FAST MOTION OF A HEAVY RIGID BODY ABOUT A FIXED POINT IN A RESISTIVE MEDIUM

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This article investigates fast rotational motion of a heavy rigid body about a fixed point when external resistance is present. The moment of the resistive forces is assumed to be a linear function of the angular velocity. The system obtained after averaging with respect to Euler-Poinsot motion is analyzed, in the case of fast rotations.

1. Consider fast motion of an asymmetrical heavy rigid body about a fixed point O in a resistive medium. Fast motions will be taken to be those for which the moment of external forces relative to the fixed point is small as compared to the instantaneous value of the kinetic energy of the rotations.

We introduce three Cartesian coordinate systems. The Ox_i $(i = 1, 2, 3)$ axes are fixed; the Ox_3 axis is vertically upward. The Oz_i axes are associated with the principal axes of inertia of the rigid body. The Oy_3 axis of the Oy_i system lies along the kinetic moment vector **G** of the rigid body relative to point O; the Oy_2 axis lies in the horizontal plane; and Oy_1 is in the vertical plane. The angles λ and δ define the direction of vector **G** in space, as shown in Fig. 1.

The formulas for the cosines of the angles between the axes are given in the accompanying table.

	Oz_1	Oz_2	Oz_3
Oy_1	$\alpha_{11} = \cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi$	$\alpha_{12} = -\sin\varphi\cos\psi - \cos\theta\cos\varphi\sin\psi$	$\alpha_{13} = \sin\theta\sin\psi$
Oy_2	$\alpha_{21} = \cos\varphi\sin\psi + \cos\theta\sin\varphi\cos\psi$	$\alpha_{22} = -\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi$	$\alpha_{23} = -\sin\theta\cos\psi$
Oy_3	$\alpha_{31} = \sin\theta\sin\varphi$	$\alpha_{32} = \sin\theta\cos\varphi$	$\alpha_{33} = \cos\theta$

Here θ, φ, ψ are the Euler angles, which define the orientation of the Oz_i axes relative to Oy_i.

We can write the equations of motion of the body relative to the fixed point as follows:

$$dG/dt = L_3;\quad d\delta/dt = L_1/G;\quad d\lambda/dt = L_2/(G\sin\delta)$$

$$\frac{d\theta}{dt} = G\sin\theta\sin\varphi\cos\varphi\left(\frac{1}{A} - \frac{1}{B}\right) + \frac{L_2\cos\psi - L_1\sin\psi}{G}$$

$$\frac{d\varphi}{dt} = G\cos\theta\left(\frac{1}{C} - \frac{\sin^2\varphi}{A} - \frac{\cos^2\varphi}{B}\right) + \frac{L_1\cos\psi + L_2\sin\psi}{G\sin\theta} \tag{1.1}$$

$$\frac{d\psi}{dt} = G\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right) - \frac{L_1\cos\psi + L_2\sin\psi}{G}\operatorname{ctg}\theta - \frac{L_2}{G}\operatorname{ctg}\delta$$

Here L_i are the moments of external forces relative to the Oy_i axes; G is the magnitude of the kinetic moment; and A, B, and C are the principal moments of inertia of the body relative to the Oz_i axes.

Projection of vector **G** onto the axes of the associated Oz_i coordinate system yields

$$Ap=G\sin\theta\sin\varphi,\ Bq=G\sin\theta\cos\varphi,\ Cr=G\cos\theta \qquad (1.2)$$

Here p, q, r are the projections of the absolute angular velocity vector ω of the body onto the Oz_1 axes.

In some cases, instead of the angle θ it is convenient to use the kinetic energy as a variable:

$$T=\frac{G^2}{2}\left[\left(\frac{\sin^2\varphi}{A}+\frac{\cos^2\varphi}{B}\right)\sin^2\theta+\frac{\cos^2\theta}{C}\right] \qquad (1.3)$$

where its derivative has the form

$$\dot{T}=\frac{2T}{G}L_3+G\sin\theta\left[\cos\theta\left(\frac{\sin^2\varphi}{A}+\frac{\cos^2\varphi}{B}-\frac{1}{C}\right)\cdot(L_2\cos\psi-L_1\sin\psi)+\right.$$
$$\left.+\sin\varphi\cos\varphi\left(\frac{1}{A}-\frac{1}{B}\right)(L_1\cos\psi+L_2\sin\psi)\right] \qquad (1.4)$$

The projections L_i of the moment of external forces, made up of the forces of gravity and forces of external resistance, onto the Oy_1 axes can be written in the following form with allowance for (1.2):

$$L_1=-mg\cos\delta\sum_{j=1}^{3}a_j\alpha_{3j}-G\sum_{i=1}^{3}\left(\frac{I_{i1}}{A}\alpha_{i1}\alpha_{11}+\frac{I_{i2}}{B}\alpha_{i2}\alpha_{12}+\frac{I_{i3}}{C}\alpha_{i3}\alpha_{13}\right)$$

$$L_2=mg\sum_{j=1}^{3}a_j(\alpha_{3j}\sin\delta+\alpha_{1j}\cos\delta)-$$
$$-G\sum_{i=1}^{3}\left(\frac{I_{i1}}{A}\alpha_{i1}\alpha_{21}+\frac{I_{i2}}{B}\alpha_{i2}\alpha_{22}+\frac{I_{i3}}{C}\alpha_{i3}\alpha_{23}\right) \qquad (1.5)$$

$$L_3=-mg\sin\delta\sum_{j=1}^{3}a_j\alpha_{2j}-G\sum_{i=1}^{3}\left(\frac{I_{i1}}{A}\alpha_{31}^2+\frac{I_{i2}}{B}\alpha_{32}^2+\frac{I_{i3}}{C}\alpha_{33}^2\right)$$

Here it is assumed that the moment of resistive forces L^* can be represented in the form $L^*=I\cdot\omega$, where the tensor I has constant components I_{ij} in the Oz_1 system associated with the body [2]. Since we are investigating rapid motion, we assume a small ratio $mga/T\sim\varepsilon\ll 1$, where a is the distance from the center of mass to the fixed point. The resistance of the medium is assumed to be weak and of the same order of smallness: $\|I\|/G\sim\varepsilon\ll 1$, where $\|I\|$ is the norm of the matrix of the resistance coefficients.

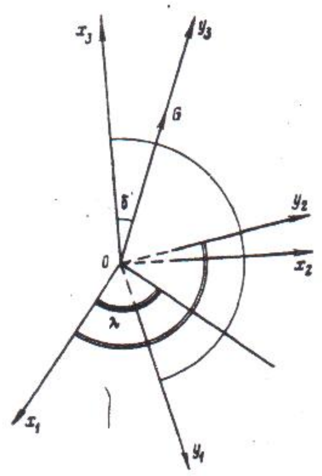

Fig. 1

Let us investigate the solution of system (1.1), (1.4) for small ε on a large time interval $t\sim\varepsilon^{-1}$. To solve the problem, we can employ the averaging method of [3]. The error in the averaged solution for the slow variables is a quantity of order ε on the time interval over which the body executes $\sim\varepsilon^{-1}$ revolutions. We perform averaging with respect to Euler-Poinsot motion using the procedure of [1,4]. The equations of averaged motion were obtained earlier in [5].

2. Let us consider unperturbed motion (ε = 0), when the moments of the external forces are equal to zero. In this case the rotation of the rigid body is Euler-Poinsot motion. The quantities G, δ, λ, T become constants, while θ, φ, ψ become functions of time t. The slow variables in perturbed motion will be G, δ, λ, T, while the fast variables will be the Euler angles θ, φ, ψ.

For the sake of being definite, we take $A>B>C$ and consider motion under the condition

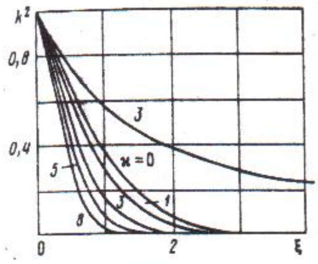

Fig. 2

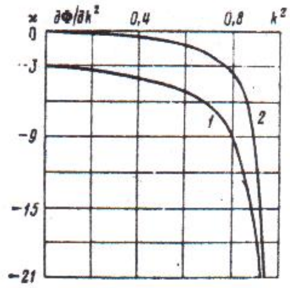

Fig. 3

$2TA \geqslant G^2 \geqslant 2TB$, corresponding to trajectories of the kinetic moment vector that encompass the Oz_1 axis [6]. We introduce the quantity

$$k^2 = \frac{(B-C)(2TA-G^2)}{(A-B)(G^2-2TC)} \quad (0 \leqslant k^2 \leqslant 1) \tag{2.1}$$

which is a constant - the modulus of the elliptic functions - in unperturbed motion.

To construct the first-approximation averaged system, we substitute the solution of unperturbed Euler-Poinsot motion [6] into the right sides of (1.1) and (1.4) and average with respect to the variable ψ, then with respect to time t, allowing for the fact that θ, φ depend on t [1]. We retain the earlier notation for the slow averaged variables. As a result we obtain [5]

$$\lambda^{\cdot} = \frac{\pi m g a_1}{2G^2 \mathbf{K}(k)} \sqrt{\frac{A(G^2-2TC)}{A-C}}, \quad \delta^{\cdot} = 0$$

$$G^{\cdot} = -\frac{G}{A(B-C)+C(A-B)k^2}\left\{I_{22}(A-C)\left[1-\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right]+ I_{33}(A-B)\left[k^2-1+\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right]+I_{11}(B-C)\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right\}$$

$$T^{\cdot} = -\frac{2T}{A(B-C)+C(A-B)k^2}\left\{I_{22}(A-C)\left[1-\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right]+ I_{33}(A-B)\left[k^2-1+\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right]+\frac{(A-B)(A-C)(B-C)}{B-C+(A-B)k^2}\times \right.$$
$$\left.\times\left\{\frac{I_{33}}{C}\left[k^2-1+\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right]+\frac{I_{22}}{B}(1-k^2)\left[1-\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right]\right\}+ \frac{I_{11}(B-C)[A(B-C)+C(A-B)k^2]}{B-C+(A-B)k^2}\cdot\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right\} \tag{2.2}$$

Here $\mathbf{K}(k)$, $\mathbf{E}(k)$ are complete elliptic integrals of the first and second kind. Differentiating (2.1) for k^2 and using the last two equations in (2.2), we obtain the differential equation

$$\frac{dk^2}{d\xi} = (1-\varkappa)(1-k^2) - [(1-\varkappa)+(1+\varkappa)k^2]\frac{\mathbf{E}(k)}{\mathbf{K}(k)}$$
$$\varkappa = (2I_{22}AC - I_{11}BC - I_{33}AB)/[(I_{33}A - I_{11}C)B] \tag{2.3}$$
$$\xi = (t-t_*)/N, \quad N = AC/(I_{33}A - I_{11}C)$$

Here t_* is a constant. The value $k^2 = 1$ corresponds to the equation $2TB = G^2$, which in turn corresponds to the separatrix for Euler-Poinsot motion. If the equation $k^2 = 1$ is attained for some solution of (2.3), then we choose t_* in such a way that $k^2 = 1$ for $\xi = 0$, $t = t_*$.

It follows from (2.2) that the presence of resistance in the medium leads to evolution of both the kinetic energy T of the body and the magnitude of the kinetic moment G. It is directly evident that in first approximation only the resistive force affects the change in T and G; the equations include only the diagonal coefficients I_{ii} of the matrix of the moment of friction. Terms containing nondiagonal components I_{ij} $(i \neq j)$ drop out upon averaging. The angular rotational velocity of the kinetic moment about the vertical $\lambda^{\cdot}$ depends on both the force of gravity and the resistive force of the medium. We should note that the action of these forces does not lead to change in the angular variable δ, and the departure from the vertical remains constant in the approximation under consideration.

Equation (2.3) describes averaged motion of the end of the kinetic moment vector **G** on a sphere of radius G. The third equation in (2.2) describes the change in the radius of the sphere over time.

The expression in braces on the right side of Eq. (2.2) for G is positive (for $A > B > C$), since we have $(1-k^2)\mathbf{K} < \mathbf{E} < \mathbf{K}$ [7]. Each coefficient for I_{11} is a nonnegative function of k^2; these coefficients cannot all simultaneously vanish. Therefore $dG/dt<0$ for $G > 0$, i.e., the variable G strictly decreases for any $k^2 \in [0,1]$. We can similarly show that the kinetic energy also strictly decreases.

Equations (2.2) and (2.3) for G, T, k^2 can be integrated in quadratures. We write them in the form

$$G^{\cdot}=-Gf_G(k^2),\quad T^{\cdot}=-Tf_T(k^2),\quad k^{2\cdot}=f_k(k^2) \qquad (2.4)$$

where f_G, f_T, and f_k are the functions defined in (2.2) and (2.3). From this we find

$$G(k^2)=G_0\exp\left[-\int_{k_0^2}^{k^2} F_G(l)\,dl\right],\quad T(k^2)=T_0\exp\left[-\int_{k_0^2}^{k^2} F_T(l)\,dl\right] \qquad (2.5)$$

$$F_{G,T}(k^2)=\frac{f_{G,T}(k^2)}{f_k(k^2)},\quad \int_{k_0^2}^{k^2}\frac{dl}{f_k(l)}=t-t_0$$

Estimating f_G from (2.4), we find that the following differential inequality is valid:

$$-f_G^- \leqslant G^{\cdot}/G \leqslant -f_G^+,\quad k^2\in[0,1] \qquad (2.6)$$

where f_G^-, f_G^+ are positive constants. Consequently, integrating (2.6), we obtain the following bound for G:

$$G_0\exp(-f_G^- t)\leqslant G\leqslant G_0\exp(-f_G^+ t). \qquad (2.7)$$

Similar inequalities are valid for T.

3. A basic stage in the investigation of the motion of the body is the analysis of Eq. (2.3). It is interesting that (2.3) coincides with the equation obtained for the case of free spatial motion of a body with a cavity filled with a high-viscosity fluid [4]. We should note that the acceleration due to gravity does not appear in (2.3). Only the resistance of the medium affects the evolution of k^2; in view of the fact that this equation can be integrated independently, the effects of resistance and gravity are partially separated. Complete separation does not occur in this case, since the slowly decreasing variables G and T appear on the right side of the expression for $\lambda^{\cdot}$. We should note that paper [8] investigated the case in which a small perturbing moment of the forces of gravity is present (no resistance); G and T remain constant in this case.

It is not hard to establish that $\varkappa$ from (2.3) can be written in the form

$$\varkappa=\frac{C\varkappa_1-A\varkappa_2}{C\varkappa_1+A\varkappa_2},\quad \varkappa_1=I_{33}A-I_{11}B,\quad \varkappa_2=I_{33}B-I_{33}C$$

Since $\varkappa_1$, $\varkappa_2$ can assume arbitrary values, the value of $\varkappa$ varies from $-\infty$ to $+\infty$ depending on the parameters of the problem. In [4], the inequalities $\varkappa_1>0$, $\varkappa_2>0$ were observed, and hence $|\varkappa|\leqslant 1$. Paper [9] considered an equation of the form (2.3) for a rigid body with a cavity of arbitrary shape, filled with a strongly viscous fluid, where $\varkappa$ varied in the range $|\varkappa|\leqslant 3$. Numerical integration of (2.3) with initial condition $k^2(0) = 1$ was performed in these studies. It was shown that k^2 decreases monotonically from 1 to 0 with increasing ξ, the decrease being more rapid, the larger $\varkappa$.

Then the family of solutions of (2.3), corresponding to different $\varkappa\in(-\infty,\infty)$, was investigated. Note that for $\varkappa<-3$ new qualitative effects appear, while for $\varkappa>3$ the nature of the solution is the same as for $|\varkappa| \leqslant 3$. Indeed, as can be seen from the graphs of $k^2(\varkappa, \xi)$, in Fig. 2 for $\varkappa = -3, 0, 1, 3, 5, 8$, large $\varkappa$ correspond to more rapidly decreasing functions of the argument ξ.

For $\varkappa < -3$, Eq. (2.3) for k^2 admits stationary points $k^2 = k_*^2$, i.e., independently of G and T the quantity k^2 remains a constant, in view of (2.3), with appropriate choice of initial conditions. Note that for $\varkappa > -3$ such stationary points do not exist (except for $k = 0$, $k = 1$).

Let us determine the quasi-steady solutions $k^2 = k_*^2$, for which we set the right side of (2.3) equal to zero. We solve the resultant equation for $\varkappa$:

$$\varkappa = \frac{k^2 - 1 + (1+k^2)\mathbf{E}(k)/\mathbf{K}(k)}{(1-k^2)[\mathbf{E}(k)/\mathbf{K}(k) - 1]} \qquad (3.1)$$

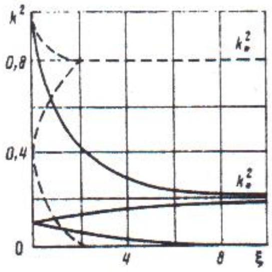

Fig. 4

The graph of $\varkappa$ as a function of k^2, determined numerically, is shown by curve 1 in Fig. 3; it follows that for any $\varkappa < -3$ there exists a unique solution $k_*^2 \in (0, 1)$, corresponding to quasi-steady motion $k^2 = k_*^2 = \text{const}$. Equation (2.3) was analyzed numerically for $\varkappa < -3$. For specified values $k_*^2 \in (0, 1)$, corresponding to quasi-steady motion, the corresponding $\varkappa$ values were determined on the basis of (3.1). Figure 4 shows typical plots of functions $k^2(\varkappa, \xi)$, obtained as a result of numerical integration of (2.3). Here $k_*^2 = 0{,}2$ for the solid curves and 0.8 for the dashed ones. Each graph contains three branches. The initial condition for the upper branches was taken to be $k^2(0) = 1$. The two lower branches on each graph were plotted for initial conditions $k^2(0) = {}^1/_2 k_*^2$. The increasing branch corresponds to integration for $\xi > 0$, while the decreasing branch is a mirror image, with respect to the axis $\xi = 0$, of the relationship $k^2(\varkappa, \xi)$, obtained for $\xi < 0$.

The curves make it possible to investigate (2.3), for the indicated parameter values, and to construct a solution for any initial condition. Indeed, in view of the self-contained nature of Eq. (2.3) for k^2 the solution $k^2(\varkappa, \xi)$ for arbitrary initial conditions is determined by shifting the origin along the ξ axis. For any initial value $k^2 = k_0^2$, therefore, by choosing the corresponding branch of the graphs, we can describe the subsequent change in k^2 by this branch. If $k_0^2 > k_*^2$, the upper branch is chosen; if ${}^1/_2 k_*^2 \leqslant k_0^2 < k_*^2$, the middle one.

If, however, $k_0^2 < {}^1/_2 k_*^2$, the lower branch is chosen; with increasing ξ, motion along this branch is in the negative direction up to $k^2 = {}^1/_2 k_*^2$, after which we change over to the middle branch. For $k_0^2 = k_*^2$ we have a stationary solution.

4. Let us consider some particular cases of motion. For $I_{22}A = I_{11}C$ we have $|N|$, $|\varkappa| \to \infty$ in (2.3). After expanding the indeterminacy, we obtain instead of (2.3)

$$\frac{dk^2}{dt} = \frac{2}{AB}(I_{11}C - I_{22}A)(1-k^2)\left[1 - \frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right] \qquad (4.1)$$

Consequently, for $I_{11}C > I_{22}A$ the variable k^2 increases and tends to unity; for $I_{11}C < I_{22}A$, k^2 decreases and tends to zero, i.e., the motion tends to rotation about the axis with moment of inertia A.

It follows from (2.3) that the solution $k^2 = 0$ satisfies the equation. This quasi-steady motion corresponds to retarded rotation about the axis with greatest moment of inertia.

Equations (2.2) yield the following expressions for the variables G and T for $k^2 = 0$:

$$G = G_0 \exp\left(-\frac{I_{11}}{A}t\right), \quad T = T_0 \exp\left(-2\frac{I_{11}}{A}t\right)$$

Formally setting $k^2 = 1$, this corresponding to motion along the separatrix of Euler-Poinsot motion, we have stable with respect to the variable k^2 (in the sense of [12]) for $\xi \geq 0$. This can also be seen from Fig. 4.

In true time $t \geq t_*$ we have stability for $N > 0$ or $I_{33}A > I_{11}C$; see (2.3). In the opposite case, for $N<0$, $I_{33}I<I_{11}C$, these quasi-steady motions are unstable.

In accordance with (4.2), quasi-steady motion $k^2 = 0$ for $\xi > 0$ is asymptotically stable for $\varkappa>-3$ and unstable for $\varkappa<-3$. In true time for $t \geq t_*$ the motion can be either asymptotically stable or unstable, depending on the value of $\varkappa$ and the sign of the parameter N.

The above analysis yields the following qualitative picture of motion. Let us first consider the case $N > 0$. Function (4.2) and formulas (2.1) for k^2, (2.2) and (2.3) describe motion for $t \geq t_*$, i.e., in the region $2TA>G^2>2TB$. For $t \leq t_*$ the inequalities $2TB>G^2>2TC$ are satisfied, these corresponding to trajectories of the kinetic moment vector that encompass the C axis. In this case we can interchange A and C and I_{11} and I_{33} in (2.1)-(2.3), and also replace a_1 by a_3 in (2.2). Then Eq. (2.3) retains its form, but we need to replace $\varkappa$ by $-\varkappa$, and N by $-N$ in it. Motion for $N < 0$ can be similarly determined. It is assumed that at $t = t_*$ the motion (one of the branches in Fig. 4) passes through the separatrix; as already noted, however, "sticking" for an indeterminately long time is possible for a set of initial data of small measure [10,11].

Figure 5 shows k^2 as a function of $\varkappa, N$ in true time t. Points corresponding to quasi-steady motions are indicated; the arrows show the direction of motion. The letters z_1, z_2, z_3 denote the axes of the body corresponding to the given value of k^2; to the left of z_2 there is a region where $2TA>G^2>2TB$, while to the right of it there is a region where $2TB>G^2>2TC$.

Our results can be interpreted as follows. We introduce the notation

$$I_{11}/A=\alpha_1,\ I_{22}/B=\alpha_2,\ I_{33}/C=\alpha_3,\ \alpha_i/\alpha_2=\beta_i \quad (i=1,2,3) \tag{4.5}$$

Rotation of the body about one of the principal axes, e.g., Oz_1, under the action of a dissipative moment is decribed by the expressions $A\dot{\omega}=-I_{11}\omega$, $\omega=\mathrm{const}\,\exp(-\alpha_1 t)$.

Therefore the α_1 in (4.5) have the meaning of the attenuation factors of the rotations about the principal axes of inertia Oz_1. The dimensionless quantities β_1 are equal to the corresponding coefficients referred to α_2; here $\beta_2 = 1$. We rewrite expressions (2.3) for $\varkappa, N$ in terms of β_1 as follows:

$$\varkappa=\frac{2-\beta_1-\beta_3}{\beta_3-\beta_1}, \quad N=\frac{1}{\beta_3-\beta_1} \tag{4.6}$$

In the β_1, β_3 plane we draw the straight line $\beta_3 = \beta_1$, on which N changes sign, and the straight lines $1+\beta_1=2\beta_3$ and $1+\beta_3=2\beta_1$, corresponding to the equations $\varkappa=\pm3$ in accordance with (4.6). These lines partition the quadrant $\beta_1>0$, $\beta_3>0$ into six regions, shown in Fig. 6. The numbers of the regions correspond to the ordinal number of the qualitative portraits of motion shown in Fig. 5. It can be seen that the number of quasi-steady modes of motion and their stability depend on the relative magnitude of the attenuation coefficients of the rotations α_1 about the principal axes of inertia.

Thus, in the approximation under consideration, perturbed motion of the body is made up of rapid Euler-Poinsot rotation about the vector **G** and slow evolution of the parameters of this motion. The magnitudes of the kinetic moment and kinetic energy strictly decrease; their change depends only on the presence of resistance of the medium. The motion of vector **G** itself in space is described by the first two equations of system (2.2), and involves a constant deviation from the vertical δ = const. Unlike the case in which only a small gravity acts [8], the rotational velocity of **G** about the vertical is variable. The evolution of the parameters of Euler-Poinsot motion in the coordinate system associated with the body is described by Eq. (2.3) and is shown qualitatively in Fig. 5.

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