a)\cos\left(m + \frac{1}{2}\right)x \tag{17}$$

where

$$\mu_k(\cos a) = \frac{P_{k-2}(\cos a) - P_k(\cos a)}{2k-1},\quad (k = 2, 3, \ldots),$$

$$\mu_0(\cos a) = 1,\quad \mu_1(\cos a) = -\cos a,$$

$P_n(\cos a)$ is Legender polynomial defined by the formula

$$P_n(\cos a) = \frac{1}{\pi}\int_{-a}^{a}\frac{\cos\left(n + \frac{1}{2}\right)x dx}{\sqrt{2(\cos x - \cos a)}} \tag{18}$$

Let us transform equation (16) and write it as

$$\phi_{-}(x) = -\frac{1}{X(x)}\left[m(x) + 2\sum_{n=1}^{+\infty}\Gamma_n V_n(x)\Phi_{n-} + a_0\cos\frac{x}{2}\right] \tag{19}$$

In order to define the coefficients $\Phi_{n-}$ we substitute in the formula

$$\Phi_{n-} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_{-}(\xi)\cos n\xi d\xi \tag{20}$$

instead of $\Phi_{-}(\xi)$ its expression in the right hand side of (19). We get the following infinite system of linear algebraic equations for $\Phi_{n-}$

$$\Phi_{n-} = M_n + 2\sum \Gamma_k N_{nk} \Phi_{k-} + a_0 R_n, \quad (n = 0, 1, 2, \ldots), \qquad (21)$$

where

$$M_n = \frac{1}{2\pi} \int_{-a}^{a} \frac{m(x)\cos nx dx}{X(x)}, \quad N_{nk} = \frac{1}{2\pi} \int_{-a}^{a} \frac{V_k(x)\cos nx dx}{X(x)},$$

$$R_n = \frac{1}{2\pi} \int_{-a}^{a} \frac{\cos nx \cos \frac{x}{2} dx}{X(x)}$$

The integrals $N_{nk}$ and $R_n$ can immediately be written down as

$$N_{nk} = \frac{1}{4} \sum_{m=0}^{n} \mu_{k-m}(\cos a)[P_{m-n}(\cos a) + P_{m+n}(\cos a)]$$
$$R_n = \frac{1}{4}[P_n(\cos a) + P_{n+1}(\cos a)] \qquad (22)$$

The infinite system of equation (21) will be solved approximately. Namely we use the method of truncation. We set up function spaces and sequence spaces. Let $f'(x) \in L_r[-a; a]$, where $r > \frac{4}{3}$, then $\phi_-(x) \in L_r[-a; a]$, where $1 < \rho < \frac{4}{3}$, $\Phi_{n-}$ are Fourier coefficients for $\phi_-(x)$ therefore $\Phi_{n-} \in l_p$, where $p = \frac{\rho}{\rho - 1}$ [5]. So we work in the space $l_p (p > 4)$ with the norm

$$\|\Phi\|_{l_p} = \left( \sum_{n=0}^{\infty} |\Phi_{n-}|^p \right)^{\frac{1}{p}}$$

where $\Phi = \{\Phi_{n-}\}_{n=0,\infty}$.

THEOREM

Suppose that:

1) The homogeneous system of equations (21) has in the space $l_p$ only zero solution.

2) $\sum_{n=0}^{\infty} \left( \sum_{k=1}^{\infty} |\Gamma_k N_{nk}|^{\frac{p}{p-1}} \right)^{p-1} < \infty,$

3) $\sum_{n=0}^{\infty}|R_n|^p < \infty$.

Then the infinite system (21) has a unique solution in $l_p$ the corresponding finite system also has a unique solution and the following estimate holds

$$\left\|\Phi - \Phi^N\right\|_{l_p} \leq Q_1\left[\sum_{n=N+1}^{\infty}\left(\sum_{k=1}^{\infty}\left|\Gamma_k N_{nk}\right|^{\frac{p}{p-1}}\right)^{p-1}\right]^{\frac{1}{p}} + Q_2\left[\frac{\sum_{n=N+1}^{\infty}|R_n|^p}{\sum_{n=0}^{\infty}|R_n|^p}\right]^{\frac{1}{p}}$$

where $Q_1$ and $Q_2$ are constants.

The proof of this theorem is similar to the proof in the space $l_2$ done in [6]. The first condition of this theorem is satisfied because the infinite system of equation (21) is equivalent to the singular integral equation which in its turn corresponds to the problem of mathematical physics having a unique solution. To check the fulfillment of the second and third conditions, we use the formula

$$N_{nk} = -\frac{1}{2}\frac{n+1}{n-k}\left[P_n(\cos a)P_{k+1}(\cos a) - P_{n+1}(\cos a)P_k(\cos a)\right], \ (k \geq 1)$$

From the estimate [7]

$$\left|P_n(\cos a)\right| \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{\sqrt{n\sin a}}, \quad (0 < a < \pi, \ n = 1, 2, \ldots)$$

it follows

$$|N_{nk}| \sim \frac{C}{\sqrt{n}}, \ n \to \infty, \ |N_{nk}| \sim \frac{C}{k^{\frac{2}{3}}}, \ k \to \infty, \ R_n \sim \frac{C}{\sqrt{n}}, \ n \to \infty,$$

therefore conditions (2) and (3) are satisfied as $p > 4$. In additions

$$\left\|\Phi - \Phi^N\right\|_{l_p} \leq \frac{c}{(N+1)^{\frac{1}{p}}}$$

After finding $a_0$ from condition (9)

$$\lambda_0^{-1}\Phi_{0-} - 2\sum_{n=1}^{N}\lambda_n^{-1}\Phi_{n-} = f(0)$$

we can write the solution of problem (1)-(4) in the form

$$T(x, y, t) = e^{-iwt}\left(\lambda_0^{-1}\Phi_{0-}e^{-\lambda_0 y} + 2\sum\lambda_n^{-1}\Phi_{n-}\cos nx\, e^{-\lambda_n y}\right) \tag{23}$$

### *Conclusion*

On choosing the appropriate number N and using the formula (23) we can get an approximate solution of the problem up to any prescribed accuracy.

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