Perturbed rotational motions of a rigid body close to regular precession in the Lagrange case

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PERTURBED ROTATIONAL MOTIONS OF A RIGID BODY THAT ARE CLOSE TO REGULAR PRECESSION IN THE LAGRANGE CASE

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In this paper we investigate perturbed rotational motions of a rigid body that are close to regular precession in the Lagrange case. It is assumed that the angular velocity of the body is large and that its direction is close to the axis of dynamic symmetry, and also that the perturbing moments are small as compared to the restoring ones. A small parameter is introduced in a special manner, and the averaging method is employed. Averaged systems of equations of motion are obtained in the first and second approximations. Particular mechanical models of perturbations are considered.

1. Statement of the problem. Consider motion of a dynamically symmetrical rigid body about fixed point O, under the action of restoring and perturbing moments. The equations of motion (the dynamic and kinematic Euler equations) have the form

$$\begin{gathered} Ap^{\cdot}+(C-A)qr=k\sin\theta\cos\varphi+M_1 \\ Aq^{\cdot}+(A-C)pr=-k\sin\theta\sin\varphi+M_2 \\ Cr^{\cdot}=M_3,\ M_i=M_i(p,\ q,\ r,\ \psi,\ \theta,\ \varphi,\ t)\ \ (i=1,\ 2,\ 3) \\ \psi^{\cdot}=(p\sin\varphi+q\cos\varphi)\operatorname{cosec}\theta,\ \theta^{\cdot}=p\cos\varphi-q\sin\varphi \\ \varphi^{\cdot}=r-(p\sin\varphi+q\cos\varphi)\operatorname{ctg}\theta \end{gathered} \tag{1.1}$$

Dynamic equations (1.1) are written in the projections onto the principal axes of inertia of the body, passing through point O. Here p, q, r are the projections of the angular velocity vector of the body onto these axes; M_i ($i = 1, 2, 3$) are the projections of the vector of the perturbing moment onto these same axes, which are 2π-periodic functions of the Euler angles ψ, θ, ϕ; A and C are the equatorial and axial moments of inertia of the body relative to point O, $A \neq C$. It is assumed that the body is acted upon by a restoring moment, whose maximum value is equal to k and which is created by a force, constant in magnitude and direction, that is applied to some fixed point of the axis of dynamic symmetry. In the case of a heavy top we have $k = mgl$, where m is the mass of the boyd; g is the acceleration due to gravity; and l is the distance from fixed point O to the center of gravity of the body.

The perturbing moments M_i in (1.1) are assumed to be known functions of their arguments. When there are no perturbations ($M_i = 0$, $i = 1, 2, 3$), Eqs. (1.1) correspond to the Lagrange case.

Equations (1.1) can describe the motions of a Lagrange top under perturbations of various physical origin, as well as the motions of a free rigid body relative to the center of mass when this body is acted upon by a restoring moment due to aerodynamic forces, and also by perturbing moments.

The following initial assumptions are made:

$$p^2+q^2\ll r^2,\ Cr^2\gg k,\ |M_i|\ll k\ \ (i=1,\ 2,\ 3) \tag{1.2}$$

Assumptions (1.2) mean that the direction of the angular velocity of the body is close to the axis of dynamic symmetry; the angular velocity is large enough that the kinetic energy of the body is much greater than the potential energy resulting from the restoring moment; and the perturbing moments are small as compared to the restoring ones. Inequalities (1.2) enable us to introduce a small parameter $\varepsilon \ll 1$ and to set

$$p=\varepsilon P,\ q=\varepsilon Q,\ k=\varepsilon K$$
$$M_i=\varepsilon^2 M_i^*(P, Q, r, \psi, \theta, \varphi, t)\ (i=1, 2, 3) \tag{1.3}$$

Paper [1] also considered motions of a heavy rigid body similar to the Lagrange case. It was assumed that the body is acted upon by small perturbing moments that satisfy certain additional conditions. Then paper [1] proceeded to average the equations of motion with respect to the angle of nutation, and to give the results of numerical integration of the resultant averaged system for the case of linear dissipative perturbing moments. In contrast to [1], we will consider the case of a body that rotates rapidly about the axis of dynamic symmetry, and therefore the generating solution is not the trajectory of motion in the Lagrange case, but rather some simpler solution. As a result, using the averaging method in the first and second approximations we are able to obtain exact analytic solutions.

In [2], as in this paper, it is assumed that the angular velocity is large, and that its direction is close to the axis of dynamic symmetry of the body. In contrast to the third inequality in (1.2), it was assumed in [2] that two projections of the vector of the perturbing moment onto the principal axes of inertia of the body are small as compared to the restoring moment, while the third is of the same order of magnitude.

The new variables P and Q, as well as the variables and constants r, ψ, θ, ϕ, K, A, C, M_1^*, are assumed to be bounded variables of order unity as $\varepsilon \to 0$. We pose the problem of investigating the asymptotic behavior of system (1.1) for small ε, if conditions (1.2) and (1.3) are observed. We will employ the averaging method [3,4] on a time interval of order ε^{-1}.

We should note that the averaging method has been extensively employed in problems of rigid-body dynamics. In [5,6] the method was used to investigate a variety of problems in dynamics, chiefly for bodies displaying dynamic symmetry; paper [7] first performed averaging with respect to Euler-Poinsot motion for an asymmetrical body; while studies [1,2,6,8,9] investigated perturbed motions similar to Lagrange motion. The ensemble of simplifying assumptions (1.2) or (1.3) enables us to obtain a relatively simple averaging scheme in the general case, and to investigate a variety of examples.

2. **Averaging procedure.** We make the change of variables (1.3) in system (1.1). Cancelling ε on both sides of the first two equations in (1.1), we obtain

$$\begin{gathered}
AP^{\cdot}+(C-A)Qr=K\sin\theta\cos\varphi+\varepsilon M_1^*\\
AQ^{\cdot}+(A-C)Pr=-K\sin\theta\sin\varphi+\varepsilon M_2^*\\
Cr^{\cdot}=\varepsilon^2 M_3^*,\ \psi^{\cdot}=\varepsilon(P\sin\varphi+Q\cos\varphi)\operatorname{cosec}\theta\\
\theta^{\cdot}=\varepsilon(P\cos\varphi-Q\sin\varphi),\ \varphi^{\cdot}=r-\varepsilon(P\sin\varphi+Q\cos\varphi)\operatorname{ctg}\theta
\end{gathered} \tag{2.1}$$

Let us first consider the zero-approximation system; we set $\varepsilon = 0$ in (2.1). Then the last four equations in (2.1) yield

$$r=r_0,\ \psi=\psi_0,\ \theta=\theta_0,\ \varphi=r_0t+\varphi_0 \tag{2.2}$$

Here r_0, ψ_0, θ_0, ϕ_0 are constants that are equal to the initial values of the corresponding variables for $t = 0$. We substitute (2.2) into the first two equations of system (2.1) for $\varepsilon = 0$, and we integrate the resultant system of two linear equations for P and Q. The solution can be written as follows:

$$\begin{gathered}
P=a\cos\gamma_0+b\sin\gamma_0+KC^{-1}r_0^{-1}\sin\theta_0\sin(r_0t+\varphi_0)\\
Q=a\sin\gamma_0-b\cos\gamma_0+KC^{-1}r_0^{-1}\sin\theta_0\cos(r_0t+\varphi_0)\\
a=P_0-KC^{-1}r_0^{-1}\sin\theta_0\sin\varphi_0,\ b=-Q_0+KC^{-1}r_0^{-1}\sin\theta_0\cos\varphi_0\\
\gamma_0=n_0t,\ n_0=(C-A)A^{-1}r_0\neq0,\ |n_0/r_0|\leqslant1
\end{gathered} \tag{2.3}$$

Here P_0, Q_0 are the initial values of the new variables P and Q, introduced in accordance with (1.3), while the variable $\gamma = \gamma_0$ is interpreted as the phase of the oscillations. System (2.1) is essentially nonlinear (the frequency of natural oscillations of the variables P and Q depends on the slow variable r), and therefore in what follows we introduce the additional variable γ, defined by the equation

$$\gamma^{\cdot}=n,\ \gamma(0)=0,\ n=(C-A)A^{-1}r \tag{2.4}$$

For $\varepsilon = 0$ we have $\gamma = \gamma_0 = n_0 t$ in accordance with (2.3). Equations (2.2) and (2.3) define the general solution of system (2.1), (2.4) for $\varepsilon = 0$. By eliminating the constants with allowance for (2.2), we can rewrite the first two equations in (2.3) in equivalent form:

$$\begin{aligned} P&=a\cos\gamma+b\sin\gamma+KC^{-1}r^{-1}\sin\theta\sin\varphi \\ Q&=a\sin\gamma-b\cos\gamma+KC^{-1}r^{-1}\sin\theta\cos\varphi \end{aligned} \tag{2.5}$$

We solve (2.5) for a and b:

$$\begin{aligned} a&=P\cos\gamma+Q\sin\gamma-KC^{-1}r^{-1}\sin\theta\sin(\gamma+\varphi) \\ b&=P\sin\gamma-Q\cos\gamma+KC^{-1}r^{-1}\sin\theta\cos(\gamma+\varphi) \end{aligned} \tag{2.6}$$

We introduce the new variable δ as follows:

$$r=r_0+\varepsilon\delta \tag{2.7}$$

Now we return to system (2.1) for $\varepsilon \neq 0$; we will consider (2.5)-(2.7) as change-of-variable formulas that define the conversion from variables P, Q, r to variables a, b, δ (these formulas also include the new variable γ). Using these formulas, we changeover in system (2.1), (2.4) from the variables P, Q, r, ψ, θ, ϕ, γ to the new variables a, b, δ, ψ, θ, α, γ, where

$$\alpha=\gamma+\varphi \tag{2.8}$$

After some manipulations, we obtain a seven-equation system (instead of the six-equation system (2.1)) that is more convenient for subsequent investigation:

$$\begin{gathered} a^{\cdot}=\varepsilon A^{-1}(M_1^0\cos\gamma+M_2^0\sin\gamma)-\varepsilon KC^{-1}r_0^{-1}\cos\theta\,(b- \\ -KC^{-1}r_0^{-1}\sin\theta\cos\alpha)+\varepsilon^2KC^{-1}r_0^{-2}\delta\cos\theta\,(b-2KC^{-1}r_0^{-1}\sin\theta\cos\alpha)+ \\ +\varepsilon^2KC^{-2}r_0^{-2}M_3^0\sin\theta\sin\alpha \\ b^{\cdot}=\varepsilon A^{-1}(M_1^0\sin\gamma-M_2^0\cos\gamma)+\varepsilon KC^{-1}r_0^{-1}\cos\theta\,(a+ \\ +KC^{-1}r_0^{-1}\sin\theta\sin\alpha)-\varepsilon^2KC^{-1}r_0^{-2}\delta\cos\theta\,(a+ \\ +2KC^{-1}r_0^{-1}\sin\theta\sin\alpha)-\varepsilon^2KC^{-2}r_0^{-2}M_3^0\sin\theta\cos\alpha \\ \delta^{\cdot}=\varepsilon C^{-1}M_3^0 \\ \psi^{\cdot}=\varepsilon\operatorname{cosec}\theta\,(a\sin\alpha-b\cos\alpha)+\varepsilon KC^{-1}r_0^{-1}-\varepsilon^2KC^{-1}r_0^{-2}\delta \\ \theta^{\cdot}=\varepsilon(a\cos\alpha+b\sin\alpha) \\ \alpha^{\cdot}=CA^{-1}r_0+\varepsilon CA^{-1}\delta-\varepsilon\operatorname{ctg}\theta\,(a\sin\alpha-b\cos\alpha)- \\ -\varepsilon KC^{-1}r_0^{-1}\cos\theta+\varepsilon^2KC^{-1}r_0^{-2}\delta\cos\theta \\ \gamma^{\cdot}=n_0+\varepsilon(C-A)A^{-1}\delta \end{gathered} \tag{2.9}$$

Here the M_i^0 denote functions obtained from M_i^* (see (1.3)) as a result of substitution (2.5)-(2.8):

$$M_i^0(a,b,\delta,\psi,\theta,\alpha,\gamma,t)=M_i^*(P,Q,r,\psi,\theta,\varphi,t)\quad(i=1,2,3) \tag{2.10}$$

It should be pointed out that the changeover from the two variables P, Q to the three variables a, b, γ stems from considerations of convenience; for $\varepsilon = 0$ the system for P, Q is linear, while change (2.5) is nonsingular for all a, b.

System (2.9) can be brought to the form

$$\begin{aligned} x^{\cdot}&=\varepsilon F_1(x,y)+\varepsilon^2F_2(x,y),\ x(0)=x_0 \\ y^{1\cdot}&=\omega_1+\varepsilon g_1(x,y)+\varepsilon^2g_2(x,y),\ y^1(0)=y^{10} \\ y^{2\cdot}&=\omega_2+\varepsilon h_1(x,y)+\varepsilon^2h_2(x,y),\ y^2(0)=y^{20} \end{aligned} \tag{2.11}$$

where vector-valued function $x=(x^1,\ldots,x^5)$ is comprised of the slow variables a, b, δ, ψ, θ; here y^1 and y^2 denote the fast variables α, γ; and ω_1, ω_2 are constant phases, equal to $CA^{-1}r_0$ and $(C-A)A^{-1}r_0$ respectively. Vector-valued functions F_i, g_i, h_i $(i=1,2)$ are defined by the right sides of (2.9).

We denote two-dimensional vector (g_1, h_1) by Z_1. Here and henceforth, we will assume that the perturbing moments M_i^* are independent of t. Since M_i^* ($i = 1, 2, 3$) are 2π-periodic in ϕ, it follows that, in accordance with change (2.5), (2.6), (2.8), functions M_i^0 from (2.10) will be 2π-periodic functions of α and γ.

In accordance with the familiar procedure of [4] for constructing the asymptotic form of system (2.11), we will seek a change of variables

$$x=x^*+\varepsilon u_1(x^*, y^*)+\varepsilon^2 u_2(x^*, y^*)+\ldots$$
$$y=y^*+\varepsilon v_1(x^*, y^*)+\varepsilon^2 v_2(x^*, y^*)+\ldots$$
$$y=(y^1, y^2),\ x^*=(x^{*1},\ldots,x^{*5}),\ y^*=(y^{*1}, y^{*2})$$

such that system (2.11) in the new variables (x^*, y^*) assumes the form

$$x^{*\cdot}=\varepsilon A_1(x^*)+\varepsilon^2 A_2(x^*)+\ldots$$
$$y^{*\cdot}=\omega+\varepsilon B_1(x^*)+\varepsilon^2 B_2(x^*)+\ldots,\ \omega=(\omega_1, \omega_2) \tag{2.12}$$

For this it is necessary to appropriately choose functions u_1, u_2, v_1, v_2 that define the change of variables. It is known [4] that the equations for vector-valued functions u_1, v_1 have the form

$$\omega\, \partial u_1/\partial y^*=F_1(x^*, y^*)-A_1(x^*)$$
$$\omega\, \partial v_1/\partial y^*=Z_1(x^*, y^*)-B_1(x^*) \tag{2.13}$$

where $(\partial f/\partial x)$ is the matrix of partial derivatives $\|\partial f_i/\partial x^j\|$ $(i, j=1,\ldots,5)$. Functions $A_1(x^*)$, $B_1(x^*)$ are given by the formulas

$$A_1(x^*)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_1(x^*, y^*)dy^{*1}\,dy^{*2},\quad B_1(x^*)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} Z_1(x^*, y^*)dy^{*1}\,dy^{*2} \tag{2.14}$$

Function $u_2(x^*, y^*)$ should be the solution of the equation

$$\frac{\partial u_2}{\partial y^*}\omega=F_2(x^*, y^*)+\frac{\partial F_1}{\partial x^*}u_1+\frac{\partial F_1}{\partial y^*}v_1-\frac{\partial u_1}{\partial x^*}A_1(x^*)-\frac{\partial u_1}{\partial y^*}B_1(x^*)-A_2(x^*) \tag{2.15}$$

Function $A_2(x^*)$ is given by the formula

$$A_2(x^*)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Big(F_2(x^*, y^*)+\frac{\partial F_1}{\partial x^*}u_1+\frac{\partial F_1}{\partial y^*}v_1-\frac{\partial u_1}{\partial x^*}A_1(x^*)-$$
$$-\frac{\partial u_1}{\partial y^*}B_1(x^*)\Big)dy^{*1}\,dy^{*2} \tag{2.16}$$

Let us determine the averaged system of first-approximation equations for the slow variables

$$x_1^{*\cdot}=\varepsilon A_1(x_1^*),\ x_1^*(0)=x_{10} \tag{2.17}$$

and also the second-approximation system for the slow variables

$$x_2^{*\cdot}=\varepsilon A_1(x_2^*)+\varepsilon^2 A_2(x_2^*),\ x_2^*(0)=x_{20} \tag{2.18}$$

and the second-approximation system for the fast variables

$$y_2^{*\cdot}=\omega+\varepsilon B_1(x_1^*(t)),\ y_2^*(0)=y^0,\ y^0=(y^{10}, y^{20}) \tag{2.19}$$

which can be integrated directly:

$$y_2^*(t)=\omega t+y^0+\varepsilon\int_0^t B_1(x_1^*(s))\,ds \tag{2.20}$$

To investigate the second-approximation system (2.18), it is convenient to make the change of independent variable $\tau = \varepsilon t$. Then system (2.18) becomes

$$dx_2^*/d\tau = A_1(x_2^*) + \varepsilon A_2(x_2^*) \tag{2.21}$$

Here the time interval $(0, T/\varepsilon)$, on which the solutions of the initial system (2.11) are being considered, becomes the interval $(0, T)$, which is independent of the small parameter ε. We will seek the solution of system (2.11) in the form

$$x_2^*(\tau) = x^{(1)}(\tau) + \varepsilon x^{(2)}(\tau) + O(\varepsilon^2) \tag{2.22}$$

Substituting expansion (2.22) into (2.21), we obtain the following systems of equations for vector-valued functions $x^{(i)}(\tau) = x_i(t)$ $(\tau = \varepsilon t,\ i = 1, 2)$:

$$dx^{(1)}/d\tau = A_1(x^{(1)}), \quad x^{(1)}(0) = x_0 \tag{2.23}$$

$$dx^{(2)}/d\tau = A_1'(x^{(1)}(\tau))x^{(2)} + A_2(x^{(1)}(\tau)), \quad x^{(2)}(0) = 0 \tag{2.24}$$

where A_1' is the matrix of partial derivatives of the components of vector-valued functions $A_1(x)$: $A_1'(x) = \|\partial A_1^i/\partial x^j\|$. System (2.23) is linear, and therefore in a number of cases it is simpler to employ it than to investigate system (2.21).

We denote by $X(\tau, c)$ the general solution of the first-approximation system (2.23):

$$X_\tau' = A_1(X), \quad X(0, c) = c = x_0 \tag{2.25}$$

Then we obtain the following expressions for functions $x^{(1)}(\tau)$, $x^{(2)}(\tau)$:

$$x^{(1)}(\tau) = X(\tau, x_0),\ x^{(2)}(\tau) = \Phi(\tau)\int_0^\tau \Phi^{-1}(\tau_1)\eta(\tau_1)\,d\tau_1 \tag{2.26}$$

Here Φ is the fundamental matrix of the homogeneous equation corresponding to the second approximation:

$$\Phi(\tau) = \|\partial X(\tau, c)/\partial c\|_{c=x_0}, \quad \eta(\tau) = A_2(x^{(1)}(\tau)) = A_2(X(\tau, x_0))$$

We define the vector-valued functions

$$\begin{gathered} x_\varepsilon^\sim(t) = x^{(1)}(\varepsilon t) + \varepsilon x^{(2)}(\varepsilon t) + \varepsilon u_1\Big(x^{(1)}(\varepsilon t),\ y^0 + \omega t + \varepsilon\int_0^t B_1(x^{(1)}(\varepsilon s))\,ds\Big) \\ y_\varepsilon^\sim(t) = y^0 + \omega t + \varepsilon\int_0^t B_1(x^{(1)}(\varepsilon s))\,ds \end{gathered} \tag{2.27}$$

Theorem. There exists a set L of measure zero on the (ω_1, ω_2) plane such that, if ω does not belong to L, then Eqs. (2.13) and (2.15) are solvable (and hence the above formal scheme for constructing functions $x_\varepsilon^\sim(t)$, $y_\varepsilon^\sim(t)$ is meaningful), and we have the inequalities

$$|x_\varepsilon^\sim(t) - x(t)| \le C_1\varepsilon^2, \quad |y_\varepsilon^\sim(t) - y(t)| \le C_1\varepsilon, \quad t \in [0, T\varepsilon^{-1}] \tag{2.28}$$

the constant $C_1 > 0$ being independent of ε. The theorem can be proved on the basis of the standard procedure of change of variables of the averaging method [4], as well as on the basis of an arithmetic lemma used to estimate the "small denominators" [10] that appear in constructing solutions of Eqs. (2.13) and (2.15) in the form of trigonometric series [4].

Thus, construction of approximate solutions $x_\varepsilon^\sim(t)$, $y_\varepsilon^\sim(t)$ that satisfy bound (2.28) reduces to the following procedure: we use Fourier series to solve Eqs. (2.13) and (2.15); then, using formula (2.16), we set up vector-valued function $A_2(x^*)$; then, in accordance with (2.26), we determine the solutions $x^{(1)}(\tau)$ and $x^{(2)}(\tau)$ of Eqs. (2.23) and (2.24); and, finally, on the basis of (2.27) we obtain the desired approximations $x_\varepsilon^\sim(t)$, $y_\varepsilon^\sim(t)$. In what follows, the procedure described will be implemented for some specific systems of equations of rigid-body dynamics.

The examples of perturbations to be considered below are such that the Fourier series expansions of the right sides of (2.13) and (2.15) contain only a finite number of terms. Therefore the solvability condition for Eqs. (2.13) and (2.15) reduces to verification of a finite number of conditions of the form $\omega_1 m_1 + \omega_2 m_2 \neq 0$. In the specific examples considered, conditions (2.30) assume the form $CA^{-1}r_0 \neq 0$, $(C-A)A^{-1}r_0 \neq 0$, and these conditions are always satisfied in view of the initial assumptions. Thus, bound (2.28) is valid without any additional assumptions regarding the frequencies ω_1, ω_2.

3. Case of a solid with a cavity filled with a high-viscosity fluid. As an example of our technique, let us consider the motion of a rigid body in the Lagrange case with a symmetrical cavity that is filled with a high-viscosity fluid. Then the moments of the forces acting on the rigid body have the form [11]:

$$\begin{aligned}
M_1 &= \rho P_{11}\nu^{-1}A^{-2}[C(A-C)pr^2+k(C-A)r\sin\theta\sin\varphi+kAp\cos\theta]\\
M_2 &= \rho P_{11}\nu^{-1}A^{-2}[C(A-C)qr^2+k(C-A)r\sin\theta\cos\varphi+kAq\cos\theta]\\
M_3 &= \rho P_{11}\nu^{-1}A^{-2}[A(C-A)(p^2+q^2)r-kA\sin\theta(p\sin\varphi+q\cos\varphi)]
\end{aligned} \tag{3.1}$$

where ρ and ν are the density and kinematic coefficient of viscosity of the fluid; P_{11} is the component of the tensor introduced in [11], in the coordinate system associated with the body. The tensor depends only on the shape of the cavity, $P_{11} > 0$; in the case of a symmetrical cavity under consideration, $P_{11} = P_{22}$. In what follows, we will assume that $\nu^{-1} \sim \varepsilon$ (the viscosity of the fluid is high). Making change (1.3) and discarding terms of order $O(\varepsilon^3)$, we obtain

$$\begin{aligned}
M_1^* &= \rho P_{11}A^{-2}[C(A-C)Pr^2+K(C-A)r\sin\theta\sin\varphi]\\
M_2^* &= \rho P_{11}A^{-2}[C(A-C)Qr^2+K(C-A)r\sin\theta\cos\varphi], \quad M_3^* = 0
\end{aligned} \tag{3.2}$$

The first three equations of system (2.9) can be written in the variables a, b, δ, ψ, θ, α, γ as follows:

$$\begin{aligned}
a^{\cdot} &= \rho P_{11}\nu^{-1}A^{-3}C(A-C)r_0^2 a-\varepsilon KC^{-1}r_0^{-1}\cos\theta(b-KC^{-1}r_0^{-1}\sin\theta\cos\alpha)+\\
&\quad +\varepsilon^2 KC^{-1}r_0^{-2}\delta\cos\theta(b-2KC^{-1}r_0^{-1}\sin\theta\cos\alpha)\\
b^{\cdot} &= \rho P_{11}\nu^{-1}A^{-3}C(A-C)r_0^2 b+\varepsilon KC^{-1}r_0^{-1}\cos\theta(a+KC^{-1}r_0^{-1}\sin\theta\sin\alpha)-\\
&\quad -\varepsilon^2 KC^{-1}r_0^{-2}\delta\cos\theta(a+2KC^{-1}r_0^{-1}\sin\theta\sin\alpha), \qquad \delta^{\cdot} = 0
\end{aligned} \tag{3.3}$$

The remaining equations of system (2.9) are unaffected.

Let us apply the above general scheme for constructing an approximate solution to the specific system (3.3). Vector-valued functions A_1 and B_1 are defined on the basis of (2.14) and have the form

$$\begin{aligned}
&A_1 = \{A_1^{(i)}\} \quad (i=\overline{1,5}), \quad B_1 = \{B_1^{(j)}\} \quad (j=1,2)\\
&A_1^{(1)} = \rho P_{11}A^{-3}C(A-C)r_0^2 a-KC^{-1}r_0^{-1}b\cos\theta\\
&A_1^{(2)} = \rho P_{11}A^{-3}C(A-C)r_0^2 b+KC^{-1}r_0^{-1}a\cos\theta\\
&A_1^{(3)} = 0, \quad A_1^{(4)} = KC^{-1}r_0^{-1}, \quad A_1^{(5)} = 0\\
&B_1^{(1)} = CA^{-1}\delta-KC^{-1}r_0^{-1}\cos\theta, \quad B_1^{(2)} = (C-A)A^{-1}\delta
\end{aligned} \tag{3.4}$$

The fourth and fifth components of vector-valued function $u_1 = \{u_1^{(i)}\}$ $(i=\overline{1,5})$ can be expressed as follows:

$$\begin{aligned}
u_1^{(4)} &= -C^{-1}Ar_0^{-1}\operatorname{cosec}\theta(a\cos\alpha+b\sin\alpha)\\
u_1^{(5)} &= C^{-1}Ar_0^{-1}(a\sin\alpha-b\cos\alpha)
\end{aligned} \tag{3.5}$$

Vector-valued function $A_2(x^*)$, after appropriate calculations involving formula (2.16), can be written in the form

$$\begin{aligned}
&A_2(x^*) = \{A_2^{(i)}\} \quad (i=\overline{1,5})\\
&A_2^{(1)} = KC^{-1}r_0^{-2}b\cos\theta[\delta-{}^1/_2 KC^{-2}Ar_0^{-1}(1+\cos\theta)]\\
&A_2^{(2)} = -KC^{-1}r_0^{-2}a\cos\theta[\delta-{}^1/_2 KC^{-2}Ar_0^{-1}(1+\cos\theta)]\\
&A_2^{(3)} = 0, \; A_2^{(4)} = -KC^{-1}\delta r_0^{-2}+K^2C^{-3}Ar_0^{-3}\cos\theta, \; A_2^{(5)} = 0
\end{aligned} \tag{3.6}$$

Let us define the solution of the averaged system of first-approximation equations (2.17) with allowance for (3.4) for the slow and fast variables:

$$
\begin{gathered}
a^{(1)}=\exp(st)(a^0\cos wt-b^0\sin wt)\\
b^{(1)}=\exp(st)(b^0\cos wt+a^0\sin wt)\\
\delta^{(1)}=0,\ \psi^{(1)}=\varepsilon KC^{-1}r_0^{-1}t+\psi_0,\ \theta^{(1)}=\theta_0\\
\alpha^{(1)}=CA^{-1}r_0t-wt+\varphi_0,\ \gamma^{(1)}=n_0t\\
s=\rho P_{11}\nu^{-1}A^{-2}C(A-C)r_0^2,\ w=\varepsilon KC^{-1}r_0^{-1}\cos\theta_0
\end{gathered}
\tag{3.7}
$$

where s, w, a^0, b^0, n_0 are defined in accordance with formula (2.3); ψ_0, θ_0, ϕ_0 are constants that are equal to the initial values of the Euler angles for t = 0.

On the basis of the above formulas, following (2.27), we can set up components of the function $x_0^{\sim}(t)$ corresponding to variables ψ and θ:

$$
\begin{gathered}
\psi_0^{\sim}(t)=\psi_0+\varepsilon KC^{-1}r_0^{-1}t+\varepsilon^2tKC^{-3}Ar_0^{-3}\cos\theta_0-\\
-\varepsilon C^{-1}Ar_0^{-1}\operatorname{cosec}\theta_0(a^{(1)}\cos\alpha^{(1)}+b^{(1)}\sin\alpha^{(1)})\\
\theta_0^{\sim}(t)=\theta_0+\varepsilon C^{-1}Ar_0^{-1}(a^{(1)}\sin\alpha^{(1)}-b^{(1)}\cos\alpha^{(1)})
\end{gathered}
\tag{3.8}
$$

The resultant formulas can conveniently be written in the form

$$
\begin{gathered}
\psi_0^{\sim}(t)=\psi_0+\varepsilon KC^{-1}r_0^{-1}t+\varepsilon^2tKC^{-3}Ar_0^{-3}\cos\theta_0+R^{(1)}\\
R^{(1)}=-\varepsilon C^{-1}Ar_0^{-1}\operatorname{cosec}\theta_0\exp(st)(a^{02}+b^{02})^{1/2}\sin(\alpha^{(1)}+\beta)\\
\theta_0^{\sim}(t)=\theta_0+\varepsilon C^{-1}Ar_0^{-1}\exp(st)(a^{02}+b^{02})^{1/2}\sin(\alpha^{(1)}-\mu)\\
\cos\beta=\sin\mu=b^{(1)}\exp(-st)(a^{02}+b^{02})^{-1/2}
\end{gathered}
\tag{3.9}
$$

In expression (3.9) for $\theta_0^{\sim}(t)$ the terms of order ε is the product of an exponentially decreasing (for A < C) or increasing (for A > C) cofactor exp(st), resulting from the presence of the cavity with viscous fluid, and an oscillating cofactor $\sin(\alpha^{(1)}-\mu)$. The magnitude of the attenuation decrement and the behavior of the slow phase change of small oscillations are directly evident from formulas (3.7) for $b^{(1)}$, $\alpha^{(1)}$.

Note that in expression (3.9) for the variable $\psi_0^{\sim}(t)$ the term $R^{(1)}(\varepsilon, t)$ is of order $O(\varepsilon)$ on the time interval $(0, T\varepsilon^{-1})$. The expression for the angular precession velocity $\omega_\psi=KC^{-1}r^{-1}$ is well known from approximate gyroscope theory [12]. The term $R^{(1)}(\varepsilon, t)$ that has been obtained refines this formula for the problem under consideration.

4. **Case of linear external dissipative moments.** Let us consider perturbed Lagrange motion with allowance for the moments acting on the rigid body from the surrounding environment. We will assume that the perturbing moments M_i (i = 1, 2, 3), with allowance for (1.3), have the form [13]:

$$
M_1=-\varepsilon^2I_1P,\ M_2=-\varepsilon^2I_1Q,\ M_3=-\varepsilon^2I_3r,\ I_1, I_3>0
\tag{4.1}
$$

where I_1 and I_3 are constant proportionality factors that depend on the properties of the medium and the shape of the body.

The first three equations of (2.9) for the problem in the variables a, b, δ, ψ, θ, α, γ assume the form

$$
\begin{gathered}
a^{\cdot}=-\varepsilon A^{-1}I_1(a+KC^{-1}r_0^{-1}\sin\theta\sin\alpha)-\varepsilon KC^{-1}r_0^{-1}\cos\theta(b-\\
-KC^{-1}r_0^{-1}\sin\theta\cos\alpha)+\varepsilon^2A^{-1}I_1KC^{-1}r_0^{-2}\delta\sin\theta\sin\alpha+\\
+\varepsilon^2KC^{-1}r_0^{-2}\delta\cos\theta(b-2KC^{-1}r_0^{-1}\sin\theta\cos\alpha)-\varepsilon^2KC^{-2}r_0^{-1}I_3\sin\theta\sin\alpha\\
b^{\cdot}=-\varepsilon A^{-1}I_1(b-KC^{-1}r_0^{-1}\sin\theta\cos\alpha)+\varepsilon KC^{-1}r_0^{-1}\cos\theta(\alpha+\\
+KC^{-1}r_0^{-1}\sin\theta\sin\alpha)-\varepsilon^2A^{-1}I_1KC^{-1}r_0^{-2}\delta\sin\theta\cos\alpha-\\
-\varepsilon^2KC^{-1}r_0^{-2}\delta\cos\theta(a+2KC^{-1}r_0^{-1}\sin\theta\sin\alpha)+\\
+\varepsilon^2KC^{-2}r_0^{-1}I_3\sin\theta\cos\alpha,\ \delta^{\cdot}=-\varepsilon C^{-1}I_3r_0-\varepsilon^2C^{-1}I_3\delta
\end{gathered}
\tag{4.2}
$$

The remaining equations of system (2.9) are unaltered.

We will employ the averaging procedure described in § 2 to set up an approximate

Let us define the solution of the averaged system of first-approximation equations (2.17) with allowance for (3.4) for the slow and fast variables:

$$
\begin{gathered}
a^{(1)}=\exp(st)(a^0\cos wt-b^0\sin wt)\\
b^{(1)}=\exp(st)(b^0\cos wt+a^0\sin wt)\\
\delta^{(1)}=0,\ \psi^{(1)}=\varepsilon KC^{-1}r_0^{-1}t+\psi_0,\ \theta^{(1)}=\theta_0\\
\alpha^{(1)}=CA^{-1}r_0t-wt+\varphi_0,\ \gamma^{(1)}=n_0t\\
s=\rho P_{11}\nu^{-1}A^{-2}C(A-C)r_0^2,\ w=\varepsilon KC^{-1}r_0^{-1}\cos\theta_0
\end{gathered}
\tag{3.7}
$$

where s, w, a^0, b^0, n_0 are defined in accordance with formula (2.3); ψ_0, θ_0, ϕ_0 are constants that are equal to the initial values of the Euler angles for t = 0.

On the basis of the above formulas, following (2.27), we can set up components of the function $x_0^{\sim}(t)$ corresponding to variables ψ and θ:

$$
\begin{gathered}
\psi_0^{\sim}(t)=\psi_0+\varepsilon KC^{-1}r_0^{-1}t+\varepsilon^2tKC^{-3}Ar_0^{-3}\cos\theta_0-\\
-\varepsilon C^{-1}Ar_0^{-1}\operatorname{cosec}\theta_0(a^{(1)}\cos\alpha^{(1)}+b^{(1)}\sin\alpha^{(1)})\\
\theta_0^{\sim}(t)=\theta_0+\varepsilon C^{-1}Ar_0^{-1}(a^{(1)}\sin\alpha^{(1)}-b^{(1)}\cos\alpha^{(1)})
\end{gathered}
\tag{3.8}
$$

The resultant formulas can conveniently be written in the form

$$
\begin{gathered}
\psi_0^{\sim}(t)=\psi_0+\varepsilon KC^{-1}r_0^{-1}t+\varepsilon^2tKC^{-3}Ar_0^{-3}\cos\theta_0+R^{(1)}\\
R^{(1)}=-\varepsilon C^{-1}Ar_0^{-1}\operatorname{cosec}\theta_0\exp(st)(a^{02}+b^{02})^{1/2}\sin(\alpha^{(1)}+\beta)\\
\theta_0^{\sim}(t)=\theta_0+\varepsilon C^{-1}Ar_0^{-1}\exp(st)(a^{02}+b^{02})^{1/2}\sin(\alpha^{(1)}-\mu)\\
\cos\beta=\sin\mu=b^{(1)}\exp(-st)(a^{02}+b^{02})^{-1/2}
\end{gathered}
\tag{3.9}
$$

In expression (3.9) for $\theta_0^{\sim}(t)$ the terms of order ε is the product of an exponentially decreasing (for A < C) or increasing (for A > C) cofactor exp(st), resulting from the presence of the cavity with viscous fluid, and an oscillating cofactor $\sin(\alpha^{(1)}-\mu)$. The magnitude of the attenuation decrement and the behavior of the slow phase change of small oscillations are directly evident from formulas (3.7) for $b^{(1)}$, $\alpha^{(1)}$.

Note that in expression (3.9) for the variable $\psi_0^{\sim}(t)$ the term $R^{(1)}(\varepsilon, t)$ is of order $O(\varepsilon)$ on the time interval $(0, T\varepsilon^{-1})$. The expression for the angular precession velocity $\omega_\psi=KC^{-1}r^{-1}$ is well known from approximate gyroscope theory [12]. The term $R^{(1)}(\varepsilon, t)$ that has been obtained refines this formula for the problem under consideration.

4. **Case of linear external dissipative moments.** Let us consider perturbed Lagrange motion with allowance for the moments acting on the rigid body from the surrounding environment. We will assume that the perturbing moments M_i (i = 1, 2, 3), with allowance for (1.3), have the form [13]:

$$
M_1=-\varepsilon^2I_1P,\ M_2=-\varepsilon^2I_1Q,\ M_3=-\varepsilon^2I_3r,\ I_1,\ I_3>0
\tag{4.1}
$$

where I_1 and I_3 are constant proportionality factors that depend on the properties of the medium and the shape of the body.

The first three equations of (2.9) for the problem in the variables a, b, δ, ψ, θ, α, γ assume the form

$$
\begin{gathered}
a^{\cdot}=-\varepsilon A^{-1}I_1(a+KC^{-1}r_0^{-1}\sin\theta\sin\alpha)-\varepsilon KC^{-1}r_0^{-1}\cos\theta(b-\\
-KC^{-1}r_0^{-1}\sin\theta\cos\alpha)+\varepsilon^2A^{-1}I_1KC^{-1}r_0^{-2}\delta\sin\theta\sin\alpha+\\
+\varepsilon^2KC^{-1}r_0^{-2}\delta\cos\theta(b-2KC^{-1}r_0^{-1}\sin\theta\cos\alpha)-\varepsilon^2KC^{-2}r_0^{-1}I_3\sin\theta\sin\alpha\\
b^{\cdot}=-\varepsilon A^{-1}I_1(b-KC^{-1}r_0^{-1}\sin\theta\cos\alpha)+\varepsilon KC^{-1}r_0^{-1}\cos\theta(a+\\
+KC^{-1}r_0^{-1}\sin\theta\sin\alpha)-\varepsilon^2A^{-1}I_1KC^{-1}r_0^{-2}\delta\sin\theta\cos\alpha-\\
-\varepsilon^2KC^{-1}r_0^{-2}\delta\cos\theta(a+2KC^{-1}r_0^{-1}\sin\theta\sin\alpha)+\\
+\varepsilon^2KC^{-2}r_0^{-1}I_3\sin\theta\cos\alpha,\ \delta^{\cdot}=-\varepsilon C^{-1}I_3r_0-\varepsilon^2C^{-1}I_3\delta
\end{gathered}
\tag{4.2}
$$

The remaining equations of system (2.9) are unaltered.

We will employ the averaging procedure described in § 2 to set up an approximate solution of system (4.2). Vector-valued function B_1 is defined by (3.4), while the components of vector-valued function A_1, after calculations based on formulas (2.14), can be written as follows:

$$A_1^{(1)}=-A^{-1}I_1a-KC^{-1}r_0^{-1}b\cos\theta,\ A_1^{(2)}=-A^{-1}I_1b+KC^{-1}r_0^{-1}a\cos\theta$$
$$A_1^{(3)}=-C^{-1}I_3r_0,\ A_1^{(4)}=KC^{-1}r_0^{-1},\ A_1^{(5)}=0 \tag{4.3}$$

The fourth and fifth components of vector-valued function u_1 can be expressed in accordance with (3.5).

We should note that in this section and the preceding one, as follows from (2.9), (3.3), and (4.2), combinations of the form $(M_1^0\cos\gamma+M_2^0\sin\gamma)$ and $(M_1^0\sin\gamma-M_2^0\cos\gamma)$ are independent of γ and the right sides of these equations depend only on the single fast variable α. This fact is analogous to the sufficient conditions, obtained in [1], for the possibility of averaging the equations of motion only with respect to the angle of nutation. As a result, the solution of Eqs. (2.13) becomes simpler.

We define function $A_2(x^*)$ in accordance with formula (2.16):

$$A_2^{(1)}=KC^{-1}r_0^{-2}[\delta b\cos\theta-{}^1/_2KC^{-2}r_0^{-1}Ab(3\cos^2\theta-1)-I_1C^{-1}a\cos\theta]$$
$$A_2^{(2)}=-KC^{-1}r_0^{-2}[\delta a\cos\theta-{}^1/_2KC^{-2}r_0^{-1}Aa(3\cos^2\theta-1)+I_1C^{-1}b\cos\theta]$$
$$A_2^{(3)}=-C^{-1}I_3\delta,\ A_2^{(4)}=KC^{-1}r_0^{-2}(-\delta+KC^{-2}r_0^{-1}\cos\theta)$$
$$A_2^{(5)}=I_1KC^{-2}r_0^{-2}\sin\theta \tag{4.4}$$

The solution of the averaged system of first-approximation equations (2.17) with allowance for (4.3) for the slow and fast variables has the form

$$a^{(1)}=\exp(-\varepsilon A^{-1}I_1t)(a^0\cos wt-b^0\sin wt)$$
$$b^{(1)}=\exp(-\varepsilon A^{-1}I_1t)(b^0\cos wt+a^0\sin wt)$$
$$\delta^{(1)}=-\varepsilon C^{-1}I_3r_0t,\ \psi^{(1)}=\varepsilon KC^{-1}r_0^{-1}t+\psi_0,\ \theta^{(1)}=\theta_0$$
$$\alpha^{(1)}=CA^{-1}r_0t-wt-{}^1/_2\varepsilon^2A^{-1}I_3r_0t^2+\varphi_0$$
$$\gamma^{(1)}=n_0t-{}^1/_2\varepsilon^2(C-A)A^{-1}C^{-1}I_3r_0t^2 \tag{4.5}$$

where, as in (3.7), $w=\varepsilon KC^{-1}r_0^{-1}\cos\theta_0$; the quantities a^0, b^0, n_0 are defined in accordance with (2.3); and ψ_0, θ_0, ϕ_0 are constants that are equal to the initial values of the Euler angles for $t = 0$. Comparison of the resultant expressions for the slow variables $a^{(1)}, b^{(1)}, \delta^{(1)}, \psi^{(1)}, \theta^{(1)}$, with allowance for (2.7), and the corresponding formulas of [2], if we formally set $I_3 = \varepsilon I_3$ in them, indicates that the expressions in question coincide.

On the basis of (2.27) and formulas (3.5), (4.4), and (4.5) we can determine the components of function $x_0^{\sim}(t)$ corresponding to variables ψ and θ:

$$\psi_0^{\sim}(t)=\psi_0+\varepsilon KC^{-1}r_0^{-1}t+\varepsilon^2tK^2C^{-3}r_0^{-3}\cos\theta_0+$$
$$+{}^1/_2\varepsilon^2KC^{-2}I_3r_0^{-1}t^2-\varepsilon C^{-1}Ar_0^{-1}\operatorname{cosec}\theta_0(a^{(1)}\cos\alpha^{(1)}+b^{(1)}\sin\alpha^{(1)})$$
$$\theta_0^{\sim}(t)=\theta_0+\varepsilon^2tI_1KC^{-2}r_0^{-2}\sin\theta_0+\varepsilon C^{-1}Ar_0^{-1}(a^{(1)}\sin\alpha^{(1)}-b^{(1)}\cos\alpha^{(1)}) \tag{4.6}$$

The resultant expressions can be written as follows:

$$\psi_0^{\sim}(t)=\psi_0+\varepsilon KC^{-1}r_0^{-1}t+S^{(1)}$$
$$S^{(1)}=\varepsilon^2tK^2C^{-3}r_0^{-3}\cos\theta_0+{}^1/_2\varepsilon^2KC^{-2}r_0^{-1}I_3t^2-$$
$$-\varepsilon C^{-1}Ar_0^{-1}\operatorname{cosec}\theta_0\exp(-\varepsilon A^{-1}I_1t)(a^{02}-b^{02})^{1/2}\sin(\alpha^{(1)}+\sigma)$$
$$\theta_0^{\sim}(t)=\theta_0+\varepsilon^2tI_1KC^{-2}r_0^{-2}\sin\theta_0+$$
$$+\varepsilon C^{-1}Ar_0^{-1}\exp(-\varepsilon A^{-1}I_1t)(a^{02}+b^{02})^{1/2}\sin(\alpha^{(1)}-\lambda)$$
$$\cos\sigma=\sin\lambda=b^{(1)}\exp(\varepsilon A^{-1}I_1t)(a^{02}+b^{02})^{-1/2} \tag{4.7}$$

In expression (4.7) for $\theta_0^{\sim}$ the term of order ε is the product of the slowly exponentially decreasing cofactor $\exp(-\varepsilon A^{-1}I_1t)$, due to energy dissipation, and the oscillating cofactor $\sin(\alpha^{(1)}-\lambda)$. The magnitude of the attenuation decrement and the behavior of the slow phase change of small oscillations are evident from formulas (4.5) for $b^{(1)}, a^{(1)}$.

In expression (4.7) for the variable $\psi_0^{\sim}(t)$ the terms $S^{(1)}(\varepsilon, t)$ are of order $O(\varepsilon)$ on the time interval $(0, T\varepsilon^{-1})$.

For the problem in question, the resultant expression for $S^{(1)}(\varepsilon, t)$ refines the formula for the angular precession velocity $\omega_p = KC^{-1}r_0^{-1}$ that obtains in approximate gyroscope theory.

5. **Case of small constant moment.** Consider motion of a rigid body in the Lagrange case under the action of a moment that is constant in the associated axes. In this case the perturbing moments M_i ($i = 1, 2, 3$) have the form $M_i = \varepsilon^2 M_i^* = \varepsilon^2 M_i^* = \mathrm{const}$. In setting up the approximate solution of system (2.9), with allowance for the expression for M_i, we employ the averaging procedure cited in § 2. Vector-valued function B_1 is defined in accordance with (3.4), while vector-valued function A_1, obtained in accordance with (2.14), has the following components:

$$A_1^{(1)} = -KC^{-1}r_0^{-1} b\cos\theta, \quad A_1^{(2)} = KC^{-1}r_0^{-1} a\cos\theta \tag{5.1}$$

$$A_1^{(3)} = C^{-1}M_3^*, \quad A_1^{(4)} = KC^{-1}r_0^{-1}, \quad A_1^{(5)} = 0$$

The fourth and fifth components of vector-valued function u_1 have the form (3.5). Function $A_2(x^*)$ is determined from formula (2.16):

$$\begin{gathered} A_2^{(1)} = KC^{-1}r_0^{-2} b[\cos\theta - {}^1/_2 KC^{-2}Ar_0^{-1}(3\cos^2\theta - 1)] \\ A_2^{(2)} = -KC^{-1}r_0^{-2} a[\delta\cos\theta - {}^1/_2 KC^{-2}Ar_0^{-1}(3\cos^2\theta - 1)] \\ A_2^{(3)} = 0, \quad A_2^{(4)} = -KC^{-1}r_0^{-2}\delta + K^2C^{-2}Ar_0^{-2}\cos\theta \\ A_2^{(5)} = 0 \end{gathered} \tag{5.2}$$

Let us obtain a solution for the averaged system of first-approximation equations (2.17) with allowance for (5.1) for the slow and fast variables:

$$\begin{gathered} a^{(1)} = a^0\cos wt - b^0\sin wt \\ b^{(1)} = b^0\cos wt + a^0\sin wt \\ \delta^{(1)} = \varepsilon C^{-1}M_3^* t, \quad \psi^{(1)} = \varepsilon KC^{-1}r_0^{-1}t + \psi_0, \quad \theta^{(1)} = \theta_0 \\ \alpha^{(1)} = CA^{-1}r_0 t - wt + {}^1/_2\varepsilon^2 A^{-1}M_3^* t^2 + \varphi_0 \\ \gamma^{(1)} = n_0 t + {}^1/_2\varepsilon^2 (C-A)C^{-1}A^{-1}M_3^* t^2 \end{gathered} \tag{5.3}$$

where $w = \varepsilon KC^{-1}r_0^{-1}\cos\theta_0$; the quantities a^0, b^0, n_0 are defined in accordance with (2.3); and ψ_0, θ_0, φ_0 are the initial values of the Euler angles for $t = 0$.

Note that the solution of the averaged first-approximation system (5.3) contains only the component of the constant moment (in the associated axes) that is applied along the axis of symmetry: M_3^*. The projections M_1^*, M_2^* of the perturbing-moment vector drop out upon averaging. Comparison of our expressions (5.3) with the slow variables with allowance for (2.7) and the corresponding formulas of [2], formally setting $M_3^* = \varepsilon M_3^*$ in them, indicates that the expressions for $a^{(1)}, b^{(1)}, \delta^{(1)}, \psi^{(1)}, \theta^{(1)}$ coincide. The components of $z_0^{\sim}(t)$ corresponding to the variables ψ and θ can be determined in accordance with (2.27) with substitution of the corresponding expressions of (3.5), (5.2), and (5.3):

$$\begin{gathered} \psi_0^{\sim}(t) = \psi_0 + \varepsilon KC^{-1}r_0^{-1}t + \varepsilon^2 tK^2C^{-2}Ar_0^{-2}\cos\theta_0 - \\ - {}^1/_2\varepsilon^2 KC^{-2}M_3^* r_0^{-2}t^2 - \varepsilon C^{-1}Ar_0^{-1}\operatorname{cosec}\theta_0 (a^{(1)}\cos\alpha^{(1)} + b^{(1)}\sin\alpha^{(1)}), \quad \theta_0^{\sim}(t) = \\ = \theta_0 + \varepsilon C^{-1}Ar_0^{-1}(a^{(1)}\sin\alpha^{(1)} - b^{(1)}\cos\alpha^{(1)}) \end{gathered}$$

The resultant expressions can be conveniently written in the form

$$\begin{gathered} \psi_0^{\sim}(t) = \psi_0 + \varepsilon KC^{-1}r_0^{-1}t + V^{(1)} \\ V^{(1)} = \varepsilon^2 tK^2C^{-2}Ar_0^{-2}\cos\theta_0 - {}^1/_2\varepsilon^2 KC^{-2}M_3^* r_0^{-2}t^2 - \\ - \varepsilon C^{-1}Ar_0^{-1}(a^{02} + b^{02})^{1/2}\sin(\alpha^{(1)} + \varkappa) \\ \theta_0^{\sim}(t) = \theta_0 + \varepsilon C^{-1}Ar_0^{-1}(a^{02} + b^{02})^{1/2}\sin(\alpha^{(1)} - \chi) \\ \cos\varkappa = \sin\chi = b^{(1)}(a^{02} + b^{02})^{-1/2} \end{gathered}$$

Here, in the expression for $\theta_0^{\sim}$ the bounded oscillating term contains nonzero initial data a^0, b^0. As in the preceding examples, the resultant term $V^{(1)}$ supplements the expression for the angular precession velocity $\omega_p = KC^{-1}r_0^{-1}$ that is known from approximate gyroscope theory.

Note that, if we confine ourselves to constructing the first approximation, then the formulas for the nutation and precession angles do not contain parameters of the perturbing moments, and therefore the effect of perturbations on regular precession of the body will not be taken into account. In this case, therefore, construction of the second approximation is essential.

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