MATEC Web of Conferences **116**, 02022 (2017) DOI: 10.1051/matecconf/201711602022
Transbud-2017

Exact solution of the differential equation of transverse oscillations of the rod taking into account own weight

Yurii Krutii[1,*], *Mykola* Suriyaninov[1], and *Victor* Vandynskyi[1]

[1]Odessa State Academy of Civil Engineering and Architecture, Structural Mechanics, Department Didrikson str., 4, Odessa, 65029, Ukraine

Abstract. The free transverse oscillations of the rod of uniform cross section taking into account its own weight are considered in this work. The appropriate partial differential equation of transverse oscillations of a rod was reduced to two ordinary differential equations for the time function and the amplitude function of deflections. Concurrent with the differential equations for the amplitude state, the equivalent system of differential equation is considered. In total, the exact solution of the initial partial differential equation of transverse oscillations of a rod, expressed in non-dimensional fundamental functions and initial parameters is attained. The method of power series was used for the construction of fundamental functions. Due to the exact solution, the formulas in an explicit form for dynamic variables of the state of a rode – deflections, angular displacement, bending moment and transverse force – were defined. The analytical form for equation of free oscillation frequency is defined. That has limited the finding of frequency to definition the unknown non-dimensional parameter through the frequency equation. As a result, the presence of derivative exact solutions provides the possibility to investigate the free oscillations of rod with various types of boundary conditions.

1 Introduction

It is generally known, that oscillation is the most widespread type of motion. There is no any branch of technology without phenomena where the vibrations take place. At this point, the most important characteristics of oscillatory system are natural frequencies and fundamental mode shapes. Problem of characterization is the actual scientific and practical question, which often reduces to the problem of the solving of appropriate differential equation of motion.

Among others, the important question is the investigation of bending vibration in various buildings and constructions taking into account the effect of axial force. Unfortunately, in the most part of the publication [1–5], where the oscillations of these

[*] Corresponding author: yurii.krutii@gmail.com

constructions are considered, an axial force is accepted constant from any height for simplification.

However, in the real structures the axial forces in different cross sections have different values. For example, columns in industrial buildings have different compression ratio at different stories due to concentrated load at floor level. The industrial high-rise buildings like smokestacks, water towers, multipurpose steel towers, used for electric-power transmission line, wind generators supports, antennas of various constructions may be added to this example.

One of the most widespread design models for investigation of transverse oscillations of mentioned buildings is the uniform cross-section rod under the action of building weight in the capacity of variable axial force. The math model of this physical phenomenon is the differential variable-coefficient equation [3, 6, 7], the exact solution is still unknown. So the investigations are usually made by variational methods. Because of this, the investigations are usually made by variational methods. However, it is evident that the most full and qualitative evaluation of mechanical system can be gained only on the basis of the exact solution of differential equation. As a matter of fact, the focus of this paper is the definition of this solution.

2 Results

2.1 General symbols and equations

Let's analyze the free bending vibrations of the rod with uniform flexural rigidity and intensity of the distributed mass taking into account own weight (Fig.1, Fig.2).

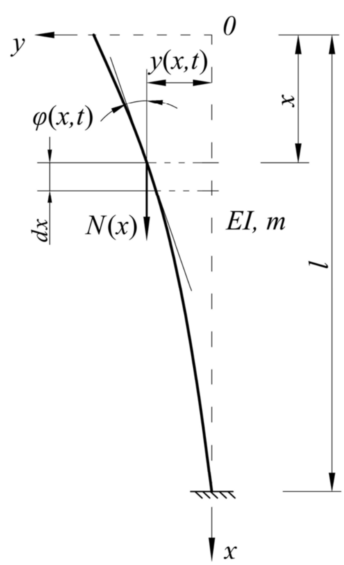

Fig. 1. The scheme of free transverse oscillations of a straight rod.

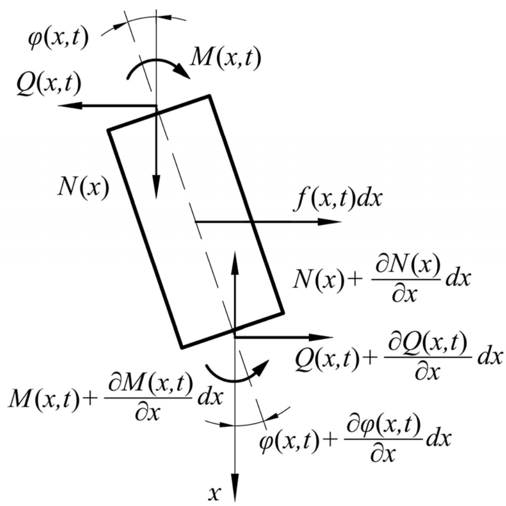

Fig. 2. The scheme of the forces acting upon the element of a rod.

List of symbols:
EI – the flexural rigidity of a rod;
m – the intensity of the distributed mass (own weight) of the rod;
$N(x) = qx$ – the variable axial (compressive) force, where q – weight per unit length of beam;
$y(x,t)$ – the cross motion of the axis point of the rod with coordinate x at time t (dynamic deflection);
$\varphi(x,t)$ – the dynamic angular displacement;
$M(x,t)$ – the dynamic bending moment;
$Q(x,t)$ – the dynamic transverse force;
$f(x,t)$ – the intensity of inertial forces that appear during oscillation (D'Alembert force).

However, we should note, that all the following formulas are valid for any boundary conditions at the ends of the rod.

It is known [3, 6], that the equation of free transverse oscillations taking into account own weight is written as:

$$EI\frac{\partial^4 y}{\partial x^4} + q\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) + m\frac{\partial^2 y}{\partial t^2} = 0\ . \tag{1}$$

This equation is valid for a model where the longitudinal displacement of cross sections and their twists and shifts are decided to ignore.

Defined dynamic deflection $y(x,t)$ from Eq.(1), other dynamic parameters of the rod state are given according to the known formulas [3, 6]:

$$\varphi(x,t) = \frac{\partial y}{\partial x};\ M(x,t) = -EI\frac{\partial^2 y}{\partial x^2};\ Q(x,t) = \frac{\partial M}{\partial x} - qx\frac{\partial y}{\partial x}. \tag{2}$$

Using Fourier method, the solution of partial derivative equation (1) is given as

$$y(x,t) = v(x)T(t)\,, \tag{3}$$

where $v(x)-$ the amplitude of the transverse displacement, which depends only on variable x; $T(t)-$ the time function t. After the implement (3) in formulas (2), we'll have the similar formulas for the other dynamic parameters:

$$\varphi(x,t)=\varphi(x)T(t);\;\; M(x,t)=M(x)T(t);\;\; Q(x,t)=Q(x)T(t)\,, \tag{4}$$

where $\varphi(x), M(x), Q(x)-$ amplitude functions, which are linked by the equalities

$$\varphi(x)=v'(x);\;\; M(x)=-EIv''(x);\; Q(x)=M'(x)-qx\,v'(x). \tag{5}$$

If we substitute (3) into Eq. (1) and separate the variables there, we will have two differential equations:

$$\ddot{T}(t)+p^2T(t)=0\,; \tag{6}$$

$$EI\,v''''(x)+q(x\,v'(x))'-p^2m\,v(x)=0\,, \tag{7}$$

where p^2- Fourier method constant.

The solution of the Eq. (6), expressed in terms of parameters of initial motion conditions $T(0), \dot{T}(0)$, is given by

$$T(t)=T(0)\cos pt+\frac{\dot{T}(0)}{p}\sin pt\,.$$

Its analysis shows that motion of mechanical system is oscillatory in nature. The constant is the frequency of free oscillations.

The fundamental mode is the solution of Eq. (7). The main difficulty of the problem is exactly the definition of this solution.

The equivalent system of differential equations is considered along with Eq. (7). The state vector of the rod is used in the function of vector of unknowns. The components of the state vector are amplitude deflections, angular displacement, bending moment and transverse force:

$$\Phi(x)=\begin{pmatrix} v(x) \\ \varphi(x) \\ M(x) \\ Q(x) \end{pmatrix}=\begin{pmatrix} v(x) \\ v'(x) \\ -EI\,v''(x) \\ -EI\,v'''(x)-qx\,v'(x) \end{pmatrix}. \tag{8}$$

Then, writing the collection of formulas (5), (7) in form of matrix we'll have:

$$\frac{d\Phi(x)}{dx}=D(x)\Phi(x)\,. \tag{9}$$

where $D(x)-$ the coefficient matrix of the system, which have the form

$$D(x)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{EI} & 0 \\ 0 & qx & 0 & 1 \\ -p^2m & 0 & 0 & 0 \end{pmatrix}.$$

2.2 The exact solution of amplitude equation of oscillation. Formulas of state parameters of the rod

For the construction of an exact solution of Eq. (7) we use integration technique proposed and developed in [8].

There are four infinite systems of functions $b_{n,0}(x),\ b_{n,k}(x)\ (n=1,2,3,4)(k=1,2,3,...)$, which are decided continuous ones with their derivatives from the first-order to fourth-order inclusive. With the help of these functions and their derivatives, we make the expansion in powers of parameter $P=p^2m$:

$$U_n(x)=b_{n,0}(x)+Pb_{n,1}(x)+P^2b_{n,2}(x)+\cdots; \tag{10}$$

$$U_n^{(\nu)}(x)=b_{n,0}^{(\nu)}(x)+Pb_{n,1}^{(\nu)}(x)+P^2b_{n,2}^{(\nu)}(x)+\cdots, \tag{11}$$

where (ν) – the order of derivate, $\nu=1,2,3,4$. At the moment we allow, that the ranges (10), (11) are uniform convergent, so it's possible to do term-by-term differentiation.

Following the terminology, accepted in [8], the functions $b_{n,0}(x)$ is called primary, and $b_{n,k}(x)\ (k=1,2,3,...)$ – generating. We are going to define them on condition that $U_n(x)$ is satisfied for Eq. (7), so

$$EI\,U_n''''(x)+q(xU_n'(x))'-PU_n(x)=0\ (n=1,2,3,4). \tag{12}$$

Substituting the values (10), (11) instead of $U_n(x),\ U_n'(x),\ U_n''(x),\ U_n''''(x)$, after transformations one arrives the equality

$$EI\,b_{n,0}''''(x)+q(xb_{n,0}'(x))'+\sum_{k=1}^{\infty}P^k\,(EI\,b_{n,k}''''(x)+q(x\,b_{n,k}'(x))'-b_{n,k-1}(x))=0. \tag{13}$$

For its satisfaction set all the coefficients at powers of parameter P, beginning from zero power:

$$EI\,b_{n,0}''''(x)+q(x\,b_{n,0}'(x))'=0 \tag{14}$$

$$EI\,b_{n,k}''''(x)+q(x\,b_{n,k}'(x))'=b_{n,k-1}(x)\ \ (k=1,2,3,...) \tag{15}$$

So, the differential equations for primary and generating function definition are attained. Use the power series method for integration this equations, previously accepted next boundary conditions:

$$\begin{pmatrix} b_{1,0}(0) & b_{2,0}(0) & b_{3,0}(0) & b_{4,0}(0) \\ b'_{1,0}(0) & b'_{2,0}(0) & b'_{3,0}(0) & b'_{4,0}(0) \\ b''_{1,0}(0) & b''_{2,0}(0) & b''_{3,0}(0) & b''_{4,0}(0) \\ b'''_{1,0}(0) & b'''_{2,0}(0) & b'''_{3,0}(0) & b'''_{4,0}(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (16)$$

$$b_{n,k}(0) = b'_{n,k}(0) = b''_{n,k}(0) = b'''_{n,k}(0) = 0 \;\; (n = 1,2,3,4)(k = 1,2,3,...). \qquad (17)$$

Integrating both parts (14), one arrives at the equation

$$EI\, b'''_{n,0}(x) + qx\, b'_{n,0}(x) = A_n \;\; (n = 1,2,3,4) \qquad (18)$$

where $A_n -$ permanent integration. Accepted the equation $x = 0$, define $A_n = EI\, b'''_{n,0}(0)$. Taking into account (16) we have: $A_1 = A_2 = A_3 = 0;\; A_4 = EI$.

Integrating both part (15) and taking into account boundary conditions (17), we will have

$$EI\, b'''_{n,k}(x) + qx\, b'_{n,k}(x) = \int_0^x b_{n,k-1}(x)\, dx \,(k = 1,2,3,...) \qquad (19)$$

Solution of Eq. (18), (19) is definite in form of series:

$$b_{n,\,0}(x) = x^{n-1} \sum_{j=0}^{\infty} C_{n,0,j} x^{3j} \qquad (20)$$

$$b_{n,\,k}(x) = x^{n+4k-1} \sum_{j=0}^{\infty} C_{n,k,j} x^{3j} \;\; (k = 1,2,3,...) \qquad (21)$$

According to this choice, the boundary conditions for generating functions (17) are satisfied in advance, and after boundary condition realization for primary function (16), we'll have:

$$C_{n,0,0} = \frac{1}{(n-1)!} \,(n = 1,2,3,4);\;\; C_{1,0,\,1} = 0 \qquad (22)$$

Based on (20), (21), we find:

$$x\, b'_{n,0}(x) = x^{n-1} \sum_{j=0}^{\infty} e_{n,0,j} C_{n,0,j} x^{3j}\,; \qquad (23)$$

$$b'''_{n,0}(x) = x^{n-1} \sum_{j=0}^{\infty} f_{n,0,j} C_{n,0,j} x^{3j-3} = (n-3)(n-2)(n-1) C_{n,0,0} x^{n-4} + \\ + x^{n-1} \sum_{j=0}^{\infty} f_{n,0,j+1} C_{n,0,j+1} x^{3j}\,; \qquad (24)$$

$$x\, b'_{n,\,k}(x) = x^{n+4k-1} \sum_{j=0}^{\infty} e_{n,k,j} C_{n,k,j} x^{3j} \;\; (k = 1,2,3,...)\,; \qquad (25)$$

$$b'''_{n,k}(x) = x^{n+4k-1}\sum_{j=0}^{\infty} f_{n,k,j} C_{n,k,j} x^{3j-3} = x^{n+4k-4} f_{n,k,0} C_{n,k,0} + x^{n+4k-1}\sum_{j=0}^{\infty} f_{n,k,j+1} C_{n,k,j+1} x^{3j} \quad (k = 1,2,3...); \tag{26}$$

$$\int_0^x b_{n,k-1}(x)\,dx = x^{n+4k-1}\sum_{j=0}^{\infty} \frac{C_{n,k-1,j}}{e_{n,k,j-1}} x^{3j-3} = \frac{C_{n,k-1,0}}{e_{n,k,-1}} x^{n+4k-4} + x^{n+4k-1}\sum_{j=0}^{\infty} \frac{C_{n,k-1,j+1}}{e_{n,k,j}} x^{3j} \quad (k = 1,2,3,...), \tag{27}$$

where

$$e_{n,k,j} = n + 4k + 3j - 1\,,\;\; f_{n,k,j} = (n + 4k + 3j - 3)(n + 4k + 3j - 2) e_{n,k,j}\,.$$

After substitution (23), (24) in Eq. (18), and (25)–(27) in Eq. (19), we arrives at the equalities:

$$(n-3)(n-2)(n-1) C_{n,0,0} x^{n-4} + x^{n-1}\sum_{j=0}^{\infty} f_{n,0,j+1} C_{n,0,j+1} x^{3j} + \frac{q}{EI} x^{n-1} \sum_{j=0}^{\infty} e_{n,0,j} C_{n,0,j} x^{3j} = \frac{A_n}{EI} \quad (n = 1,2,3,4); \tag{28}$$

$$\begin{gathered} f_{n,k,0} C_{n,k,0} x^{n+4k-4} + x^{n+4k-1}\sum_{j=0}^{\infty} f_{n,k,j+1} C_{n,k,j+1} x^{3j} + \frac{q}{EI} x^{n+4k-1}\sum_{j=0}^{\infty} e_{n,k,j} C_{n,k,j} x^{3j} = \\ = \frac{1}{EI}\frac{C_{n,k-1,0}}{e_{n,k,-1}} x^{n+4k-4} + \frac{1}{EI} x^{n+4k-1}\sum_{j=0}^{\infty}\frac{C_{n,k-1,j+1}}{e_{n,k,j}} x^{3j} \quad (k = 1,2,3,...). \end{gathered} \tag{29}$$

So the coefficients at the same powers are equated, we'll have:

$$C_{n,0,j+1} = -\frac{q}{EI}\frac{e_{n,0,j}}{f_{n,0,j+1}} C_{n,0,j} \quad (j = 0,1,2,...) \tag{30}$$

$$C_{n,k,0} = \frac{1}{EI}\frac{C_{n,k-1,0}}{f_{n,k,0}\, e_{n,k,-1}} \quad (k = 1,2,3,...) \tag{31}$$

$$C_{n,k,j+1} = \frac{1}{EI\, f_{n,k,j+1}}\left(\frac{C_{n,k-1,j+1}}{e_{n,k,j}} - q e_{n,k,j} C_{n,k,j}\right) \quad (k = 1,2,3,...)\,(j = 0,1,2,...). \tag{32}$$

Formulas (30), (31) taking into account (22) are possible to be written in the explicit form. On the whole, the collection (22), (30) will be equal to formulas:

$$C_{n,0,0} = \frac{1}{(n-1)!} \; (n = 1,2,3,4) \tag{33}$$

$$C_{n,0,j+1} = (-1)^{j+1}\left(\frac{q}{EI}\right)^{j+1} \frac{(n-1)(n+2)...(n+3j-1)}{(n+3j+2)!} \quad (j = 0,1,2,...), \tag{34}$$

based on formula (31) we define

$$C_{n,k,0} = \frac{1}{(EI)^k} \frac{1}{(n+4k-1)!} \; (k=1,2,3,...) \tag{35}$$

Thus, the coefficients of series (20), (21) are totally definite by formulas (33)–(35) and recurrence formula (32). Disadvantage of these formulas is that they are depending on initial dimensional problem parameters EI and q. This disadvantage can be avoided making the next substitution

$$C_{n,k,j} = \frac{(-1)^j}{(EI)^k} \left(\frac{q}{EI} \right)^j c_{n,k,j} \quad (n=1,2,3,4)(k=1,2,3,...)(j=0,1,2,...). \tag{36}$$

Substituted (36) in (32)–(35) and lowered the j index in the finally formulas by one, we'll have next formulas for definition of new non-dimensional coefficients $c_{n,k,j}$:

$$c_{n,0,0} = \frac{1}{(n-1)!} \; (n=1,2,3,4); \tag{37}$$

$$c_{n,0,j} = \frac{(n-1)(n+2)...(n+3j-4)}{(n+3j-1)!} \; (j=1,2,3,...); \tag{38}$$

$$c_{n,k,0} = \frac{1}{(n+4k-1)!} (k=1,2,3,...); \tag{39}$$

$$c_{n,k,j} = \frac{1}{f_{n,k,j}} \left(\frac{c_{n,k-1,j}}{e_{n,k,j-1}} + e_{n,k,j-1} \, c_{n,k,j-1} \right) \quad (k=1,2,3,...)(j=1,2,3,...). \tag{40}$$

As a result, formulas (20), (21) take the form:

$$b_{1,0}(x) = 1; b_{n,0}(x) = x^{n-1} \left(c_{n,0,0} + \sum_{j=1}^{\infty} (-1)^j \left(\frac{q}{EI} \right)^j c_{n,0,j} \, x^{3j} \right) \; (n=2,3,4); \tag{41}$$

$$b_{n,k}(x) = \frac{x^{n+4k-1}}{(EI)^k} \left(c_{n,k,0} + \sum_{j=1}^{\infty} (-1)^j \left(\frac{q}{EI} \right)^j c_{n,k,j} \, x^{3j} \right) \; (k=1,2,3,...). \tag{42}$$

So four solutions $U_n(x)$ $(n=1,2,3,4)$ are defined by formulas (10), (37)–(42) Eq.(15). Accordingly each of these solutions by formula (8) generates the solution of system (9)

$$\Phi_n(x) = \begin{pmatrix} U_n(x) \\ U_n'(x) \\ -EI\,U_n''(x) \\ -EI\,U_n'''(x) - qx\,U_n'(x) \end{pmatrix} (n=1,2,3,4) \tag{43}$$

Then the matrix $\Omega(x) = \left\| \Phi_1(x) \; \Phi_2(x) \; \Phi_3(x) \; \Phi_4(x) \right\|$, consisted of these vectors, also will satisfy the system (9).

Based on formulas (10), (11) and boundary conditions (17) for vector $\Phi_n(0)$ $(n=1,2,3,4)$ we'll have

$$\Phi_n(0)=\begin{pmatrix} U_n(0)\\ U_n'(0)\\ -EI\,U_n''(0)\\ -EI\,U_n'''(0)\end{pmatrix}=\begin{pmatrix} b_{n,0}(0)\\ b_{n,0}'(0)\\ -EI\,b_{n,0}''(0)\\ -EI\,b_{n,0}'''(0)\end{pmatrix}.$$

Therefore taking into account (16) we find:

$$\Phi_1(0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\Phi_2(0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\Phi_3(0)=\begin{pmatrix}0\\0\\-EI\\0\end{pmatrix},\Phi_4(0)=\begin{pmatrix}0\\0\\0\\-EI\end{pmatrix},$$

i.e.

$$\Omega(0)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-EI&0\\0&0&0&-EI\end{pmatrix}.$$

Once $|\Omega(0)|=(EI)^2\neq 0$, that the vectors (43) and the functions $U_n(x)$ $(n=1,2,3,4)$ are linearly independent[9]. Thus, the matrix $\Omega(x)$ is the fundamental matrix of the system (9), and $U_n(x)$ $(n=1,2,3,4)-$ are the fundamental solutions of Eq. (7).

Multiplied matrix on the right $\Omega(x)$ by the constant matrix $\Omega^{-1}(0)$ we'll get new fundamental matrix of the system (9) $\Lambda(x)=\Omega(x)\Omega^{-1}(0)$, and this

$$\Lambda(0)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}. \tag{44}$$

Fundamental matrix, which satisfies the condition (44), is uniquely determined and called matrizant [9].

Thus, allowing for the uniform convergence at the beginning (10), (11), we arrive at a conclusion that the matrix $\Lambda(x)$, which consists of the sums of these series $U_n(x), U_n'(x), U_n''(x), U_n'''(x)$ $(n=1,2,3,4)$, is the matrizant of the system (9). On the other hand, in the differential equations theory it is proved that matrizant of equation system with continuous coefficients is always absolutely and uniform convergent matrix series [9]. It follows that because of the uniqueness of matrizant the series (10) and (11) are absolutely and uniform convergent for $\nu=1,2,3$. As for series (11) for $\nu=4$, its uniform convergence follows immediately from identity (12). So, previously accepted designations $U_n^{(\nu)}$ $(\nu=1,2,3,4)$ for series (11) are correct.

General solution of differential equation system with the presence of matrizant is given by known [9] formula $\Phi(x)=\Lambda(x)\Phi(0)$. Writing it in expanded form we get the formulas for amplitude deflections and internal forces:

$$v(x) = v(0)U_1(x) + \varphi(0)U_2(x) - M(0)\frac{1}{EI}U_3(x) - Q(0)\frac{1}{EI}U_4(x); \tag{45}$$

$$\varphi(x) = v(0)U_1'(x) + \varphi(0)U_2'(x) - M(0)\frac{1}{EI}U_3'(x) - Q(0)\frac{1}{EI}U_4'(x); \tag{46}$$

$$M(x) = -v(0)EI\,U_1''(x) - \varphi(0)EI\,U_2''(x) + M(0)U_3''(x) + Q(0)U_4''(x); \tag{47}$$

$$\begin{aligned} Q(x) = & -v(0)(EI\,U_1'''(x) + qxU_1'(x)) - \varphi(0)(EI\,U_2'''(x) + qxU_2'(x)) + \\ & + M(0)\left(U_3'''(x) + \frac{qx}{EI}U_3'(x)\right) + Q(0)\left(U_4'''(x) + \frac{qx}{EI}U_4'(x)\right). \end{aligned} \tag{48}$$

2.3 Parameters of free transverse oscillation of the rod, expressed in terms of non-dimensional functions

Pass from the fundamental system of solutions Eq. (7) $U_n(x)\;(n=1,2,3,4)$ to a new fundamental system of non-dimensional functions $X_n(x)\;(n=1,2,3,4)$. To this end, the non-dimensional multipliers in the formulas of primary and generating functions (41), (42) must be identified. As a result, we'll have:

$$b_{n,0}(x) = l^{n-1}\beta_{n,0}(x);\;\; b_{n,k}(x) = \frac{l^{n+4k-1}}{(EI)^k}\beta_{n,k}(x)\;\;(n=1,2,3,4)\,(k=1,2,3,...), \tag{49}$$

where $\beta_{n,0}(x),\, \beta_{n,k}(x)\;(n=1,2,3,4)\;(k=1,2,3,...)-$ new non-dimensional primary and generating functions, are given by:

$$\beta_{1,0}(x) = 1; \beta_{n,0}(x) = \left(\frac{x}{l}\right)^{n-1}\left(c_{n,0,0} + \sum_{j=1}^{\infty}(-1)^j\alpha^j c_{n,0,j}\left(\frac{x}{l}\right)^{3j}\right)\;\;(n=2,3,4); \tag{50}$$

$$\beta_{n,k}(x) = \left(\frac{x}{l}\right)^{n+4k-1}\left(c_{n,k,0} + \sum_{j=1}^{\infty}(-1)^j\alpha^j c_{n,k,j}\left(\frac{x}{l}\right)^{3j}\right)\;\;(k=1,2,3,...), \tag{51}$$

where $\alpha = \dfrac{ql^3}{EI} -$ non-dimensional parameter.

Using (49), we comes to new primary and generating functions in the formula (10) . As a result, we have:

$$U_n(x) = l^{n-1}X_n(x)\;\;(n=1,2,3,4), \tag{52}$$

where $X_n(x)\;\;(n=1,2,3,4)-$ new fundamental functions, are given by:

$$X_n(x) = \beta_{n,0}(x) + K^2\beta_{n,1}(x) + K^4\beta_{n,2}(x) + ...\quad (n=1,2,3,4);$$

$$K = p\;l^2\sqrt{\frac{m}{EI}} \tag{53}$$

The zero dimension of functions $\beta_{n,0}(x)$, $\beta_{n,k}(x)\ (n=1,2,3,4)\quad (k=1,2,3,...)$, and the zero dimension of the parameter K provide evidently dimensionless character of functions $X_n(x)\ (n=1,2,3,4)$.

Differentiating (52) we'll also have

$$U_n'(x)=l^{n-2}\tilde{X}_n(x)\,,\ U_n''(x)=l^{n-3}\overset{\frown}{X}_n(x)\,,\ U_n'''(x)=l^{n-4}\hat{X}_n(x)\ (n=1,2,3,4), \tag{54}$$

where

$$\tilde{X}_n(x)=lX_n'(x),\ \overset{\frown}{X}_n(x)=l\tilde{X}_n'(x),\ \hat{X}_n(x)=l\overset{\frown}{X}_n'(x)\quad (n=1,2,3,4).$$

It proves that the functions $\tilde{X}_n(x),\ \overset{\frown}{X}_n(x),\ \hat{X}_n(x)\ (n=1,2,3,4)$ are expressed in the form of series:

$$\tilde{X}_n(x)=\tilde{\beta}_{n,0}(x)+K^2\tilde{\beta}_{n,1}(x)+K^4\tilde{\beta}_{n,2}(x)+K^6\tilde{\beta}_{n,3}(x)+...;$$

$$\overset{\frown}{X}_n(x)=\overset{\frown}{\beta}_{n,0}(x)+K^2\overset{\frown}{\beta}_{n,1}(x)+K^4\overset{\frown}{\beta}_{n,2}(x)+K^6\overset{\frown}{\beta}_{n,3}(x)+...;$$

$$\hat{X}_n(x)=\hat{\beta}_{n,0}(x)+K^2\hat{\beta}_{n,1}(x)+K^4\hat{\beta}_{n,2}(x)+K^6\hat{\beta}_{n,3}(x)+...,$$

where

$$\tilde{\beta}_{n,k}(x)=l\beta'_{n,k}(x),\ \overset{\frown}{\beta}_{n,k}(x)=l\tilde{\beta}'_{n,k}(x),\hat{\beta}_{n,k}(x)=l\overset{\frown}{\beta}'_{n,k}(x)\ \ (n=1,2,3,4)\ (k=0,1,2,...). \tag{55}$$

Based on (50), (51), (55), it can be seen that $\tilde{\beta}_{n,k}(x),\ \overset{\frown}{\beta}_{n,k}(x),\ \hat{\beta}_{n,k}(x)$ $(n=1,2,3,4)\ (k=0,1,2,...)$ functions are non-dimensional. So the functions $\tilde{X}_n(x),\overset{\frown}{X}_n(x)$, $\hat{X}_n(x)\ (n=1,2,3,4)$ will also be non-dimensional.

Using (52), (54) we'll get the final formulas for amplitude parameters of free transverse oscillation of the rod, expressed in terms of initial parameters and non-dimensional function:

$$v(x)=v(0)X_1(x)+\varphi(0)lX_2(x)-M(0)\frac{l^2}{EI}X_3(x)-Q(0)\frac{l^3}{EI}X_4(x); \tag{56}$$

$$\varphi(x)=v(0)\frac{1}{l}\tilde{X}_1(x)+\varphi(0)\tilde{X}_2(x)-M(0)\frac{l}{EI}\tilde{X}_3(x)-Q(0)\frac{l^2}{EI}\tilde{X}_4(x); \tag{57}$$

$$M(x)=-v(0)\frac{EI}{l^2}\overset{\frown}{X}_1(x)-\varphi(0)\frac{EI}{l}\overset{\frown}{X}_2(x)+M(0)\overset{\frown}{X}_3(x)+Q(0)l\overset{\frown}{X}_4(x); \tag{58}$$

$$\begin{aligned}Q(x)=-v(0)\left(\frac{EI}{l^3}\hat{X}_1(x)+\frac{qx}{l}\tilde{X}_1(x)\right)-\varphi(0)\left(\frac{EI}{l^2}\hat{X}_2(x)+qx\,\tilde{X}_2(x)\right)+\\+M(0)\left(\frac{1}{l}\hat{X}_3(x)+\frac{ql\,x}{EI}\tilde{X}_3(x)\right)+Q(0)\left(\hat{X}_4(x)+\frac{ql^2x}{EI}\tilde{X}_4(x)\right).\end{aligned} \tag{59}$$

It is important to note, that dimensional of constant coefficients in non-dimensional functions in the right parts of formulas (56)–(59) are same as the appropriate left parts.

Realized specified boundary condition with the help of formulas (56)–(59), we'll get the frequency equation in terms of unknown non-dimensional parameter K . Found out the frequency equation roots $K_1, K_2, K_3, \ldots$, we'll have the frequency spectrum in according to formula (53)

$$p_j = \frac{K_j}{l^2}\sqrt{\frac{EI}{m}}\ (j = 1, 2, 3, \ldots) .$$

Appropriate fundamental modes are given by formula (56).

3 Conclusions

In this work, the exact solution of differential partial derivative equation of free transverse oscillation of a rod has been attained. Consequently, the formulas for dynamic variables of rode state (deflections, angular displacement, bending moment and transverse force) expressed in terms of non-dimensional fundamental solutions, have been defined. It made possible to attain the analytical form for equation of oscillation frequency.

In prospect, the exact solutions will allow to investigate the free oscillations of the rod taking into account own weight with any boundary conditions on their ends.

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