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OPTIMAL CONTROL

Quasi-Optimal Deceleration of Rotations of an Asymmetric Body in Resistive Medium

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Abstract—A minimum-time problem on deceleration of rotations of a free rigid body affected by a small control torque with close but not identical coefficients is studied; such a problem can be considered as a quasi-optimal control problem. In addition, the rigid body is affected by a small deceleration viscous friction torque. The body is assumed to be dynamically asymmetric. A quasi-optimal feedback control for the deceleration of rotations of the rigid body is constructed, the optimal control time, and phase trajectories are found. The quasi-stationary trajectories are analyzed.

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INTRODUCTION

The development of research in dynamics and control of motion of bodies about a fixed point goes in the direction of taking into account the fact that these bodies are not perfectly rigid but are rather close to perfect models. The need for the analysis of the influence of various deviations from perfectness is caused by growing accuracy requirements in space exploration, gyroscopy, etc. The influence of imperfections can be revealed using asymptotic methods of nonlinear mechanics (singular perturbations, averaging, and others). This influence reduces to additional terms in the Euler equations of motion of a fictitious rigid body. Passive motions of rigid bodies in a resistive medium were studied in [1–5]. The important practical problem of controlling rotation of quasi-rigid bodies (the body contains a cavity filled with viscous fluid whose influence is taken into account by means of the internal torque created by the viscous fluid) using concentrated torques received less attention.

In this paper, we consider the problem of quasi-optimal deceleration of rotations of a dynamically asymmetric body to which the decelerating torque created by the linear drag force is applied. The rotations are controlled by a bounded torque. The components of the control torques are represented by the products $\varepsilon b_i u_i$ ($i = 1, 2, 3$), where b_i ($i = 1, 2, 3$) have the dimension of torque, ε is a small parameter, and $u_i \sim 1$ ($i = 1, 2, 3$) are dimensionless control functions to be determined. Note that a similar problem with $b_1 = b_2 = b_3 = b \neq 0$ was considered in [6]. Approximate solutions of perturbed problems of time-optimal deceleration of rotations of rigid bodies about the center of mass with applications to spacecraft dynamics were obtained in the monograph [7]. Stabilization problems for bodies with internal degrees of freedom were studied. The deceleration of rotations of almost spherical rigid bodies under the action of the torque exerted by the linear resistance of the medium was analyzed.

1. STATEMENT OF THE PROBLEM

We consider a dynamically asymmetric rigid body with moments of inertia satisfying, for definiteness, the inequalities $A_1 > A_2 > A_3$. Based on the approach described in [7], the equations of controlled rotations projected on the axes of the body-related reference frame (the Euler equations) can be expressed as (see [2, 3, 7])

$$\dot{\mathbf{G}} + \boldsymbol{\omega} \times \mathbf{G} = \mathbf{M}^u + \mathbf{M}^r. \tag{1.1}$$

Here, $\mathbf{G}$ is the body angular momentum, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^{\mathrm{T}}$ is the vector of absolute angular velocity, $\mathrm{J} = \mathrm{diag}(A_1, A_2, A_3)$ is the tensor of the body inertia, $\mathbf{M}^u$ is the vector of control torque, and $\mathbf{M}^r$ is the dissipation torque.

The angular momentum of the body is determined in the standard way as

$$\mathbf{G} = J\boldsymbol{\omega}, \quad \mathbf{G} = (G_1, G_2, G_3)^{\mathrm{T}}, \quad G_i = A_i\omega_i, \quad i = 1,2,3, \quad G = |\mathbf{G}| = (G_1^2 + G_2^2 + G_3^2)^{1/2}. \tag{1.2}$$

The dissipation torque is assumed to be proportional to the angular momentum:

$$\mathbf{M}^r = -\lambda'\mathbf{G}. \tag{1.3}$$

Here, λ' is a constant coefficient depending on the medium properties. The resistance of the medium is assumed to be low and have the order of smallness $\varepsilon : \lambda' = \varepsilon\lambda$. In this case, the projections of the momentum on the major axes of body inertia are given by $\varepsilon\lambda G_i$ ($i = 1, 2, 3$), where $\varepsilon \ll 1$ (see [2, 3]).

The magnitude of the control torque $\mathbf{M}^u$ is assumed to be small of the order of ε. The components of the control torque are represented by the products of the constant b_i, which have the dimension of torque, by the small parameter ε and by the dimensionless control functions u_i to be determined:

$$M_i^u = \varepsilon b_i u_i. \tag{1.4}$$

The products εb_i ($i = 1, 2, 3$) characterize the efficiency of the control system with respect to the corresponding axes of the body reference frame.

In terms of projections on the principal central axes of inertia, equations of controlled motion (1.1) in this problem statement are

$$\begin{aligned} \dot{G}_1 + G_3\omega_2 - G_2\omega_3 &= \varepsilon b_1 u_1 - \varepsilon\lambda G_1, \\ \dot{G}_2 + G_1\omega_3 - G_3\omega_1 &= \varepsilon b_2 u_2 - \varepsilon\lambda G_2, \\ \dot{G}_3 + G_2\omega_1 - G_1\omega_2 &= \varepsilon b_3 u_3 - \varepsilon\lambda G_3. \end{aligned} \tag{1.5}$$

For system (1.5), it is required to find the optimal controls $u_i = u_i(t, \omega_1, \omega_2, \omega_3)$, ($i = 1, 2, 3$), that satisfy the constraint

$$u_1^2 + u_2^2 + u_3^2 \le 1 \tag{1.6}$$

and steer system (1.5) from an arbitrary initial state $\boldsymbol{\omega}(t_0) = \boldsymbol{\omega}^0$ into the state of rest $\boldsymbol{\omega}(T) = 0$ in a minimum amount of time.

In the case $b_1 = b_2 = b_3 = b$ ($b > 0$), where b may be a function of time, the optimal control has the form $\mathbf{u} = \mathbf{G}/G$, where $\mathbf{u}$ is a vector whose projections on the principal axes of inertia are u_1, u_2, u_3 (see [7, 8]). If b_i are close to each other, then this control can be considered as a quasi-optimal one [7, 9].

For applications, it is of interest to investigate the motion of rigid bodies with the simple control defined as (see [7, 9])

$$M_i^u = \varepsilon b_i u_i, \quad u_i = -\frac{G_i}{G}, \quad i = 1, 2, 3. \tag{1.7}$$

Substitute (1.7) into (1.5) to obtain a closed system of equations of controlled motion in terms of projections on the principal central axes of inertia; for that reason, we do not write out the kinematic relations here.

2. SOLUTION OF THE QUASI-OPTIMAL DECELERATION PROBLEM

Taking into account (1.7), multiply the first equation in (1.5) by G_1, the second equation by G_2, and the third one by G_3 and add the products to obtain the dot product $\dot{\mathbf{G}} \cdot \mathbf{G}$. Taking into account the property of the derivative of the dot product squared $\dot{\mathbf{G}} \cdot \mathbf{G} = d|\mathbf{G}|^2/(2\,dt) = G\dot{G}$, we divide it by G to obtain the scalar equation

$$\dot{G} = -\varepsilon\lambda G - \frac{\varepsilon}{G^2}\sum_{i=1}^{3} b_i G_i^2. \tag{2.1}$$

Taking into account (1.7), multiply the first equation in (1.5) by ω_1, the second equation by ω_2, and the third one by ω_3 and add the products. By the well-know property of the energy integral, the kinetic energy of the rigid body is determined by the equality $H = (A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2)/2$. As a result, we have the following expression for the derivative of the kinetic energy:

$$\dot{H} = -2\varepsilon\lambda H - \frac{\varepsilon}{G}\sum_{i=1}^{3} b_i A_i \omega_i^2. \tag{2.2}$$

Consider the unperturbed motion ($\varepsilon = 0$). In the absence of disturbances, the body rotation is the Euler–Poinsot motion. The variables G and H are constant, and the Eulerian angles φ, ψ, and θ are functions of the time t. In the perturbed motion ($\varepsilon \neq 0$), G and H are slow variables, and the Eulerian angles are fast variables. As has already been mentioned above, we consider the statement of the problem of rotation deceleration in which there are no angle variables—they can be calculated by simultaneous integration of dynamic and kinematic equations.

Consider the motion under the condition $2HA_1 \geq G^2 > 2HA_2$, which corresponds to trajectories of the angular momentum vector that encircle the axes of the greatest moment of inertia Oz_1. Define the quantity

$$k^2 = \frac{(A_2 - A_3)[2H(t)A_1 - G^2(t)]}{(A_1 - A_2)[G^2(t) - 2H(t)A_3]} \quad (0 \leq k^2 \leq 1), \tag{2.3}$$

which is constant in the case of unperturbed motion; it is the elliptic modulus describing this motion, and it is uniquely associated with the angular momentum G and the kinetic energy H.

To construct the averaged first approximation system, we substitute the functions ω_i ($i = 1, 2, 3$) from the unperturbed Euler–Poinsot motion [10] into the right-hand sides of Eqs. (2.1), (2.2) and then average over the period of this motion. For the slow variables G and H, we use the same notation. As a result, for $\tau = \varepsilon t \in [0, T]$, we obtain

$$\begin{aligned}
\frac{dH}{d\tau} &= -2\lambda H - \frac{G}{S(k)}\left\{b_1(A_2 - A_3)\frac{E(k)}{K(k)} + b_2(A_1 - A_3)W(k) + b_3(A_1 - A_2)(k^2 - W(k))\right\},\\
\frac{dG}{d\tau} &= -\lambda G - \frac{1}{S(k)}\left\{b_1A_1(A_2 - A_3)\frac{E(k)}{K(k)} + b_2A_2(A_1 - A_3)W(k) + b_3A_3(A_1 - A_2)(k^2 - W(k))\right\},\\
S(k) &= A_1(A_2 - A_3) + A_3(A_1 - A_2)k^2, \quad W(k) = 1 - \frac{E(k)}{K(k)}.
\end{aligned} \tag{2.4}$$

Here, $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds, respectively (see [11]). The first equation in (2.4) implies that the kinetic energy H of the body evolves under the influence of the medium drag and the control torque. The expression in braces on the right-hand side of the first equation in (2.4) is positive (for $A_1 > A_2 > A_3$) due to the inequalities (see [11])

$$(1 - k^2)K \leq E \leq K. \tag{2.5}$$

Therefore, $dH/d\tau < 0$ because $H > 0$; hence, H is strictly decreasing for any $k^2 \in [0,1]$. It can be similarly shown that the angular momentum also decreases.

3. INVESTIGATION OF QUASI-STEADY MOTIONS

Differentiating expression (2.3) for k^2 with regard to (2.4), we obtain the differential equation

$$\frac{dk^2}{d\tau} = \frac{2}{G}\left\{b_1k^2\frac{E(k)}{K(k)} + b_2(k^2 - 1)W(k) + b_3(W(k) - k^2)\right\}. \tag{3.1}$$

Note that as $G \to 0$, Eq. (3.1) has an essential singularity. The value $k^2 = 1$ is associated with the equality $2HA_2 = G^2$, which corresponds to the separatrix for the Euler–Poinsot motion. Equation (3.1) describes the averaged motion of the endpoint of the angular momentum vector on the sphere of radius G.

Note that the evolution of k^2 is affected by the control torque and by the torque created by dissipation forces. Equation (3.1) for $k^2(\tau)$ has stationary points k_*^2; in addition to $k_*^2 = 0$ and $k_*^2 = 1$, there are also

$$k_*^2 = (b_2 - b_3) W(k_*) \left[(b_1 - b_2) \frac{E(k_*)}{K(k_*)} + (b_2 - b_3) \right]^{-1}. \tag{3.2}$$

In the case of quasi-stationary motions of the rigid body corresponding to the stationary points k_*^2, the motion of the vector **G** is generally composed only of the motion along the Euler—Poinsot trajectory and decrease in the length of **G** with time.

For each k_*^2 different from 0 and 1, we can introduce the following notation for the dimensionless quantities

$$\chi_1 = \frac{b_1}{b_3}, \quad \chi_2 = \frac{b_2}{b_3} \tag{3.3}$$

and rewrite (3.2) as

$$k_*^2 = (\chi_2 - 1) W(k_*) \left[(\chi_1 - \chi_2) \frac{E(k_*)}{K(k_*)} + (\chi_2 - 1) \right]^{-1}. \tag{3.4}$$

This implies that

$$\chi_1 = \chi_2 \frac{W(k_*^2) + k_*^2 F(k_*^2) + k_*^2}{k_*^2 F(k_*^2)} + \frac{k_*^2 - W(k_*^2)}{k_*^2 F(k_*^2)}, \tag{3.5}$$

where $F(k_*^2) = E(k_*^2)/K(k_*^2)$. Expression (3.5) is a linear function for which the conditions $\chi_1 > 0$ and $\chi_2 > 0$ must hold. If the second inequality is fulfilled, then we have

$$\chi_2 > \frac{W(k_*^2) - k_*^2}{W(k_*^2) + k_*^2 F(k_*^2) + k_*^2}. \tag{3.6}$$

The expression on the right-hand side of (3.6) is positive for any k_*^2 due to inequalities (2.5). Therefore, for all χ_2 satisfying (3.6), the necessary condition $\chi_1 > 0$ also holds.

The left-hand side of Eq. (3.4) must satisfy the inequality $0 < k_*^2 < 1$, whence we obtain necessary conditions for the existence of quasi-stationary solutions for χ_1 and χ_2. Two domains where quasi-stationary solutions exist were obtained (Fig. 1). The boundary lines in the construction of these domains are line *2*, which corresponds to $\chi_2 = 1$, line *3*, which corresponds to $\chi_1 = 1$, and line *4* determined by the equation

$$\chi_1 = 1 - \chi_2 \left(1 - \frac{E(k_*)}{K(k_*)} \right).$$

In Fig. 1 line *1* is constructed by formula (3.5) for $k_*^2 = 0.4$. It is seen that a quasi-stationary motion does exist not for all values of the dimensionless coefficients of the control torque projections χ_1 and χ_2; linear dependence (3.5) can hold only in two quadratic domains described above.

Consider the equation governing the change in the angular momentum of system (2.4) and Eq. (3.1). We analyze the deceleration time of the rigid body depending on the magnitude of the control torque coefficients b_i ($i = 1, 2, 3$). Figure 2 illustrates two cases. Curve *1* corresponds to $b_2 = 0.1$ and $b_3 = 0.1$ for $b_1 \in [0.1; 1.2]$; curve *2* corresponds to $b_1 = 0.1$ and $b_3 = 0.1$ for $b_2 \in [0.1; 1.2]$. The curves in Fig. 2 show that the greater the control torque coefficient, the shorter is the deceleration time. It is seen that the function in both cases is exponential-like. The behavior of the function $T = T(b_i)$ obtained in [12] is similar.

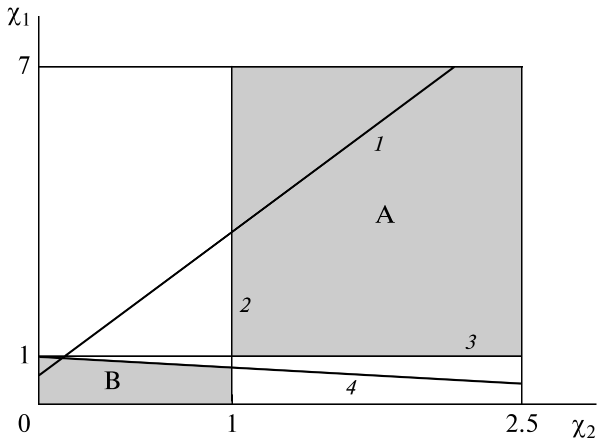

Fig. 1.

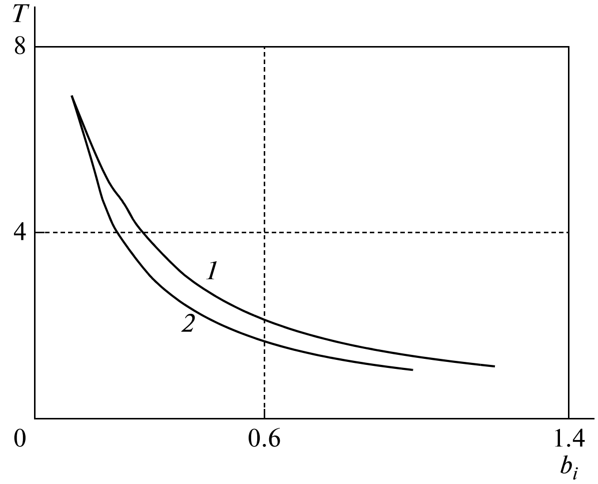

Fig. 2.

4. NUMERICAL RESULTS

In the general case, system (2.4), (3.1) can be solved numerically. To this end, we reduce this system to dimensionless form by choosing as the characteristic parameters of the problem the unknown deceleration time T, the coefficient of the control torque b_3, and the value of the angular momentum at the initial time G_0. The dimensionless quantities are

$$\tilde{t} = \frac{\tau}{T}, \quad \tilde{\lambda} = \lambda T, \quad \tilde{H} = \frac{H}{b_3}, \quad \tilde{A}_i = \frac{A_i}{G_0 T}, \quad \tilde{G} = \frac{G}{G_0}.$$

Define the characteristic number

$$\sigma = \frac{b_3 T}{G_0}, \tag{4.1}$$

which determines the basic process—deceleration of the rigid body under the action of the control torque in the minimum amount of time T.

The system of equations in dimensionless form is

$$\begin{aligned}
\frac{d\tilde{H}}{d\tilde{t}} &= -2\tilde{\lambda}\tilde{H} - \frac{\tilde{G}}{\tilde{S}(k)}\left\{\chi_1\left(\tilde{A}_2 - \tilde{A}_3\right)\frac{E(k)}{K(k)} + \chi_2\left(\tilde{A}_1 - \tilde{A}_3\right)W(k) + \left(\tilde{A}_1 - \tilde{A}_2\right)(k^2 - W(k))\right\},\\
\frac{d\tilde{G}}{d\tilde{t}} &= -\tilde{\lambda}\tilde{G} - \frac{\sigma}{\tilde{S}(k)}\left\{\chi_1\tilde{A}_1\left(\tilde{A}_2 - \tilde{A}_3\right)\frac{E(k)}{K(k)} + \chi_2\tilde{A}_2\left(\tilde{A}_1 - \tilde{A}_3\right)W(k) + \tilde{A}_3\left(\tilde{A}_1 - \tilde{A}_2\right)(k^2 - W(k))\right\},\\
\frac{dk^2}{d\tilde{t}} &= \frac{2\sigma}{\tilde{G}}\left\{\chi_1 k^2\frac{E(k)}{K(k)} + \chi_2(k^2 - 1)W(k) + (W(k) - k^2)\right\},\\
\tilde{S}(k) &= \tilde{A}_1\left(\tilde{A}_2 - \tilde{A}_3\right) + \tilde{A}_3\left(\tilde{A}_1 - \tilde{A}_2\right)k^2, \quad W(k) = 1 - \frac{E(k)}{K(k)}.
\end{aligned} \tag{4.2}$$

The integration was performed for the initial conditions $k^2(0) = 0.9999$ and $\tilde{G}(0) = 1$; the kinetic energy at the initial time was determined from the equation

$$\tilde{H}(0) = \frac{\tilde{G}^2(0)(\tilde{A}_2 - \tilde{A}_3 + (\tilde{A}_1 - \tilde{A}_2)k^2(0)))}{2\sigma\tilde{S}(k^2(0))}. \tag{4.3}$$

The third equation in (4.2) describes the variation of k^2; therefore, for the initial condition $k^2 \approx 1$, the right-hand side of this equation must be negative. The second and the third terms in braces are negative; therefore, the condition

$$\chi_1 < \frac{\chi_2(1 - k^2)W(k) + k^2 - W(k)}{k^2 F(k)} \tag{4.4}$$

for the dimensionless coefficients of the control torque must be fulfilled.

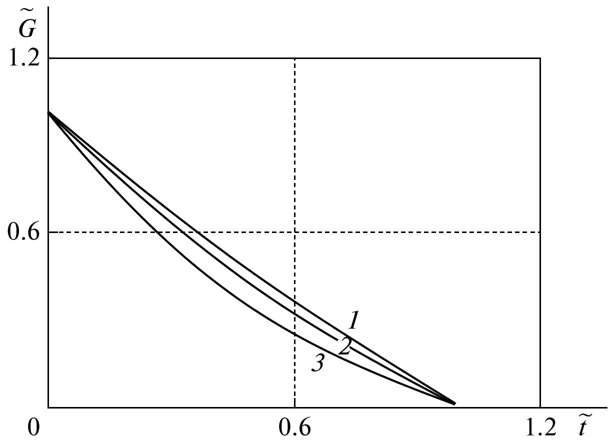

Fig. 3.

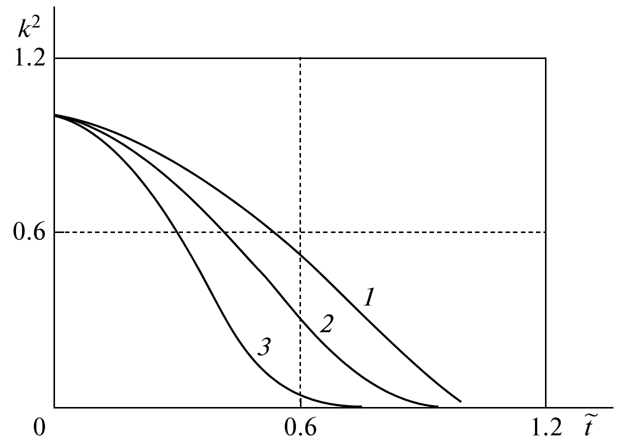

Fig. 4.

Numerical computations were performed for various values of χ_1, χ_2, and σ. For different values of the characteristic number σ, there exist values of the dimensionless coefficients of the control torque χ_1 and χ_2 at which the deceleration of the body is quasi-optimal. The deceleration process can be different.

We investigate the case $\sigma = 1.3$. Fig. 3 shows the plot of changes in the angular momentum of the body with the reduced tensor of mass inertia $\tilde{A}_1 = 8$, $\tilde{A}_2 = 6$, $\tilde{A}_3 = 4$ and the dimensionless coefficient of the medium drag torque $\tilde{\lambda} = 0.1$; curve *1* corresponds to $\chi_1 = 0.5$ and $\chi_2 = 0.8$; curve *2* corresponds to $\chi_1 = 0.6$ and $\chi_2 = 1$; and curve *3* corresponds to $\chi_1 = 0.7$ and $\chi_2 = 1.4$.

It is seen from Fig. 3 that the curvature of the curve increases with increasing χ_1 and χ_2. In the first case, the function $\tilde{G} = \tilde{G}(\tilde{t})$ is almost linear. For other values of the characteristic number σ, the angular momentum function is similar.

Fig. 4 shows the result of numerical integration for the rigid body with the same mass geometry and in the same resistive medium for the characteristic number $\sigma = 1.4$. Curve *1* corresponds to $\chi_1 = 0.6$ and $\chi_2 = 0.8$; curve *2* corresponds to $\chi_1 = 0.7$ and $\chi_2 = 1.1$; and curve *3* corresponds to $\chi_1 = 0.8$ and $\chi_2 = 1.8$. We see that the greater the control torque, the faster is the deceleration of the rigid body and the plots of $k^2 = k^2(\tilde{t})$ are more intricate with clearly seen concave and convex segments.

The behavior of the kinetic energy function in the case of the quasi-optimal deceleration of the body is illustrated in Fig. 5. The numerical investigation was performed for the body with the same mass geometry and in the same resistive medium. It is seen from this figure that in all the examined cases, the body comes to rest in the quasi-optimal deceleration amount of time. Curve *1* was obtained for the dimensionless coefficients of the control torque $\chi_1 = 0.2$ and $\chi_2 = 1.6$ and the characteristic number $\sigma = 0.7$. Curve *2* was obtained for $\sigma = 0.8$ and $\chi_1 = 1.0$, $\chi_2 = 1.2$. In the case of $\sigma = 1.1$, the quasi-optimal deceleration was obtained for $\chi_1 = 0.5$ and $\chi_2 = 1.1$, which corresponds to curve *3*. Curve *4* shows the shape of the function $\tilde{H} = \tilde{H}(\tilde{t})$ corresponding to curve *2* in Fig. 5. Note that in all the numerically investigated cases, the function monotonically decreases to zero in the quasi-optimal amount of time T.

5. INFLUENCE OF A SMALL PERTURBATION

The optimal deceleration of the rotations of an asymmetric body in resistive medium in the case

$$b_1 = b_2 = b_3 = b \tag{5.1}$$

was studied in [6]. In that paper, an analytical solution for the variation of the magnitude of the angular momentum vector and the kinetic energy of the rigid body was obtained. The numerical computation for system (4.2) under condition (5.1) gives the result coinciding with that obtained in [6] up to 10 decimal digits. Let us investigate the behavior of these functions under a small variation of the coefficients of the control torque. According to (3.3), we introduce the dimensionless coefficients χ_1 and χ_2, and we can introduce χ_3 that is always equal to unit.

Consider small increments

$$\chi_i = 1 + \mu_i, \tag{5.2}$$

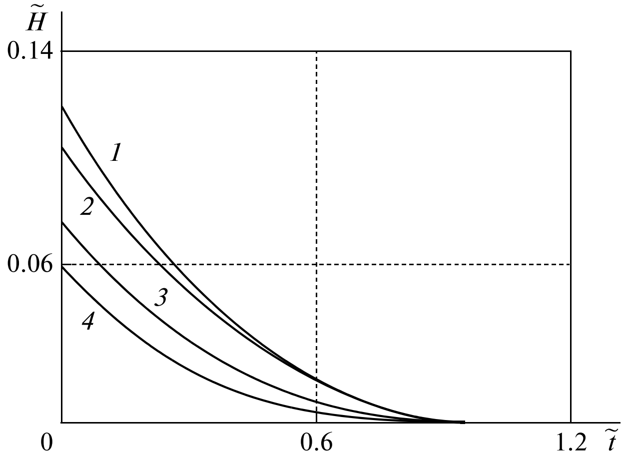

Fig. 5.

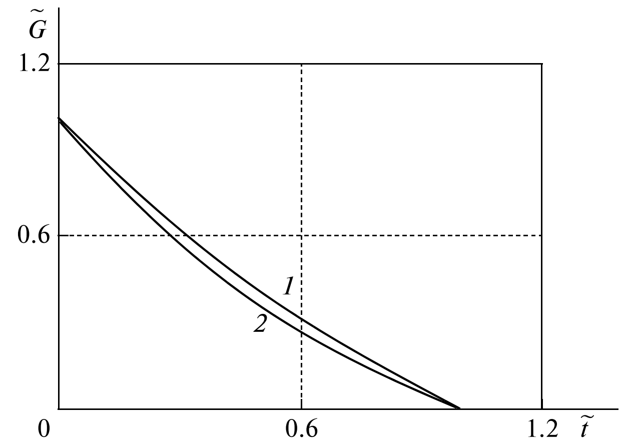

Fig. 6.

where $|\mu_i| \ll 1$. First, numerical integration was performed for $\mu_i = 0$, which corresponds to case (5.1). Next, the numerical solution was found for various values of μ_i.

Fig. 6 shows the results of numerical computations: curve *1* corresponds to case (5.1), and curve *2* was obtained for small increments of the control torque coefficients. A similar pattern is observed for the kinetic energy of the rigid body.

The numerical solutions allow us to conclude that small increments of one of the coefficients cause a small increase in the gradients of the function of the rigid body deceleration.

CONCLUSIONS

The synthesis problem of a time quasi-optimal deceleration of rotations of a dynamically asymmetric rigid body in resistive medium is investigated analytically and numerically. In the framework of the asymptotic approach, the control, the optimal time (Bellman function), evolution of the elliptic modulus squared k^2, and the dimensionless kinetic energy and angular momentum are determined. Qualitative properties of the quasi-optimal motion are established. Quasi-stationary motions are investigated.

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