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# ON THE JUSTIFICATION OF APPROXIMATE SOLUTIONS OF INTEGRAL EQUATIONS IN THE CONTACT PROBLEMS OF THE THEORY OF ELASTICITY

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**The approximate unbounded solution of a class of singular integral equations is fully justified and the error is estimated. Thus is the class to which lot of mixed plane boundary value problems are reducible.**

## 1. Introduction

The method of integral equation formulation of mixed boundary value problems consists of defining the extension of one partially imposed condition (of one of the boundaries). This extension most be compatible with the other where it is imposed. It constitute the unknown of the integral equation and the points where the boundary conditions change define the end points of the principal interval. At such points some physical quantities like the heat flow or the normal stress (in the contact problems of the theory of elasticity) are permitted to be unbounded. In fact, the solutions that provide these singularities are great physical importance. For example it is unbounded normal stress at these points which causes the wear in the contact problems. On searching for such solutions, it had been shown [1, 2] that lot of mixed plane problems can be reduced to a discrete Riemann problem of the form

$$n\Phi_{n+} + nF_{n-} = \operatorname{sgn}\left(n+\tfrac{1}{2}\right)\Phi_{n-} + \Gamma_n\Phi_{n-}, \qquad n \in Z - \{0\}, \qquad (1.1)$$

where $\Phi_{n-}$ are the Fourier components of the required unbounded extension $\varphi_-(x)$ that differs than zero on an open interval $\Delta_1 \subset [-\pi,\pi]$, $\Phi_{n+}$ are those of $\varphi_+(x)$ the extensions of the complementary condition which vanishes on $\Gamma_1 = \Delta_1$ and usually is continuous on

$\Delta_2 \subset [-\pi,\pi]/\Delta_1$, $F_{n-}$ are the components of the external source $f(x)$ at $\Delta_1$ which is externded to zero on $\Delta_2$ and the factors $\Gamma_n$ differ from problem to another but always

$$\Gamma_n = o\left(\frac{1}{n^2}\right) \tag{1.2}$$

If the problem is a mixed Sturm-Liouville one, $F_{n-}$ are simply zeroes while $\varphi_{\pm}$, $\Phi_{n\pm}$ are functions of $\gamma$, the parameter of the separation of variables. The restriction of the inverse Fourier transform of (1.1) leads in general to the singular integral equation

$$\frac{1}{\pi}\int_{-c}^{c}\frac{\varphi_-(t)}{1-e^{i(x-t)}}\,dt = \frac{1}{2\pi}\int_{-c}^{c}\gamma(x-t)\varphi_-(t)dt + i\,f'(x), \tag{1.3}$$

where

$$\frac{1}{2\pi}\int_{-c}^{c}\gamma(x-t)\varphi_-(t)dt = \sum_{n=-\infty}^{\infty}{}'\,\Gamma_n\Phi_{n-}e^{inx}, \tag{1.4}$$

and the prime over the summation symbols means that the value $n=0$ is not included. Without loss of generalisation, $\Delta_1$ is taken such that $\pm c$ designate its end points. It should be noted that equation (1.1) does not involve $\Phi_{o-}$. In fact, the quantity $\operatorname{sgn}\left(n+\frac{1}{2}\right)-\Gamma_n$ is always proportional to $n$ [1, 2]. This means that equation (1.1) sufficient to define at most the derivative $\varphi_-(x)$. Indeed, the solution of equation (1.3) includes an arbitrary constant $a_o$ as a reflection and consequently, different values of the constant $a_o$ lead to different values for the component $\Phi_{o-}$. To fix this indeterminacy we choose the value of $a_o$ such that the solution of equation (1.3) is equivalent to that would be obtained originally through the solution of the mixed problem. To illustrate, if $u$ is the required solution of the mixed problem and $f(x)$ is the Dirichlet data prescribed on $\Delta_1$, then our condition of equivalency becomes

$$u|_{x\in\Delta_1} = u(\varphi_-(x)) = f(x) \qquad (1.5)$$

In practice, condition (1.4) is usually expressed in terms of just one point on $\Delta_1$, namely the point involved by all problems similar to that considered and obtained by changing the points where the boundary condition changes on $\Delta = \Delta_1 \cup \Delta_2 = [-\pi, \pi]$.

It is clear that the inversion of equation (1.3) defines $\varphi_-(x)$ in terms of an infinite number of unknowns: its Fourier components and therefore this equation can be general be only approximately solved. This approximation is usually achieved by truncating the summation in equation (1.4). Thus, it is necessary to justify the truncation and estimate the resulting error.

In section II of this paper we use a theorem of Cherskii [2] to justify the approximate unbounded solution obtained by truncating equation (1.3), to show that this approximate solution approaches the unique exact solution when the order of the truncation increases indefinitely, and estimate the error resulting due to the truncation. This theorem can be summarised as follows:

THEOREM. *Let the following conditions be fulfilled*

1. *The approximate equation* $\widetilde{k}\widetilde{\varphi} = \widetilde{f}$ *has the unique solution* $\widetilde{\varphi}$.
2. $f - \widetilde{f} \in Y_o$, *where* $Y_o$ *is a linear subset,* $Y_o \subset Y$.
3. *Operator* $K - \widetilde{K}$ *is acting from* $X$ *into* $Y$.
4. *The inverse operator* $\widetilde{K}^{-1}$, *acting from* $Y_o$ *into* $X_o \subset X$, *is determined.*
5. $\| \widetilde{K}^{-1}(K - \widetilde{K}) \| < 1$.

*Then the equation* $K\varphi = f$, *where* $K$ *acts from* $X$ *into* $Y$, *has the unique solution*

$$\varphi = \widetilde{\varphi} + [I + \widetilde{K}^{-1}(K - \widetilde{K})]^{-1}\widetilde{K}^{-1}(f - K\varphi),$$

*and the following estimate holds*

$$\| \varphi - \widetilde{\varphi} \|_{X_o} \le \frac{\| \widetilde{K}^{-1}(f - K\varphi \|_{X_o}}{1 - \| \widetilde{K}^{-1}(K - \widetilde{K})}.$$

In section III we consider a simple example, reduce it to a form of equation (3.1) and carry out the procedures right to the numerical results.

## II. The Unbounded Solution

Equation (1.3) has a solution in the class of integrable functions, that is for which $\varphi_-(x)$ is subject at $\pm c$ to the estimation

$$|\varphi_-(x)| < \frac{\text{const}}{|(e^{ix}-e^{ic})(e^{ix}-e^{-ic})|^{\lambda}}, \qquad 0 \le \lambda < 1. \tag{2.1}$$

This solution [4] can be written in the form

$$\varphi_-(t) = \frac{1}{R(x)}\left[ m(x) + a_o + \frac{1}{2\pi^2}\int_{-c}^{c}\frac{R(t)e^{it}\,dt}{e^{it}-e^{ix}}\int_{-c}^{c}\gamma(t-y)\varphi_-(y)dy \right], \tag{2.2}$$

where

$$R(x) = -e^{ix/2}\sqrt{2(\cos x - \cos c)}, \tag{2.3}$$

$$m(x) = -\frac{1}{\pi i}\int_{-c}^{c}\frac{f'(t)R(t)e^{it}}{e^{it}-e^{ix}}dt, \qquad a_o \text{ is a constant} \tag{2.4}$$

Equation (2.2) is in turn an integral one that can be rewritten in the operator form

$$K\varphi_- \equiv \varphi_-(x) - \frac{1}{2\pi^2}\frac{1}{R(x)}\int_{-c}^{c}\frac{R(t)e^{it}\,dt}{e^{it}-e^{ix}}\gamma(t-y)\varphi_-(y)dy = g(x), \tag{2.5}$$

where

$$g(x) = \frac{1}{R(x)}[m(x) + a_o], \tag{2.6}$$

but since every solution of the equation

$$\frac{1}{\pi i}\int_{\Gamma}\frac{\psi(\tau)d\tau}{\tau - t} = h(t)$$

belongs to the class $L_p(\Gamma)$ where $p<\frac{4}{3}$, if $h(t)\in L_r(\Gamma)$; $r>\frac{4}{3}$ [5], then if $f'(x)\in L_r[-c,c]$; $r>\frac{4}{3}$ it follows that $g(x)\in L_p[-c,c]$; $1<p<\frac{4}{3}$ and we can therefore set

$$X = X_o = Y = Y_o = L_p[-c,c]; \qquad\qquad 1<p<\tfrac{4}{3} \tag{2.7}$$

The "approximate" operator $\widetilde{K}$ is determined by the equality

$$\widetilde{K}\widetilde{\varphi}_- = \widetilde{\varphi}_-(x) - \frac{1}{2\pi^2}\frac{1}{R(x)}\int_{-c}^{c}\frac{R(t)e^{it}dt}{e^{it}-e^{ix}}\int_{-c}^{c}\sum_{n=-N}^{N}\Gamma_n e^{in(t-y)}\widetilde{\varphi}_-(y)dy, \tag{2.8}$$

LEMMA. *For any positive* $\varepsilon$ *we have* $\| K-\widetilde{K}\|<\varepsilon$ *provided* $N$ *is appropriately chosen.*

$$\begin{aligned}
\|(K-\widetilde{K})\varphi_-\|_{L_p}^p &= \int_{-c}^{c}\left|\frac{1}{2\pi^2}\frac{1}{R(x)}\int_{-c}^{c}\frac{R(t)e^{it}dt}{e^{it}-e^{ix}}\int_{-c}^{c}\sum_{|k|>N}\Gamma_k e^{ik(x-y)}\varphi_-(y)dy\right|^p dx\\
&\le (2\pi^2)^{-p}\int_{-c}^{c}\left[\left|\frac{1}{R(x)}\int_{-c}^{c}\frac{R(t)e^{it}dt}{e^{it}-e^{ix}}\right|\sum_{|k|>N}|\Gamma_k|\int_{-c}^{c}|\varphi_-(y)|dy\right]^p dx\\
&\le (2c)^{p/q}(2\pi)^{(-p)}\left(\sum_{|k|>N}|\Gamma_k|\right)^p\int_{-c}^{c}\left|\frac{\cos c-e^{ix}}{R(x)}\right|^p dx\,\|\varphi_-\|_{L_p}^p\\
&\le (2c)^{p/q}(2\pi)^{-p}(1+|\cos c|)^p\left(\sum_{|k|>N}|\Gamma_k|\right)^p\\
&\quad\times\int_{-c}^{c}\frac{dx}{|R(x)|^p}\|\varphi_-\|_{L_p}^p,
\end{aligned}$$

where $q=\frac{p}{p-1}$, moreover we have

$$
\begin{aligned}
I_p &= \int_{-c}^{c}\frac{dx}{|R(x)|^p} = \int_{-c}^{c}\frac{dx}{2^p\left(\sin\frac{c-x}{2}\sin\frac{c+x}{2}\right)^{p/2}} \\
&\leq \left(\frac{\pi}{2}\right)^p\int_{-c}^{c}\frac{dx}{(e^2-x^2)^{p/2}} \\
&\leq 2\left(\frac{\pi}{2}\right)^p\int_{0}^{\pi/2}\frac{a\cos t dt}{(e^2-c^2\sin^2 t)^{p/2}} \\
&= 2\left(\frac{\pi}{2}\right)^p c^{1-p}\int_{0}^{\pi/2}(\cos t)^{1-p}dt,
\end{aligned}
$$

but since $1 < p < 2$, then [6]

$$
\int_{0}^{\pi/2}(\cos t)^{1-p}dt = \frac{\pi\Gamma(2-p)}{2^{2-p}\left[\Gamma\left(\frac{3}{2}-\frac{p}{2}\right)\right]^2},
$$

and we finally find

$$
\|K-\widetilde{K}\| \leq \frac{(1+|\cos c|)\pi^{1/p}\Gamma^{1/p}(2-p)}{2^{2/p}\Gamma^{2/p}\left(\frac{3}{2}-\frac{p}{2}\right)}\sum_{|k|>N}|\Gamma_k| = \gamma(N), \tag{2.9}
$$

Which can made arbitrary small in view of (1.2). The lemma has been proved.

The exact solution of the approximate equation

$$
\widetilde{K}\widetilde{\varphi}_- = \widetilde{g}(x), \tag{2.10}
$$

can be written in the form

$$
\widetilde{\varphi}_-(x) = \frac{1}{R(x)}\sum_{k=-N}^{N}\Gamma_k\widetilde{\Phi}_k\alpha_k(x) + \widetilde{g}(x), \tag{2.11}
$$

where

$$
\left.\begin{array}{lcl}
\alpha_k(x) & = & \frac{1}{\pi}\int_{-c}^{c}\frac{R(t)e^{i(k+1)t}}{e^{it}-e^{ix}}dt \\
\widetilde{g}(x) & = & \frac{1}{R(x)}\,|\,m(x)+\widetilde{a}_o\,|.
\end{array}\right\} \tag{2.12}
$$

Applying the finite Fourier transform to equation (2.11) we get the following algebraic system

$$
\widetilde{\Phi}_{k-} = M_n + \sum_{k=-N}^{N}\Gamma_k\widetilde{\Phi}_{k-}N_{kn} + \widetilde{a}_o R_N, \qquad (n=0,\pm1,\ldots,\pm N), \tag{2.13}
$$

where

$$
\left.\begin{array}{lcl}
M_n & = & \frac{1}{2\pi}\int_{-c}^{c}\frac{m(x)e^{-inx}}{R(x)}dx, \\
N_{kn} & = & \frac{1}{2\pi}\int_{-c}^{c}\frac{\alpha_k(x)e^{-inx}}{R(x)}dx, \\
R_n & = & \frac{1}{2\pi}\int_{-c}^{c}\frac{e^{-inx}}{R(x)}dx
\end{array}\right\} \tag{2.14}
$$

The explicit expressions of the above integrals are [7]

$$
\alpha_k(x) = \begin{cases} -e^{ikx}\sum_{m=0}^{k+1}\mu_m(\cos c)e^{-i(m-1)x}, & k\geq 0 \\ e^{ikx}\sum_{m=0}^{-k-1}\mu_m(\cos c)e^{imx}, & k<0 \end{cases} \tag{2.15}
$$

$$
N_{kn} = \begin{cases} -\frac{1}{2}\sum_{m=0}^{k+1}\mu_{k-m+1}(\cos c)P_{m-n+1}(\cos c), & k\geq 0 \\ \frac{1}{2}\sum_{m=0}^{-k-1}\mu_{-k-m+1}(\cos c)P_{m+n+1}(\cos c), & k<0 \end{cases} \tag{2.16}
$$

$$
R_n = \tfrac{1}{2}P_n(\cos c), \tag{2.17}
$$

where $P_n(\cos c)$ are Legendre polynomials and

$$\mu_n(\cos c) = P_n(\cos c) - 2\cos c P_{n-1}(\cos c) + P_{n-2}(\cos c), \qquad (2.18)$$

Rewriting system (2.13) in the form

$$\widetilde{\Phi}_{k-} - \sum_{k=-N}^{N} \Gamma_k \widetilde{\Phi}_{k-} N_{kn} = \widetilde{G}_n, \qquad (n = 0, \pm 1, \dots, \pm N), \qquad (2.19)$$

its solution can be written down as

$$\widetilde{\Phi}_{k-} = \frac{\Delta^{(n)}}{\Delta} = \sum_{j=-N}^{N} \widetilde{G}_j \Delta_j^{(n)} = \frac{1}{2\pi\Delta} \sum_{j=-N}^{N} \widetilde{g}(x) e^{-ijx} dx, \qquad (2.20)$$

where $\Delta$ is the determinant of the system and $\Delta_j^{(n)}$ is the cofactor of the element $\widetilde{G}_j$ in the determinant $\Delta^{(n)}$.

LEMMA. *If* $\Delta \neq 0$, *then the inverse operator* $\widetilde{K}^{-1}$ *is bounded and moreover*

$$\| \widetilde{K}^{-1} \| \leq 1 + \frac{(2e)^{1/q} I_p^{1/p}}{2\pi |\Delta|} \left[ \sum_{n=0}^{N} |\Gamma_n| \sum_{m=0}^{n+1} |\mu_m(\cos c)| \right.$$

$$\left. + \sum_{n=-N}^{-1} |\Gamma_n| \sum_{j=-N}^{N} |\Delta_j^{(n)}| \sum_{m=0}^{-n-1} \mu_m(\cos c) | \right] = R(N). \qquad (2.21)$$

To verify we have

$$\| \widetilde{\varphi}_- \|_{L_p} = \| \widetilde{K}^{-1} g \|_{L_p} = \left( \int_{-c}^{c} \left| \sum_{n=-N}^{N} \Gamma_n \widetilde{\Phi}_{n-} \frac{\alpha_n(x)}{R(x)} + \widetilde{g}(x) \right|^p dx \right)^{1/p}$$

$$\leq \sum_{n=-N}^{N} \left( \left( \int_{-c}^{c} \left| \Gamma_n \widetilde{\Phi}_n \frac{\alpha_n(x)}{R(x)} \right|^p dx \right) + \left( \int_{-c}^{c} | \widetilde{g}(x) |^p dx \right) \right)^{1/p}$$

$$\leq \sum_{n=-N}^{N} |\Gamma_n| \| \widetilde{\Phi}_{n-} | \left( \int_{-c}^{c} \left| \frac{\alpha_n(x)}{R(x)} \right|^p dx \right)^{1/p} + \| \widetilde{g} \|_{L_p}$$

$$\leq \; I_p^{1/p}\left[\sum_{n=0}^{N}|\Gamma_n\|\widetilde{\Phi}_{n-}|\sum_{m=0}^{n+1}|\mu_m(\cos c)|\right.$$

$$\left.+\;\sum_{n=-N}^{-1}|\Gamma_n\|\widetilde{\Phi}_n|\sum_{m=0}^{-n-1}|\mu_m(\cos c|\right]+\|\widetilde{g}\|_{L_p},$$

using formula (2.20) we obtain

$$\begin{aligned}
\|\widetilde{\Phi}_{n-}\| &\leq \frac{1}{2\pi|\Delta|}\sum_{j=-N}^{N}|\Delta_j^{(n)}|\int_{-c}^{c}|\widetilde{g}(x)|\,dx \\
&\leq \frac{(2e)^{1/q}}{2\pi|\Delta|}\sum_{j=-N}^{N}|\Delta_j^{(n)}|\left(\int_{-c}^{c}|\widetilde{g}(x)|^p\,dx\right)^{1/p} \\
&\leq \frac{(2e)^{1/q}}{2\pi|\Delta|}\sum_{j=-N}^{N}|\Delta_j^{(n)}\|\|\widetilde{g}\|_{L_p} \\
&= \frac{a^{1-1/p}}{2^{1/p}\pi|\Delta\|}\sum_{j=-N}^{N}|\Delta_j^{(n)}\|\|\widetilde{g}\|_{L_p},
\end{aligned}$$

therefore

$$\|\widetilde{K}^{-1}\widetilde{g}\|_{L_p} \leq \frac{(2a)^{1/q}I_p^{1/p}}{2\pi|\Delta|}\left[\sum_{n=0}^{N}|\Gamma_n|\sum_{j=-N}^{N}|\Delta_j^{(n)}|\sum_{m=0}^{n+1}|\mu_m(\cos c)|\right.$$

$$\left.+\;\sum_{n=-N}^{-1}|\Gamma_n|\sum_{j=-N}^{N}|\Delta_j^{(n)}|\sum_{m=0}^{-n-1}|\mu_m(\cos c)|\right]\|\widetilde{g}\|_{L_p}+\|\widetilde{g}\|_{L_p},$$

and inequality (2.21) is proved.

Additionally we have

$$\begin{aligned}
|\alpha_n| &\leq \sum_{m=0}^{n+1}|\mu_m(\cos c)|\leq 1|+|\cos c|+\sum_{m=2}^{n+1}\left|\frac{P_{m-2}(\cos c)-P_m(\cos c)}{2m-1}\right| \\
&\leq 1+|\cos c|+2\sum_{m=2}^{n+1}\frac{1}{m}\leq 1+|\cos c|+2\ln(n+1)-2\ln 2 \\
&\leq B+2\ln(n+1); \qquad\qquad (n\geq 1),
\end{aligned}$$

$$|\alpha_o(x)| \le 1, \; |\alpha_{-1}(x)| \le 1 \text{ and } |\alpha_{-2}(x)| \le 1 + |\cos c|.$$

Similarly

$$\sum_{m=0}^{-n-1} |\mu_m(\cos c)| \le B + 2\ln(-n-1), \qquad (n \le -3), \tag{2.22}$$

where

$$B = 1 + |\cos c| + 2\ln 2. \tag{2.23}$$

Thus

$$\|\widetilde{K}^{-1}\| \le \Bigg\{ 1 + \frac{(2c)^{1/q} I_p^{1/p}}{2\pi|\Delta|} \Bigg[ (1+|\cos c|) \Bigg( \sum_{j=-N}^{N} |\Delta_j^{(o)}| + |\Gamma_{-2}| \sum_{j=-N}^{N} |\Delta_j^{(-2)}| \Bigg)$$

$$+ |\Gamma_{-1}| \sum_{j=-N}^{N} |\Delta_j^{(-1)}| + \sum_{n=1}^{N} |\Gamma_n| [B + 2\ln(n+1)] \sum_{j=-N}^{N} |\Delta_j^{(n)}|$$

$$+ \sum_{n=-N}^{-3} |\Gamma_n| [B + 2\ln(-n-1)] \sum_{j=-N}^{N} |\Delta_j^{(n)}| \Bigg\} \Bigg]. \tag{2.24}$$

further $|N_{kn}|$ can be estimated by quantities of the same order like that given in relation (2.22), thus expanding the determinants $\Delta_j^{(n)}$ in products of $\Gamma_k$ we come to the following estimation [7]

$$\sum_{j=-N}^{N} |\Delta_j^{(n)}| \le \text{const}, \qquad (\forall n), \tag{2.25}$$

moreover this constant does not depend on $N$. These above results together with (2.9) lead simply to the results that

$$\|\widetilde{K}^{-1}\| \|K - \widetilde{K}\| < 1, \tag{2.26}$$

holds true under appropriate choice of $N$.

The above conclusions can be summarised in the following theorem.

THEOREM. *Let in equation* (1.3) *the function* $f'(x) \in L_r[-c,c]$;; $r > \frac{4}{3}$ *and condition* (1.4) *is fulfilled. Then this equation has a unique solution in the class* $L_p$ $1 < p < \frac{4}{3}$. *The function* $\widetilde{\varphi}(x)$ *defined by formula* (2.11) *constitutes an approximate solution of equation* (1.3) *as long as* $R(N)\gamma(N) < 1$ *and the following estimate holds*

$$\| \varphi_- - \widetilde{\varphi}_- \|_{L_p} \leq \frac{R(N) \| g - K\widetilde{\varphi}_- \|_{L_p}}{1 - R(N)\gamma(N)}. \quad (2.27)$$

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