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CONTROL IN DETERMINISTIC SYSTEMS

Quasi-Optimal Braking of Rotations of a Body with a Moving Mass Coupled to It through a Quadratic Friction Damper in a Resisting Medium

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Abstract—This paper addresses the problem of the time-optimal braking of rotations of a dynamically symmetric rigid body under a small control moment in the ellipsoidal range with close unequal values of the ellipsoid's semiaxes. This problem is considered a problem of quasi-optimal control. The body is assumed to have a moving mass connected to it through elastic coupling with quadratic dissipation. In addition, the body is exposed to a small braking moment of the linear resistance of the medium. The problem of synthesizing the quasi-optimal braking of the rotations of a dynamically symmetric body in a resisting medium is investigated analytically and numerically. An approximate solution is found by the phase-averaging of the processional motion. The qualitative properties of quasi-optimal motion are analyzed and the corresponding graphs are presented.

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INTRODUCTION

Investigation of the dynamics and control of rigid bodies moving about a fixed point implies that the bodies are not absolutely rigid but are close to ideal models. The effect of nonidealities can be analyzed by singular perturbation methods, averaging, or other asymptotic methods of nonlinear mechanics. It is reduced to the presence of additional terms in Euler's dynamic equations for a fictitious rigid body. For instance, the motion of rigid bodies with internal degrees of freedom was investigated in [1–8].

A great deal of attention has been paid to analyzing the uncontrolled motion of rigid bodies in a resisting medium [3, 5, 9–11]. However, the problem of controlling the rotation of quasi-rigid bodies by concentrated (applied to a rigid case, or to a carrying rigid body) moments of force has not been investigated as much [4, 5, 12].

We consider a problem of the quasi-optimal (close to optimal) braking of the rotations of a dynamically symmetric rigid body with a moving point mass fixed to a point on its symmetry axis. It is assumed that, when in relative motion, the point experiences a returning elastic force and resistive force proportional to its squared velocity (quadratic friction). In addition, the body is exposed to the braking moment of the linear resistance of the medium.

1. MODELING

Consider the controlled rotational motion of a dynamically symmetric rigid body with the moving mass of relatively small linear dimensions that is connected to it (at a point on its symmetry axis) through elastic coupling with quadratic dissipation. In addition, the body is exposed to a small braking moment of the linear resistance of the medium.

Based on the approach [4], asymptotically approximate equations of controlled rotational motion in the coordinate system associated with the body (Euler's dynamic equations) are written as follows:

$$\dot{\mathbf{G}} + \boldsymbol{\omega} \times \mathbf{G} = \mathbf{M}^u + \mathbf{M}^v + \mathbf{M}^r. \quad (1.1)$$

Here, $\mathbf{M}^{\mathbf{u}}$ is a vector of the external control moment, $\mathbf{M}^{\mathbf{v}}$ is a vector of the internal disturbance moment (which is due to the elasticity and quadratic friction of the damper), and $\mathbf{M}^{\mathbf{r}}$ is the moment of resistance of the medium [9–11]. The vector $\mathbf{G} = \mathbf{J}\boldsymbol{\omega}$ is a kinetic momentum of the body, where $\mathbf{J} = \mathrm{diag}(A_1, A_1, A_3)$ is a constant inertia tensor of the undisturbed body and $\boldsymbol{\omega} = (p, q, r)$ is an angular velocity vector given by its projections onto the body-fixed axes. The modulus of the kinetic momentum has the form

$$G = |\mathbf{G}| = [A_1^2(p^2+q^2) + A_3^2r^2]^{1/2} \equiv [A_1^2\omega_\perp^2 + A_3^2r^2]^{1/2}, \quad A_1 \neq A_3, \quad \omega_\perp^2 = p^2 + q^2.$$

The magnitude of the control moment is assumed to be small, on the order of ε. The components of the control moment are represented as products b_iu_i, $i = 1,2,3$ [4, 13]:

$$M_i^u = b_iu_i, \quad u_i = -G_iG^{-1}, \quad i = 1,2,3, \quad |\mathbf{u}| \le 1, \tag{1.2}$$

where the constants b_i have the dimension of the moment of force and are quite close (they characterize the efficiency of the control system with respect to each body-fixed axis), while u_i are dimensionless control functions to be determined.

To simplify the solution of the optimal control problem, a structural constraint is introduced in system (1.1). It is assumed that the moment of resistance of the medium is proportional to the kinetic momentum of the body [3, 9, 10]:

$$\mathbf{M}^r = -\lambda\mathbf{J}\boldsymbol{\omega}, \tag{1.3}$$

where λ is a certain constant of proportionality that has the dimension of angular velocity and depends on the properties of the medium and the shape of the body.

Taking into account (1.2) and (1.3), the approximate system of equations for the controlled motion (1.1) in the projections onto the principal central axes of the inertia of the body is written as follows [2–5, 9, 10, 13]:

$$\begin{aligned} A_1\dot{p} + (A_3 - A_1)qr &= -b_1A_1pG^{-1} + FG^2qr + Spr^6\omega_\perp - \lambda A_1p, \\ A_1\dot{q} + (A_1 - A_3)pr &= -b_2A_1qG^{-1} - FG^2pr + Sqr^6\omega_\perp - \lambda A_1q, \\ A_3\dot{r} = -b_3A_3rG^{-1} - A_1A_3^{-1}&Sr^5\omega_\perp^3 - \lambda A_3r, \quad 0 < A_3 \le 2A_1, \quad A_3 \neq A_1. \end{aligned} \tag{1.4}$$

Note that, for $b_1 = b_2 = b_3 = b$, where b can be a function of time, control (1.2) is optimal. If b_i are close, then this law is quasi-optimal. The designations F and S introduced into (1.4) are expressed in terms of the system's parameters as follows [2, 3]:

$$F = m\rho^2\Omega^{-2}A_1^{-3}A_3, \quad S = m\rho^3\Lambda\Omega^{-3}d|d|A_1^{-4}A_3^4, \quad d = 1 - A_3A_1^{-1}. \tag{1.5}$$

The coefficients F and S characterize the moments of force that are due to the elastic element. Here, m is the mass of the moving point and ρ is a radius vector for the fixing point of the moving mass on the axis of dynamic symmetry of the body. The constants $\Omega^2 = c/m$, $\lambda_1 = \mu/m = \Lambda\Omega^3$, and $\Omega \gg \omega_0$ determine the frequency of oscillations and their rate of decay; c is rigidity; μ is the coefficient of quadratic friction; and ω_0 is the modulus of the initial angular velocity.

Consider the case where the coupling coefficients λ_1 and Ω are such that the free oscillations of the point, which are due to the initial deviations, decay much faster than the body can complete its rotation. In this case, the motion of the rigid body is similar to the Euler–Poinsot motion, and the relative oscillations caused by this motion are small.

The inequality $\Omega \gg \omega_0$ allows us to introduce a small parameter into (1.5) and assume that disturbance moments are small in order to apply the averaging method. In this case, we do not exclude the possibility of the initial transition.

The problem of the quasi-optimal (in terms of time) braking of rotations is formulated as follows:

$$\boldsymbol{\omega}(T) = 0, \; T \to \min_{|\mathbf{u}|\le 1}. \tag{1.6}$$

The parameters b_i are assumed to be close ($b_i \approx b$, $|b_i - b| \ll b$).

2. ASYMPTOTIC APPROACH TO THE PROBLEM OF QUASI-OPTIMAL BRAKING

First, make the problem dimensionless. For definiteness, select the moment of inertia of the body with respect to the axis $x_1 - A_1 = A_2$ and the variable on the order of its initial velocity ω_0 as characteristic parameters for the problem at hand. Introduce the dimensionless inertia coefficients $\tilde{A}_i = A_i/A_1$ and dimensionless time $\tau = \omega_0 t$.

System (1.4) is rewritten as follows:

$$\begin{gathered}\frac{d\tilde{p}}{d\tau} = -(\tilde{A}_3 - 1)\tilde{q}\tilde{r} - \varepsilon\frac{\tilde{b}_1\tilde{p}}{\tilde{G}} + \varepsilon\tilde{F}\tilde{G}^2\tilde{q}\tilde{r} + \varepsilon\tilde{S}\tilde{p}\tilde{r}^6\tilde{\omega}_\perp - \varepsilon\tilde{\lambda}\tilde{p},\\ \frac{d\tilde{q}}{d\tau} = -(1 - \tilde{A}_3)\tilde{p}\tilde{r} - \varepsilon\frac{\tilde{b}_2\tilde{q}}{\tilde{G}} - \varepsilon\tilde{F}\tilde{G}^2\tilde{p}\tilde{r} + \varepsilon\tilde{S}\tilde{q}\tilde{r}^6\tilde{\omega}_\perp - \varepsilon\tilde{\lambda}\tilde{q},\\ \frac{d\tilde{r}}{d\tau} = -\varepsilon\frac{\tilde{b}_3\tilde{r}}{\tilde{G}} - \varepsilon\tilde{S}\tilde{A}_3^{-2}\tilde{r}^5\tilde{\omega}_\perp^3 - \varepsilon\tilde{\lambda}\tilde{r}.\end{gathered} \tag{2.1}$$

Here, taking into account the assumptions made above, the following designations are introduced:

$$\begin{gathered}\varepsilon\tilde{F} = m\rho^2\Omega^{-2}A_1^{-1}\tilde{A}_3\omega_0^2, \quad \varepsilon\tilde{S} = m\rho^3\Lambda\Omega^{-3}(1 - \tilde{A}_3)\left|1 - \tilde{A}_3\right|A_1^{-1}\tilde{A}_3^4\omega_0^6,\\ \varepsilon\tilde{b}_i = b_i/A_1\omega_0^2, \quad \varepsilon\tilde{\lambda} = \lambda/\omega_0, \quad \tilde{G} = G/A_1\omega_0, \quad \tilde{A}_1 = \tilde{A}_2 = 1.\end{gathered} \tag{2.2}$$

Below, we write the dimensionless variables without ~.

Using the general generating solution of system (2.1) for $\varepsilon = 0$, we have

$$p = a\cos\psi, \quad q = a\sin\psi, \quad a > 0, \quad r = \text{const} \neq 0. \tag{2.3}$$

Here, $\psi = (A_3 - 1)r\tau + \psi_0$ is the oscillation phase of the equatorial component of the angular velocity vector.

Substitute (2.3) into the third equation of system (2.1). For the first two equations of (2.1), we take into account that $a^2 = p^2 + q^2$ and $\dot{a} = \dot{p}\cos\psi + \dot{q}\sin\psi$. Average the obtained system of equations for a and r over the phase ψ. Upon introducing the slow argument $\theta = \varepsilon\tau$ and averaging, the system takes the form

$$\begin{gathered}a' = -\frac{a}{2}[G^{-1}(b_1 + b_2) - 2Sr^6a + 2\lambda],\\ r' = -r(b_3G^{-1} + SA_3^{-2}r^4a^3 + \lambda), \quad ' = d/d\theta.\end{gathered} \tag{2.4}$$

The expressions containing F have a zero mean. Note that, with $b_1 = b_2 = b_3 = b$, the equations for a and r are fully integrated; this optimal control problem was analytically solved in [5, 12].

3. APPROXIMATE SOLUTION

Consider a special case

$$0.5(b_1 + b_2) = b_3 = b. \tag{3.1}$$

Multiply the first equation in (2.1) by p, the second equation by q, and the third one by A_3^2r; then, sum them up. Upon averaging, obtain

$$G' = -b - \lambda G. \tag{3.2}$$

The initial and final conditions have the form

$$G(0) = G^0, \quad G(T, \theta_0, G^0) = 0, \quad T = T(\theta_0, G^0). \tag{3.3}$$

Taking into account conditions (3.3), the solution to Eq. (3.2) is written as

$$G(\theta) = -\frac{b}{\lambda} + \left(G^0 + \frac{b}{\lambda}\right)\exp(-\lambda\theta), \quad \Theta = \frac{1}{\lambda}\ln\left(G^0\frac{\lambda}{b} + 1\right). \tag{3.4}$$

Note that $\Theta \to \infty$ at $G^0/b \to \infty$ for different λ; in turn, $\Theta \to 0$ at $G^0\lambda/b \to 0$ (λ is arbitrary) or at $\lambda \to \infty$.

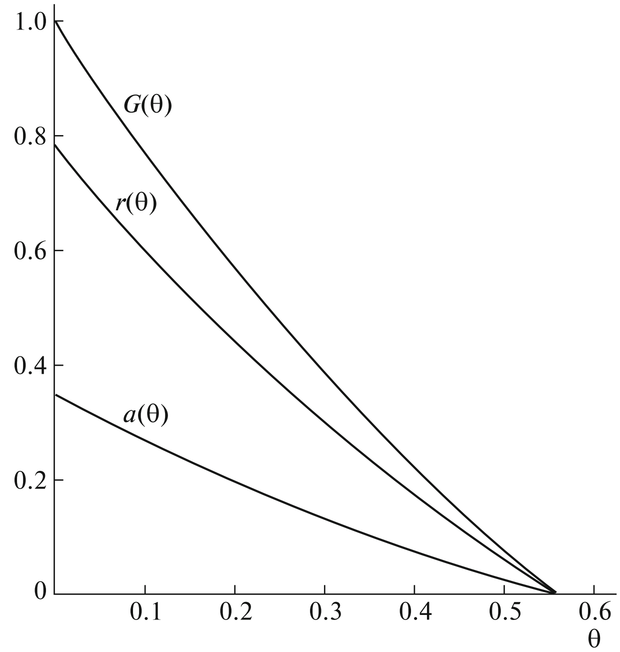

Fig. 1.

For system (2.4), under condition (3.1), substitute $r = \eta G$ and $a = \alpha G$. In this case, the equations of system (2.4) take the following form:

$$\alpha' = S\alpha^2\eta^6G^7, \quad \eta' = -A_3^{-2}S\alpha^3\eta^5G^7. \tag{3.5}$$

Divide the first equation by the second one to obtain

$$\frac{d\alpha}{d\eta} = -\frac{A_3^2\eta}{\alpha}.$$

Find the first integral C_1:

$$\eta^2 = 2C_1 - A_3^{-2}\alpha^2, \quad C_1 = \frac{1}{2}A_3^{-2}. \tag{3.6}$$

Substitute η^2 from (3.6) into the first equation of system (3.5):

$$\frac{d\alpha}{d\theta} = SA_3^{-6}G^7(1-\alpha^2)^3\alpha^2, \quad G_0 = 1. \tag{3.7}$$

When substituting the expression for G (3.4) into Eq. (3.7) for α, the latter is integrated, and its solution is written as follows [14]:

$$\begin{gathered} A_3^6\left[-\frac{1}{\alpha} + \frac{\alpha}{4(1-\alpha^2)^2} + \frac{7\alpha}{8(1-\alpha^2)} + \frac{15}{16}\ln\left|\frac{1+\alpha}{1-\alpha}\right|\right] \\ = S\left[-\frac{b^7}{\lambda^7}\theta - \frac{7b^6}{\lambda^7}b_*\exp(-\lambda\theta) + \frac{21b^5}{2\lambda^6}b_*^2\exp(-2\lambda\theta)\right. \\ -\frac{35b^4}{3\lambda^5}b_*^3\exp(-3\lambda\theta) + \frac{35b^3}{4\lambda^4}b_*^4\exp(-4\lambda\theta) - \frac{21b^2}{5\lambda^3}b_*^5\exp(-5\lambda\theta) \\ \left. + \frac{7b}{6\lambda^2}b_*^6\exp(-6\lambda\theta) - \frac{1}{7\lambda}b_*^7\exp(-7\lambda\theta)\right] + C_2, \\ b_* = G_0 + \frac{b}{\lambda} = 1 + \frac{b}{\lambda}. \end{gathered} \tag{3.8}$$

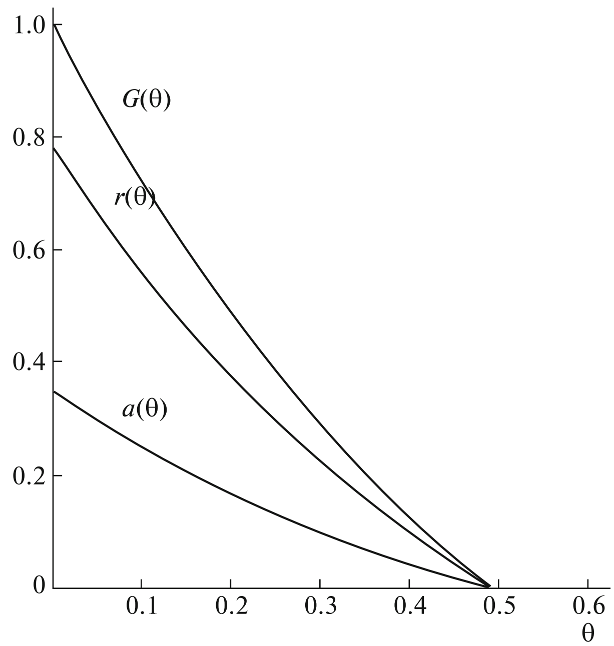

Fig. 2.

The second integration constant C_2 is found from the initial condition ($\theta = 0$ and $\alpha = \alpha_0$):

$$C_2 = \left[-\frac{1}{\alpha_0} + \frac{\alpha_0}{4(1-\alpha_0^2)^2} + \frac{7\alpha_0}{8(1-\alpha_0^2)} + \frac{15}{16}\ln\left|\frac{1+\alpha_0}{1-\alpha_0}\right|\right]A_3^6$$
$$- S\left[-\frac{7b^6}{\lambda^7}b_* + \frac{21b^5}{2\lambda^6}b_*^2 - \frac{35b^4}{3\lambda^5}b_*^3 + \frac{35b^3}{4\lambda^4}b_*^4 - \frac{21b^2}{5\lambda^3}b_*^5 + \frac{7b}{6\lambda^2}b_*^6 - \frac{1}{7\lambda}b_*^7\right].$$

4. NUMERICAL SOLUTION

To solve system (2.4), we carried out some numerical investigations for the renormalized initial conditions $G_0 = 1$, $A_3 = 1.2$, $a_0 = 0.35$, and $r_0 = (1 - a_0^2)^{1/2}/A_3$; two values of the renormalized resistance coefficient $\lambda = 1.2$ and 1.8; the coefficients of the control moment $b_1 = 1.625$, $b_2 = 1$, and $b_3 = 1.25$, where $0.5(b_1 + b_2) \neq b_3$; and the coefficient $S = 1$. The parameters were selected in such a way so as to satisfy the conditions $A_3 \leq 2$ and $a_0 < r_0$. To plot the modulus of the kinetic momentum, we used the expression $G = |\mathbf{G}| = (a^2 + A_3^2 r^2)^{1/2}$.

Figures 1 and 2 show the behavior of the functions a, r, and G for $\lambda = 1.2$ and $\lambda = 1.8$, respectively. It can be seen that the rigid body brakes faster with increasing resistance (see Fig. 2). The braking time is $T \simeq 0.55$ in the first case and $T \simeq 0.49$ in the second case.

CONCLUSIONS

In this paper, we have addressed the problem of the quasi-optimal (in terms of time) braking of the rotations of a dynamically symmetric rigid body with a moving mass coupled to it through a quadratic-dissipation damper in a resisting medium. Using the asymptotic approach, we have derived the averaged system of equations, determined the braking time for the adopted numerical values of the dimensionless parameters, and plotted the behavior of the kinetic momentum and variables a and r for the equatorial and axial components of the angular velocity vector of the quasi-rigid body.

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